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FIXED POINT THEORY FOR CONTRACTIVE MAPPINGS SATISFYING Φ- MAPS IN GENERALIZED CONE D-METRIC SPACES

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ABSTRACT

In this paper we introduce cone D-metric spaces under some contractive conditions related to a nondecreasing map $\phi:[0,+\infty)\to[0,+\infty)$ *with* $\lim_{n\to+\infty}\phi^n(t)=0$, *for all* $t\in(0,+\infty)$. *Also we prove some fixed point theorems on the cone D-metric spaces with* ϕ *maps.*

Mathematics subject classification: *54H25, 55H20.*

Keywords: *Cone metric, D-metric spaces, Cone metric space, mapping* ϕ.

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1. INTRODUCTION:

A generalized metric space or D-metric space introduced by Dhage in [2] and [3]. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. By increasing the number of factors Rhoades [4] generalized Dhage's contractive condition and proved the existence of a unique fixed point of a self-map in a D-metric space. Recently, Huang and Zhang [1] defined cone metric spaces and generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space E and obtained some fixed point theorems for mappings satisfying different contractive conditions. Our main aim is to prove some results on cone D-metric spaces under some contractive conditions related to a nondecreasing map $\phi:[0,+\infty)\to[0,+\infty)$ with $\lim_{n\to+\infty}\phi^n(t)=0$, for all $t\in(0,+\infty)$.

2. PRELIMINARIES:

Definition 2.1: Let E always be a real Banach space and P a subset of E. Then P is called a cone if
(i) P is closed, non-empty and $P\neq 0$,
(ii) $ax+by\in P$ for all $x,y\in P$ and non-negative real numbers a, b.
(iii) $P\cap(-P)=0$.

For a given cone $P\subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x\leq y$ if and only if $x-y\in P$. $x<<y$ will stand for $x-y\in \mathrm{int}(P)$, where $\mathrm{int}(P)$ denotes the interior of P [1].

Definition 2.2: The cone P is called normal if there is a number M > 0 such that for all x, y in E, $0\leq x\leq y$ implies

$$\|x\|\leq M\|y\|$$

The least positive number satisfying above is called the normal constant of P [1]. It is clear that $M\geq 1$.

In the following, let E be a normed linear space, P be a cone in E satisfying $int(p)\neq\phi$ and $\geq$' denote the partial ordering on E with respect to P.

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Definition 2.3: Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:
(a) $0 \le d(x, y)$ for all x, y in X and d(x, y) = 0 if and only if x = y,
(b) d(x, y) = d(y, x) for all x, y in X ,
(c) d(x, y) ≤ d(x, z) + d(z, y) for all x, y, z in X. Then d is called a cone metric on X, and (X, d) is called a cone metric space [1].

Example 2.4: Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \ge 0$ is constant. Then (X, d) is a cone metric space [1].

Definition 2.5: [2] Let X be a nonempty set, a D-metric space is a function $D : X \times X \times X \to R^+$ defined on X such that for any x, y, z, a in X
(i) D(x, y, z) = 0 if and only if x = y = z for each x, y, z in X,
(ii) D(x, y, z) = D(p(x, y, z)), p is a permutation,
(iii) $D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$.

Definition 2.6: Let X be a nonempty set, a strong D-metric space is a function $D : X \times X \times X \to R^+$ defined on X such that for any x, y, z, a in X
(i) D(x, y, z) = 0 if and only if x = y = z for each x, y, z in X,
(ii) D(x, y, z) = D(p(x, y, z)), p is a permutation,
(iii) $D(x, y, z) \le D(x, y, a) + D(x, a, z)$.

Lemma 2.9: Any Strong cone D-metric space is a cone D-metric space but the converse is not true in general. Since the strong cone D-metric leads to the cone D-metric, so in the rest of the article we consider both the (strong) cone D-metric space and the cone D-metric space to prove the main results.

Example 2.10: Let $E = R^3$, $P = \{(x, y, z) \in E : x, y, z \geq 0\}$, $X = R$. Define $D : X \times X \times X \to E$, by

$$D(x, y, z) = (|x - y|, |y - z|, |x - z|)$$

a cone D-metric space.

Example 2.11: Let (X, d) denotes cone metric space on X and define

$$D(x, y, z) = (d(x - y), d(y - z), d(x - z))$$

So, (X, D) is a cone D-metric space on X.

