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# ANALYTICAL SOLUTION OF TWO-POINT NON-LINEAR BOUNDARY VALUE PROBLEMS IN POROUS CATALYST PARTICLES

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*ABSTRACT*

*Systems of simultaneous second-order non-linear ordinary differential equations with boundary conditions at two points are solved. The non-linear equations in porous catalyst particles are solved analytically using Homotopy perturbation method. Herein, we report the approximat analytical expression of the concentration of the components in terms of dimensionless parameters.*

***Keywords:*** *Mathematical modelling, Non-linear equations, Homotopy perturbation method, Boundary value problems, Porous catalyst.*

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## 1. INTRODUCTION

In the study of problems with mass and heat diffusion, one frequently desires to solve second order non-linear ordinary differential equations involving conditions at two points. Well two point boundary value problems are more difficulties in analytical solution that initial value problems do. The object of this paper is to derive the analytical solution of simultaneous second-order non-linear ordinary equations with boundary conditions at two points.

Three simultaneous non-linear ordinary differential equations describing the steady state of two consecutive reactions occurring non-isothermally within porous catalyst particles. The numerical solution to the corresponding parabolic partial differential equations was based on the one proposed by Saul'yev [2]. In the study of stability problems of reactors and catalyst particles, one has to use the actual capacity terms and the transient solutions are as important as the steady-state solutions.

There are a number of numerical methods for finding the solution of two ordinary differential equations describing a simple reaction in porous particles. This problem is relatively simple, because, as shown by Damkohler [4] and Prater [5], the concentration of reactant can be expressed as a function of temperature, so one only has to solve one differential equation for energy balance. This problem was solved by Weisz and Hicks [6], and Carberry [7] who used digital computers, and by Tinkler and Metzner [8] who used an analog computer. Schilson and Amundson [9] used iterative methods, which require fairly good initial approximations (one or two straight lines) to be heat generation functions in order to obtain the final results. Recently, Shean-Lin Liu [1] used the relaxation method of Henyey [10] to solve a problem of two consecutive first-order irreversible chemical reactions occurring at steady-state with porous catalyst particles. Henyey [10] introduced a relaxation method for the numerical solution of the first-order non-linear partial differential equations describing stellar evolution. J. H. He solved the corresponding difference equations by the Newton-Raphson method.

## 2. MATHEMATICAL FORMULATION OF THE PROBLEM

Suppose that two consecutive first-order reactions, $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ take place in a porous catalyst pellet of spherical shape. The steady-state intraparticle concentrations of $A$ and $B$, and the temperature $T$, can be described by the following dimensionless differential equations [1].

$$\frac{d^2A}{dx^2} + \frac{2}{x}\frac{dA}{dx} = \phi_1^{\,2} A \exp\left[\frac{\alpha_1 T}{(1+T)}\right] \tag{1}$$

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$$\frac{d^2B}{dx^2}+\frac{2}{x}\frac{dB}{dx}=\phi_2^{\,2}B\exp\left[\frac{\alpha_2T}{(1+T)}\right]-\ \phi_1^{\,2}D^{'}A\exp\left[\frac{\alpha_1T}{(1+T)}\right] \tag{2}$$

$$\frac{d^2T}{dx^2}+\frac{2}{x}\frac{dT}{dx}=\beta_1\phi_1^{\,2}A\exp\left[\frac{\alpha_1T}{(1+T)}\right]-\ \beta_2\ \phi_2^{\,2}D^{'}A\exp\left[\frac{\alpha_2T}{(1+T)}\right] \tag{3}$$

Where $A$ and $B$ are the dimensionless concentrations of components $A$ and $B$, respectively, $T$ is the dimensionless temperature, $x$ is the dimensionless radius, and $\phi_i$, $\alpha_i$ and $\beta_i$ $(i=1,2)$ are the kinetic parameters as defined in the notation. The boundary conditions are

$$A=1,\quad B=0,\quad T=0\ ,\quad \text{when}\quad x=1 \tag{4}$$

$$\frac{dA}{dx}=0,\ \frac{dB}{dx}=0,\ \frac{dT}{dx}=0\quad \text{when}\quad x=0 \tag{5}$$

