FIXED POINT THEORY FOR CONTRACTIVE MAPPINGS SATISFYING $\Phi$-MAPS IN GENERALIZED CONE D-METRIC SPACES

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Abstract

In this paper we introduce cone D-metric spaces under some contractive conditions related to a nondecreasing map $\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\lim_{t \rightarrow +\infty} \phi(t) = 0$, for all $t \in (0, +\infty)$. Also we prove some fixed point theorems on the cone D-metric spaces with $\phi$ maps.


Keywords: Cone metric, D-metric spaces, Cone metric space, mapping $\phi$.

1. INTRODUCTION:

A generalized metric space or D-metric space introduced by Dhage in [2] and [3]. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. By increasing the number of factors Rhoades [4] generalized Dhage’s contractive condition and proved the existence of a unique fixed point of a self-map in a D-metric space. Recently, Huang and Zhang [1] defined cone metric spaces and generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space $E$ and obtained some fixed point theorems for mappings satisfying different contractive conditions. Our main aim is to prove some results on cone D-metric spaces under some contractive conditions related to a nondecreasing map $\phi: [0, +\infty) \rightarrow [0, +\infty)$ with $\lim_{t \rightarrow +\infty} \phi(t) = 0$, for all $t \in (0, +\infty)$.

2. PRELIMINARIES:

Definition 2.1: Let $E$ always be a real Banach space and $P$ a subset of $E$. Then $P$ is called a cone if
(i) $P$ is closed, non-empty and $P \neq 0$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b$.
(iii) $P \cap (-P) = 0$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $x - y \in P$. $x \ll y$ will stand for $x - y \in \text{int}(P)$, where $\text{int}(P)$ denotes the interior of $P$ [1].

Definition 2.2: The cone $P$ is called normal if there is a number $M > 0$ such that for all $x, y$ in $E$, $0 \leq x \leq y$ implies $\|x\| \leq M \|y\|$
The least positive number satisfying above is called the normal constant of $P$ [1]. It is clear that $M \geq 1$.

In the following, let $E$ be a normed linear space, $P$ be a cone in $E$ satisfying $\text{int}(P) \neq \phi$ and $\geq$ denote the partial ordering on $E$ with respect to $P$.

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Definition 2.3: Let $X$ be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:
(a) $0 \leq d(x, y)$ for all $x, y$ in $X$ and $d(x, y) = 0$ if and only if $x = y$,
(b) $d(x, y) = d(y, x)$ for all $x, y$ in $X$,
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z$ in $X$. Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space $[1]$.

Example 2.4: Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by
$$d(x, y) = (|x - y|, \alpha|x - y|),$$
where $\alpha \geq 0$ is constant. Then $(X, d)$ is a cone metric space $[1]$.

Definition 2.5: $[2]$ Let $X$ be a nonempty set, a D-metric space is a function $D : X \times X \times X \to R$ defined on $X$ such that for any $x, y, z, a$ in $X$
(i) $D(x, y, z) = 0$ if and only if $x = y = z$ for each $x, y, z$ in $X$,
(ii) $D(x, y, z) = D(\pi(x, y, z))$, $\pi$ is a permutation,
(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$.

Definition 2.6: Let $X$ be a nonempty set, a strong D-metric space is a function $D : X \times X \times X \to R^+$ defined on $X$ such that for any $x, y, z, a$ in $X$
(i) $D(x, y, z) = 0$ if and only if $x = y = z$ for each $x, y, z$ in $X$,
(ii) $D(x, y, z) = D(\pi(x, y, z))$, $\pi$ is a permutation,
(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z)$.

Lemma 2.9: Any Strong cone D-metric space is a cone D-metric space but the converse is not true in general. Since the strong cone D-metric leads to the cone D-metric, so in the rest of the article we consider both the (strong) cone D-metric space and the cone D-metric space to prove the main results.

Example 2.10: Let $E = R^3$, $P = \{(x, y, z) \in E : x, y, z \geq 0\}$, $X = R$. Define $D : X \times X \times X \to E$, by
$$D(x, y, z) = (|x - y|, |y - z|, |x - z|)$$
a cone D-metric space.

Example 2.11: Let $(X, d)$ denotes cone metric space on $X$ and define
$$D(x, y, z) = (d(x - y), d(y - z), d(x - z))$$
So, $(X, D)$ is a cone D-metric space on $X$.