Definition 2.12: Let (X, D) be a cone D-metric space on X, $x \in X$, and $\{x_n\}$ be a sequence in X then $\{x_n\}$ is called converge sequence to some fixed $x \in X$ if for each $c \in E$, $c << 0$ and N be natural number, $D(x_n, x_m, x) << c$ for all n, m > N. We can write $x_n \to x$, if $\{x_n\}$ converge to x. And $\{x_n\}$ is called a Cauchy sequence if $D(x_n, x_m, x_p) << c$ for all n, m, p > N.

Definition 2.13: A cone D-metric space on (X, D) is complete if every Cauchy sequence in X is convergent.

Proposition 2.14: Let (X, D) be a cone D-metric space on X, then the following are equivalent
(1) $\{x_n\}$ is convergent to x,
(2) $D(x_n, x_m, x) << c$ for each n, m > N,
(3) $D(x_n, x_n, x) << c$ for each n > N.

MAIN RESULTS:

Following to Matkowski [6], let Φ be the set of all functions ϕ such that $\phi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function with $\lim_{n \to +\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\Phi \in \phi$, then ϕ is called Φ-map. If ϕ is Φ-map, then it is an easy matter to show that

(1) $\phi(t) < t$, for all $t \in (0, +\infty)$,
(2) $\phi(0) = 0$.

In this section we prove some fixed point theorems on the D-cone metric spaces with ϕ-maps.

Lemma 2.15: For any natural numbers l, n and m, we get $D(x_l, x_m, x_n) \le D(x_l, x_l, x_n)$.

Theorem 2.16: Let (X, D) be a complete cone D-metric space, P be a cone normal, suppose the mapping $T:X \to X$ satisfy the contractive condition, $D(Tx, Ty, Tz) \le \phi(D(x, y, z))$, for all x, y, z in X. Then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X, define the iterative sequence $\{x_n\}$ by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$, ..., $x_{n+1} = Tx_n = T^{n+1} x_0$. So, we have

$$\begin{aligned} D(x_{n+1}, x_{n+1}, x_n) &= D(Tx_n, Tx_n, Tx_{n-1}) \\ &\le \phi(D(x_n, x_n, x_{n-1})) \\ &\le \phi^2(D(x_{n-1}, x_{n-1}, x_{n-2})) \\ &\le .... \\ &\le \phi^n(D(x_1, x_1, x_2)) \end{aligned}$$

given $\varepsilon > 0$, since $\lim_{n\to+\infty} \phi^n(D(x_1, x_1, x_0)) = 0$ and $\phi(\varepsilon) < \varepsilon$, there is an integer k_0 such that

$$\phi^n(D(x_1, x_1, x_0)) < \varepsilon - \phi(\varepsilon), \text{ for all } n \ge k_0.$$

Hence

$$\phi^n(D(x_{n+1}, x_{n+1}, x_n)) < \varepsilon - \phi(\varepsilon), \text{ for all } n \ge k_0. \tag{1}$$

For $m, n \in N$ with n > m, we claim that

$$\phi^n(D(x_n, x_n, x_m)) < \varepsilon - \phi(\varepsilon), \text{ for all } n \ge m > k_0. \tag{2}$$

We prove Inequality (2) by induction on n. Inequality (2) holds for n = m+1 by using Inequality (1) and the fact that $\varepsilon - \phi(\varepsilon) < \varepsilon$. Assume Inequality (2) holds for n = k. For n = k + 1, we have

$$\begin{aligned} D(x_{k+1}, x_{k+1}, x_m) &< D(x_{m+1}, x_{m+1}, x_m) + D(x_{k+1}, x_{k+1}, x_{m+1}) \\ &< \varepsilon - \phi(\varepsilon) + \phi(D(x_k, x_k, x_m)) \\ &< \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) \\ &= \varepsilon \end{aligned}$$

By induction on n, we conclude that inequality (2) holds for all $n \ge m \ge k_0$. So $\{x_n\}$ is Cauchy since X is completemetric space, there exist a point x in X such that $x_n \to x$.