The concentration of component $C(x)$ can be written as

$$C(x)=1-A(x)-B(x) \tag{6}$$

## 3. ANALYTICAL SOLUTION OF THE PROBLEM

Recently, many authors have applied the Homotopy perturbation method (HPM ) to solve the non-linear problem in physics and engineering sciences [16-19]. Recently this method is also used to solve some of the non-linear problem in physical sciences [20-23]. This method is a combination of homotopy in topology and classic perturbation techniques. Ji-Huan He used to solve the Lighthill equation [24], the Diffusion equation [25] and the Blasius equation [26]. The HPM is unique in its applicability, accuracy and efficiency. The HPM uses the imbedding parameter *p* as a small parameter, and only a few iterations are needed to search for an asymptotic solution. Solving equations (1) to (5) using Homotopy-perturbation method (Appendix A) we get the solution as

$$A(x)=\frac{y_1(x)}{x}=\frac{1}{x}\left[\frac{\sinh(\phi_1x)}{\sinh(\phi_1)}+\frac{\phi_1k\alpha_1}{\sinh^2(\phi_1)}\left[\sinh(\phi_1x)-x\sinh(\phi_1)\right]\right] \tag{7}$$

$$B(x)=\frac{y_2(x)}{x}=\frac{1}{x}\left[\begin{array}{l}\dfrac{\phi_1^{\,2}D^{'}}{(\phi_1^{\,2}-\phi_2^{\,2})}\left[\dfrac{\sinh(\phi_2x)}{\sinh(\phi_2)}-\dfrac{\sinh(\phi_1x)}{\sinh(\phi_1)}\right]+\left[x-\dfrac{\sinh(\phi_2x)}{\sinh(\phi_2)}\right]\\ \left[\begin{array}{l}\dfrac{\phi_1^{\,3}D^{'}k\alpha_1(\phi_1-\sinh(\phi_1))}{\phi_2^{\,2}\sinh^2(\phi_1)}+\dfrac{\phi_1^{\,3}D^{'}k\alpha_1}{\phi_2^{\,2}(k+1)\sinh(\phi_1)}\\ -\left(\dfrac{\phi_2}{\sinh(\phi_2)}-\dfrac{\phi_1}{\sinh(\phi_1)}\right)\left(\dfrac{\phi_1^{\,2}D^{'}k\alpha_2}{(k+1)(\phi_1^{\,2}-\phi_2^{\,2})}\right)\end{array}\right]\end{array}\right] \tag{8}$$

$$\begin{aligned}T(x)=\frac{y_3(x)}{x}&=(\beta_1+\beta_2D^{'})x-\frac{\beta_1\sinh(\phi_1x)}{\sinh(\phi_1)}+\frac{\beta_2D^{'}}{(\phi_1^{\,2}-\phi_2^{\,2})}\left[\frac{\phi^2\sinh(\phi_1x)}{\sinh(\phi_1)}-\frac{\phi_1^{\,2}\sinh(\phi_2x)}{\sinh(\phi_2)}\right]+\\ &l\left[\left(\frac{\sinh(\phi_1)}{\phi_1^{\,2}}-\frac{\sinh(\phi_1)}{6}\right)x-\frac{\sinh(\phi_1x)}{\phi_1^{\,2}}+\frac{x^3\sinh(\phi_1x)}{6}\right]+\\ &m\left[\left(\frac{1}{6}-\frac{1}{\phi_2^{\,2}}\right)x-\frac{x^3}{6}+\frac{\sinh(\phi_2x)}{\phi_2^{\,2}\sinh(\phi_2)}\right]+n\left[\frac{x-x^3}{6}\right]\end{aligned} \tag{9}$$

where the constants $k,\ l,\ m,$ and $n$ are given in the equations (A35) – (A38).