Definition 2.12: Let $(X, D)$ be a cone D-metric space on $X$, $x \in X$, and $\{x_n\}$ be a sequence in $X$ then $\{x_n\}$ is called converge sequence to some fixed $x \in X$ if for each $c \in E : c << 0$ and $N$ be natural number, $D(x_n, x, x) << c$ for all $n, m > N$. We can write $x_n \to x$, if $\{x_n\}$ converge to $x$. And $\{x_n\}$ is called a Cauchy sequence if $D(x_n, x_m, x_p) << c$ for all $n, m, p > N$.

Definition 2.13: A cone D-metric space on $(X, D)$ is complete if every Cauchy sequence in $X$ is convergent.

Proposition 2.14: Let $(X, D)$ be a cone D-metric space on $X$, then the following are equivalent
(1) $\{x_n\}$ is convergent to $x$,
(2) $D(x_n, x, x) << c$ for each $n, m > N$,
(3) $D(x_n, x, x) << c$ for each $n > N$.

MAIN RESULTS:

Following to Matkowski $[6]$, let $\Phi$ be the set of all functions $\phi$ such that $\phi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function with $\lim_{t \to +\infty} \phi'(t) = 0$ for all $t \in (0, +\infty)$. If $\Phi \in \phi$, then $\phi$ is called $\Phi$-map. If $\phi$ is $\Phi$-map, then it is an easy matter to show that
(1) $\phi(t) < t$, for all $t \in (0, +\infty)$,
(2) $\phi(0) = 0$. 

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In this section we prove some fixed point theorems on the D-cone metric spaces with \( \varphi \)-maps.

**Lemma 2.15:** For any natural numbers \( l, n \) and \( m \), we get \( D(x_l, x_n, x_m) \leq D(x_l, x_n, x_m) \).

**Theorem 2.16:** Let \((X, D)\) be a complete cone D-metric space, \( P \) be a cone normal, suppose the mapping \( T : X \to X \) satisfies the contractive condition, \( D(Tx, Ty, Tz) \leq \phi(D(x, y, z)) \), for all \( x, y, z \in X \). Then \( T \) has a unique fixed point in \( X \).

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \), define the iterative sequence \( \{x_n\} \) by \( x_1 = Tx_0 \), \( x_2 = T^2x_0 \), ..., \( x_{n+1} = T^nx_0 = T^{n+1}x_0 \). So, we have \( D(x_{n+1}, x_{n+1}, x_n) = D(Tx_n, Tx_n, Tx_{n+1}) \leq \phi(D(x_n, x_n, x_{n+1})) \). \[ \leq \varphi(D(x_{n+1}, x_{n+1}, x_{n+2})) \] \[ \leq \ldots \] \[ \leq \varphi(D(x_1, x_1, x_2)) \] given \( \varepsilon > 0 \), since \( \lim_{x \to c} \varphi(D(x, x, x)) = 0 \) and \( \phi(\varepsilon) < \varepsilon \), there is an integer \( k_0 \) such that \( \varphi(D(x_1, x_1, x_2)) < \varepsilon - \phi(\varepsilon) \), for all \( n \geq k_0 \).

Hence \[ \varphi(D(x_{n+1}, x_{n+1}, x_{n+2})) < \varepsilon - \phi(\varepsilon) \], for all \( n \geq k_0 \). \[ (1) \]

For \( m, n \in N \) with \( n > m \), we claim that \[ \varphi(D(x_n, x_n, x_m)) < \varepsilon - \phi(\varepsilon) \], for all \( n \geq m > k_0 \). \[ (2) \]

We prove Inequality (2) by induction on \( n \). Inequality (2) holds for \( n = m+1 \) by using Inequality (1) and the fact that \( \varepsilon - \phi(\varepsilon) < \varepsilon \). Assume Inequality (2) holds for \( n = k \). For \( n = k + 1 \), we have \[ D(x_{k+1}, x_{k+1}, x_m) \leq D(x_{m+1}, x_{m+1}, x_m) + D(x_{k+1}, x_{k+1}, x_{m+1}) \] \[ < \varepsilon - \phi(\varepsilon) + \varphi(D(x_1, x_k, x_m)) \] \[ < \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) \] \[ = \varepsilon \]

By induction on \( n \), we conclude that Inequality (2) holds for all \( n \geq m \geq k_0 \). So \( \{x_n\} \) is Cauchy since \( X \) is complete metric space, there exist a point \( x \in X \) such that \( x_n \to x \).