To show x is fixed point of the mapping T. Consider

$$D(Tx, Tx, x) < D(Tx, Tx, Tx_n) + D(Tx_n, Tx, x)$$

By lemma 2.15, we get

$$D(Tx, Tx, x) < D(Tx_n, Tx, Tx) + D(Tx_n, Tx_n, x).$$

This gives

$$\begin{aligned} D(Tx, Tx, x) &\le D(x_n, x, x) + D(x_{n+1}, x_{n+1}, x) \\ &\le D(x_n, x, x) + \phi(D(x_n, x_n, x)) \\ &< D(x_n, x, x) + D(x_n, x_n, x) \end{aligned}$$

since $\{x_n\}$ is a Cauchy sequence in the complete D-cone metric space, there exist $c << 0$ such that $D(x_n, x, x) << c$ and so $D(x_n, x, x) \to 0$, *as* $n \to \infty$ and similarly $D(x_n, x_n, x) << c$, $D(x_n, x_n, x) \to 0$, *as* $n \to \infty$, then $D(Tx, Tx, x) \to 0$ and we have Tx = x. This show that x is a fixed point of T.

UNIQUENESS: If y is another fixed point,

$$\begin{aligned} D(x,y,y) &= D(Tx, Ty, y) \\ &\le \phi(D(x,y,y)) \\ &< D(x, y, y) \end{aligned}$$

which is a contradiction. So x = y, and hence T has a unique fixed point.

Corollary 2.17: Let (X,D) be a complete cone D-metric space, P be a cone normal, suppose the mapping $T : X \to X$ satisfy the contractive condition, thus for $m \in N$, $D(T^m x, T^m x, T^m x) \le \phi(x, y, z)$, for all $x, y, z \in X$. Then T has a unique fixed point in X.

Proof: From Theorem 2.16, we get T^m has a unique fixed point say x. Since $T(x) = T(T^m x) = T^{m+1} x = T^m (Tx)$, also we have Tx is a fixed point for T^m. By uniqueness of x, we get Tx = x.

Corollary 2.18: Let (X,D) be a complete cone D-metric space, P be a cone normal, suppose the mapping $T : X \to X$ satisfy the contractive condition, thus for $m \in N$, $D(Tx,Ty,Tz) \le \dfrac{D(x,y,z)}{1 + D(x,y,z)}$, for all $x, y, z \in X$. Then T has a unique fixed point in X.

Proof: Define $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(t) = \dfrac{t}{1+t}$. Then it is clear that $\phi(t) = \dfrac{t}{1+t}$ is a nondecreasing function with $\lim_{n\to\infty} \phi^n(t) = 0$, for all t > 0. Since $D(Tx,Ty,Tz) \le \phi(D(x,y,z))$, for all $x, y, z \in X$, the result follows from Theorem 2.16.

Theorem 2.19: Let (X, D) be a complete cone D-metric space, P be a cone normal, suppose the mapping $T : X \to X$ satisfy the contractive condition, $D(Tx,Ty,Tz) \le \phi(\max\{D(x, y, z), D(Tx, Tx, x), D(Ty, Ty, y), D(Tx, y, z)\})$, for all $x, y, z \in X$. Then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X, define the iterative sequence $\{x_n\}$ by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$, ..., $x_{n+1} = Tx_n = T^{n+1} x_0$. So, we have

$$\begin{aligned} D(x_n, x_{n+1}, x_{n+1}) &= D(Tx_{n-1}, Tx_n, Tx_n) \\ &\le \phi(\max\{D(x_{n-1}, x_n, x_n), D(x_{n-1}, Tx_{n-1}, Tx_{n-1}), D(x_n, Tx_n, Tx_n), D(Tx_{n-1}, x_n, x_n)\}) \\ &\le \phi(\max\{D(x_{n-1}, x_n, x_n), D(x_{n-1}, x_n, x_n), D(x_n, x_{n+1}, x_{n+1}), D(x_n, x_n, x_n)\}) \end{aligned}$$

If

$$\max\{D(x_{n-1}, x_n, x_n), D(x_n, x_{n+1}, x_{n+1}), D(x_n, x_n, x_n)\} = D(x_n, x_{n+1}, x_{n+1}),$$

Then

$$\begin{aligned} D(x_n, x_{n+1}, x_{n+1}) &\le \phi(D(x_n, x_{n+1}, x_{n+1})) \\ &< D(x_n, x_{n+1}, x_{n+1}), \end{aligned}$$

which is impossible. So we must have

$$\max\{D(x_{n-1}, x_n, x_n), D(x_n, x_{n+1}, x_{n+1}), D(x_n, x_n, x_n)\} = D(x_{n-1}, x_n, x_n),$$

and hence

$$D(x_n, x_{n+1}, x_{n+1}) \le \phi(D(x_{n-1}, x_n, x_n)),$$

for $n \in N$

$$\begin{aligned} D(x_{n+1}, x_{n+1}, x_n) &= D(Tx_{n-1}, Tx_n, Tx_n) \\ &\le \phi(D(x_{n-1}, x_n, x_n)) \\ &\le \phi^2(D(x_{n-2}, x_{n-1}, x_{n-1})) \\ &\le \ldots. \\ &\le \phi^n(D(x_0, x_1, x_1)). \end{aligned}$$