## 4. DISCUSSION

Figures 1(a) and (b) represents the dimensionless concentration $A(x)$ of the component $A$ for different values of dimensionless parameter $\alpha_1$ and $\alpha_2$. From these figures, it is evident that the values of the concentration decreases when dimensionless parameters $\alpha_1$ and $\alpha_2$ increases. Figures 2(a) and (b) shows the concentration $B(x)$ of the component $B$ versus the dimensionless radius $x$ for various values of dimensionless parameters $\alpha_1$ and $\alpha_2$. From these figures, it is obvious that the values of the concentration decreases when dimensionless parameters $\alpha_1$ and $\alpha_2$ increases. The dimensionless temperature $T(x)$ versus the dimensionless radius $x$ for various values of dimensionless parameters $\alpha_1$ and $\alpha_2$ is plotted in figure 3 (a) and (b). Initially the temperature $T$ increases and reaches the maximum value at $x = 7.5$ and then decreases. In this figure, it is inferred that the value of the temperature increases when the parameter $\alpha_1$ and $\alpha_2$ increases. Figure (4) show the dimensionless concentrations of components $A$ and $B$ versus the dimensionless radius $x$ using the equation (5) for the fixed values of the parameters.

## 5. CONCLUSIONS

The time independent non-linear reaction-diffusion equation in membrane has been solved analytically. Analytical expressions for the concentrations are derived by using the HPM. The primary result of this work is simple approximate calculations of concentration for all values of dimensionless parameter $\alpha_1$ and $\alpha_1$. The HPM is an extremely simple method and it is also a promising method to solve other non-linear equations. This method can be easily extended to find the solution of all other non-linear equations.

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**Appendix A**

In this Appendix, we indicate how the equations (7), (8) and (9) in this paper are derived. To find the solution of equations (1) - (3), then it can be transformed into simple forms by putting

$$y_1 = Ax,\ \mathrm{y}_2 = Bx,\ \ y_3 = Tx \tag{A1}$$

We obtain

$$\frac{d^2 y_1}{dx^2} - \phi_1^{\ 2} y_1 \exp\left[\frac{\alpha_1 y_3}{(x+y_3)}\right] = 0 \tag{A2}$$

$$\frac{d^2 y_2}{dx^2} - \phi_2^{\ 2} y_2 \exp\left[\frac{\alpha_2 y_3}{(x+y_3)}\right] + \phi_1^{\ 2} y_1 D' \exp\left[\frac{\alpha_1 y_3}{(x+y_3)}\right] = 0 \tag{A3}$$

$$\frac{d^2 y_3}{dx^2} + \beta_1\phi_1^{\ 2} y_1 \exp\left[\frac{\alpha_1 y_3}{(x+y_3)}\right] + \beta_2\phi_2^{\ 2} y_2 \exp\left[\frac{\alpha_2 y_3}{(x+y_3)}\right] = 0 \tag{A4}$$

The boundary conditions becomes

$$y_1 = 1, \quad y_2 = 0, \quad y_3 = 0, \quad \text{when} \quad x = 1 \tag{A5}$$

$$y_1 = 0, \quad y_2 = 0, \quad y_3 = 0, \quad \text{when} \quad x = 0 \tag{A6}$$

When $\frac{\alpha_1 y_3}{(x+y_3)}$ , $\frac{\alpha_2 y_3}{(x+y_3)}$ be small, then equations (A2), (A3) and (A4) reduces to

$$\frac{d^2 y_1}{dx^2} - \phi_1^{\,2} y_1 \left[1 + \frac{\alpha_1 y_3}{(x+y_3)}\right] = 0 \tag{A7}$$

$$\frac{d^2 y_2}{dx^2} - \phi_2^{\,2} y_2 \left[1 + \frac{\alpha_2 y_3}{(x+y_3)}\right] + \phi_1^{\,2} y_1 D' \left[1 + \frac{\alpha_1 y_3}{(x+y_3)}\right] = 0 \tag{A8}$$

$$\frac{d^2 y_3}{dx^2} + \beta_1 \phi_1^{\,2} y_1 \left[1 + \frac{\alpha_1 y_3}{(x+y_3)}\right] + \beta_2 \phi_2^{\,2} y_2 \left[1 + \frac{\alpha_2 y_3}{(x+y_3)}\right] = 0 \tag{A9}$$