To show \( x \) is a fixed point of the mapping \( T \). Consider \[ D(Tx, Tx, x) < D(Tx, Tx, Tx) + D(Tx, Tx, x) \]

By lemma 2.15, we get \[ D(Tx, Tx, x) < D(Tx, Tx, Tx) + D(Tx, Tx, x) \] .

This gives \[ D(Tx, Tx, x) \leq D(x_n, x, x) + D(x_{n+1}, x_{n+1}, x) \] \[ \leq D(x_n, x, x) \] \[ < D(x_n, x, x) + \varphi(D(x_n, x, x)) \] \[ < D(x_n, x, x) + \varphi(D(x_n, x, x)) \] \[ = \varepsilon \]

since \( \{x_n\} \) is a Cauchy sequence in the complete D-cone metric space, there exist \( c \ll 0 \) such that \( D(x_n, x, x) < c \) and so \( D(x_{n+1}, x, x) \to 0 \), as \( n \to \infty \) and similarly \( D(x_n, x, x) < c \), \( D(x_n, x, x) \to 0 \), as \( n \to \infty \), then \( D(Tx, Tx, x) \to 0 \) and we have \( Tx = x \). This show that \( x \) is a fixed point of \( T \).
UNIQUENESS: If $y$ is another fixed point,

\[ D(x, y, y) = D(Tx, Ty, y) \leq \phi(D(x, y, y)) < D(x, y, y) \]

which is a contradiction. So $x = y$, and hence $T$ has a unique fixed point.

**Corollary 2.17:** Let $(X, D)$ be a complete cone $D$-metric space, $P$ be a cone normal, suppose the mapping $T : X \to X$ satisfy the contractive condition, thus for $m \in N$, $D(T^n x, T^n x, T^n x) \leq \phi(x, y, z)$, for all $x, y, z \in X$. Then $T$ has a unique fixed point in $X$.

**Proof:** From Theorem 2.16, we get $T^m$ has a unique fixed point say $x$. Since $T(x) = T(T^n x) = T^{n+1} x = T^n (Tx)$, also we have $Tx$ is a fixed point for $T^m$. By uniqueness of $x$, we get $Tx = x$.

**Corollary 2.18:** Let $(X, D)$ be a complete cone $D$-metric space, $P$ be a cone normal, suppose the mapping $T : X \to X$ satisfy the contractive condition, thus for $m \in N$, $D(Tx, Ty, Tz) \leq \phi(D(x, y, z))$, for all $x, y, z \in X$. Then $T$ has a unique fixed point in $X$.

**Proof:** Define $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(t) = \frac{t}{1+t}$. Then it is clear that $\phi(t) = \frac{t}{1+t}$ is a nondecreasing function with $\lim_{n \to \infty} \phi^n(t) = 0$, for all $t > 0$. Since $D(Tx, Ty, Tz) \leq \phi(D(x, y, z))$, for all $x, y, z \in X$, the result follows from Theorem 2.16.

**Theorem 2.19:** Let $(X, D)$ be a complete cone $D$-metric space, $P$ be a cone normal, suppose the mapping $T : X \to X$ satisfy the contractive condition, $D(Tx, Ty, Tz) \leq \phi(\max\{D(x, y, z), D(Tx, Tx, x), D(Ty, Ty, y), D(Tx, y, z)\})$, for all $x, y, z \in X$. Then $T$ has a unique fixed point in $X$.

**Proof:** Let $x_0$ be an arbitrary point in $X$, define the iterative sequence \( \{x_n\} \) by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$, ..., $x_{n+1} = Tx_n = T^{n+1} x_0$. So, we have

\[
D(x_n, x_{n+1}, x_{n+1}) = D(Tx_n, Tx_n, Tx_n) \\
\leq \phi(\max\{D(x_n, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1})\}) \\
\leq \phi(\max\{D(x_n, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1})\})
\]