Using proof of Theorem 2.16, we can show that $\{x_n\}$ is a Cauchy sequence, and since X is complete cone D-metric space, there exist a point x in X such that $x_n \to x$. For $n \in N$, we have

$$D(x, x, Tx) = D(x, x, x_n)+D(x_n, x_n, Tx)$$
$$< D(x, x, x_n)+\phi(\max\{D(x_{n-1}, x_{n-1}, x), D(x_{n-1}, x_n, x_n), D(x_{n-1}, x_n, x_n), D(x_n, x_{n-1}, x)\}).$$

Case 1:
$\max\{D(x_{n-1}, x_{n-1}, x), D(x_{n-1}, x_n, x_n), D(x_n, x_{n-1}, x)\} = D(x_{n-1}, x_n, x_n)$. we get

$$D(x, x, Tx) < D(x, x, x_n)+D(x_{n-1}, x_n, x_n).$$

Letting $n \to \infty$, we conclude that D(x, x, Tx) = 0, and hence Tx = x.

Case 2:
$\max\{D(x_{n-1}, x_{n-1}, x), D(x_{n-1}, x_n, x_n), D(x_n, x_{n-1}, x)\} = D(x_{n-1}, x_{n-1}, x_n)$. we get

$$D(x, x, Tx) < D(x, x, x_n)+D(x_{n-1}, x_{n-1}, x_n).$$

Letting $n \to \infty$, we conclude that D(x, x, Tx) = 0, and hence Tx = x.

Case 3:
$\max\{D(x_{n-1}, x_{n-1}, x), D(x_{n-1}, x_n, x_n), D(x_n, x_{n-1}, x)\} = D(x_n, x_{n-1}, x)$. we get

$$D(x, x, Tx) < D(x, x, x_n)+D(x_n, x_{n-1}, x)$$
$$< D(x, x, x_n) + D(x_n, x_{n-1}, x_{n-1}) + D(x_{n-1}, x_{n-1}, x).$$

Letting $n \to \infty$, we conclude that D(x, x, Tx) = 0, and hence Tx = x. In all cases, we conclude that x is a fixed point of T.
For Uniqueness: Let y be any other fixed point of T such that $x \neq y$. Then

$$D(x,y,y) = (\max\{D(x, y, y), D(x, x, x), D(y, y, y), D(x, y, y)\})$$
$$\leq \phi(D(x,y,y))$$
$$<D(x, y, y).$$

which is a contradiction since $\phi(D(x, y, y)) < D(x, y, y)$. Therefore, D(x, y, y) = 0 and hence x = y.

Corollary 2.20: Let (X, D) be a complete cone D-metric space, P be a cone normal, suppose there is $k \in [0,1)$ such that the map the mapping $T : X \to X$ satisfy the contractive condition,

$D(Tx,Ty,Tz) \leq k(\max\{D(x, y, z), D(x, Tx, Tx), D(y, Ty, Ty), D(Tx, y, z)\})$, for all $x, y, z \in X$. Then T has a unique fixed point in X.

Proof: Define $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(t) = kt$. Then it is clear that $\phi(t)$ is a nondecreasing function with $\lim_{n\to\infty} \phi^n(t) = 0$, for all t > 0. Since $D(Tx,Ty,Tz) \leq \phi((\max\{D(x, y, y) + D(x, Tx, Tx), D(y, Ty, Ty), D(Tx, y, z)\})$, for all $x, y, z \in X$, then result follows from Theorem 2.19.

Corollary 2.21: Let (X, D) be a complete cone D-metric space, P be a cone normal, suppose the map the mapping $T : X \to X$ satisfy the contractive condition,

$$D(Tx,Ty,Ty) \leq k(\max\{D(x, y, y), D(x, Tx, Tx), D(y, Ty, Ty), D(Tx, y, y)\}),$$

for all $x, y, z \in X$. Then T has a unique fixed point in X.

Proof: It follows from Theorem 2.19 by replacing z = y.

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