If $\max\{D(x_n, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1})\} = D(x_n, x_{n+1}, x_{n+1})$, then

\[
D(x_n, x_{n+1}, x_{n+1}) \leq \phi(D(x_n, x_{n+1}, x_{n+1})) \\
< D(x_n, x_{n+1}, x_{n+1}),
\]

which is impossible. So we must have

\[
\max\{D(x_n, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1}), D(x_{n+1}, x_{n+1}, x_{n+1})\} = D(x_n, x_{n+1}, x_{n+1}),
\]

and hence

\[
D(x_n, x_{n+1}, x_{n+1}) \leq \phi(D(x_n, x_{n+1}, x_{n+1})),
\]

for $n \in N$.

\[
D(x_{n+1}, x_{n+1}, x_{n+1}) = D(Tx_n, Tx_n, Tx_n) \\
\leq \phi(D(x_{n+1}, x_{n+1}, x_{n+1})) \\
\leq \phi(\phi(D(x_n, x_{n+1}, x_{n+1}))) \\
\leq \phi^{2}(D(x_n, x_{n+1}, x_{n+1})) \\
\leq \phi^{n}(D(x_0, x_1, x_1)).
\]

Using proof of Theorem 2.16, we can show that \( \{x_n\} \) is a Cauchy sequence, and since $X$ is complete cone $D$-metric space, there exist a point $x$ in $X$ such that $x_n \to x$. For $n \in N$, we have
\[ D(x, x, Tx) = D(x, x, x) + D(x, x, x) + \phi(\max\{D(x, x, x), D(x, x, x), D(x, x, x), D(x, x, x)\}). \]

**Case 1:**
\[
\max\{D(x, x, x), D(x, x, x), D(x, x, x)\} = D(x, x, x).
\]
Letting \( n \to \infty \), we conclude that \( D(x, x, Tx) = 0 \), and hence \( Tx = x \).

**Case 2:**
\[
\max\{D(x, x, x), D(x, x, x), D(x, x, x), D(x, x, x)\} = D(x, x, x).
\]
Letting \( n \to \infty \), we conclude that \( D(x, x, Tx) = 0 \), and hence \( Tx = x \).

**Case 3:**
\[
\max\{D(x, x, x), D(x, x, x), D(x, x, x)\} = D(x, x, x).
\]
Letting \( n \to \infty \), we conclude that \( D(x, x, Tx) = 0 \), and hence \( Tx = x \).

In all cases, we conclude that \( x \) is a fixed point of \( T \).

**For Uniqueness:** Let \( y \) be any other fixed point of \( T \) such that \( x \neq y \). Then
\[
D(x, y, y) = (\max\{D(x, y, y), D(x, x, x), D(y, y, y), D(x, y, y)\}) 
\leq \phi(D(x, y, y))
\leq D(x, y, y).
\]
which is a contradiction since \( \phi(D(x, y, y)) < D(x, y, y) \). Therefore, \( D(x, y, y) = 0 \) and hence \( x = y \).

**Corollary 2.20:** Let \( (X, D) \) be a complete cone D-metric space, \( P \) be a cone normal, suppose there is \( k \in [0, 1) \) such that the map mapping \( T : X \to X \) satisfy the contractive condition,
\[
D(Tx, Ty, Tz) \leq k(\max\{D(x, y, z), D(x, Tx, Ty), D(y, Ty, Tz), D(Tx, y, z)\}) \text{, for all } x, y, z \in X \text{. Then } T \text{ has a unique fixed point in } X \text{.}
\]

**Proof:** Define \( \phi : [0, +\infty) \to [0, +\infty) \) with \( \phi(t) = kt \). Then it is clear that \( \phi(t) \) is a nondecreasing function with \( \lim_{t \to \infty} \phi(t) = 0 \), for all \( t > 0 \). Since \( D(Tx, Ty, Tz) \leq \phi(\max\{D(x, y, z), D(x, Tx, Ty), D(y, Ty, Tz), D(Tx, y, z)\}) \), for all \( x, y, z \in X \), then result follows from Theorem 2.19.

**Corollary 2.21:** Let \( (X, D) \) be a complete cone D-metric space, \( P \) be a cone normal, suppose the map the mapping \( T : X \to X \) satisfy the contractive condition,
\[
D(Tx, Ty, Tz) \leq k(\max\{D(x, y, z), D(x, Tx, Ty), D(y, Ty, Tz), D(Tx, y, z)\}) \text{, for all } x, y, z \in X \text{. Then } T \text{ has a unique fixed point in } X \text{.}
\]

**Proof:** It follows from Theorem 2.19 by replacing \( z = y \).

**REFERENCES:**


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