We construct the Homotopy for the above three equations as follows:

$$(1-p)\left[\frac{d^2 y_1}{dx^2} - \phi_1^{\,2} y_1\right] + p\left[\frac{d^2 y_1}{dx^2} - \phi_1^{\,2} y_1 - \frac{\phi_1^{\,2} y_1 \alpha_1 y_3}{(x+y_3)}\right] = 0 \tag{A10}$$

$$(1-p)\left[\frac{d^2 y_2}{dx^2} - \phi_2^{\,2} y_2 + \phi_1^{\,2} y_1 D'\right] + p\left[\begin{array}{l} \dfrac{d^2 y_2}{dx^2} - \phi_2^{\,2} y_2 + \phi_1^{\,2} y_1 D' - \dfrac{\phi_2^{\,2} y_2 \alpha_2 y_3}{(x+y_3)} \\ + \dfrac{\phi_1^{\,2} y_1 D' \alpha_1 y_3}{(x+y_3)} \end{array}\right] = 0 \tag{A11}$$

$$(1-p)\left[\frac{d^2 y_3}{dx^2} + \beta_1 \phi_1^{\,2} y_1 + \beta_2 \phi_2^{\,2} y_2\right] + p\left[\begin{array}{l} \dfrac{d^2 y_3}{dx^2} + \beta_1 \phi_1^{\,2} y_1 + \beta_2 \phi_2^{\,2} y_2 \\ + \dfrac{\alpha_1 \beta_1 \phi_1^{\,2} y_1 y_3}{(x+y_3)} + \dfrac{\alpha_2 \beta_2 \phi_2^{\,2} y_2 y_3}{(x+y_3)} \end{array}\right] = 0 \tag{A12}$$

The approximate solutions of (A7), (A8) and (A8) are

$$y_1 = y_{10} + p y_{11} + p^2 y_{12} + .......... \tag{A13}$$

$$y_2 = y_{20} + p y_{21} + p^2 y_{22} + .......... \tag{A14}$$

$$y_3 = y_{30} + p y_{31} + p^2 y_{32} + .......... \tag{A15}$$

Substituting the equations (A13) to (A15) into equations (A10) to (A12) we have

$$(1-p)\left[\frac{d^2 (y_{10} + p y_{11} + ...)}{dx^2} - \phi_1^{\,2} (y_{10} + p y_{11} + .....)\right] + p\left[\begin{array}{l} \dfrac{d^2 (y_{10} + p y_{11} + ...)}{dx^2} - \phi_1^{\,2} (y_{10} + p y_{11} + ....) \\ - \dfrac{\alpha_1 \phi_1^{\,2} (y_{10} + p y_{11} + ....)(y_{30} + p y_{31} + ....)}{x + (y_{30} + p y_{31} + ....)} \end{array}\right] = 0 \tag{A16}$$

 

$$(1-p)\left[\frac{d^2(y_{20}+py_{21}+..)}{dx^2}-\phi_2{}^2(y_{20}+py_{21}+....)+\phi_1{}^2D^{'}(y_{10}+py_{11}+....)\right]$$
$$+p\left[\begin{array}{l}\frac{d^2(y_{20}+py_{21}+..)}{dx^2}-\phi_2{}^2(y_{20}+y_{21}+......)+\phi_1{}^2D^{'}(y_{10}+y_{11}+.....)-\\ \frac{\phi_2{}^2\alpha_2(y_{20}+y_{21}+...)(y_{30}+y_{31}+.....)}{x+(y_{30}+y_{31}+....)}+\frac{\phi_1{}^2\alpha_1D^{'}(y_{10}+y_{11}+...)(y_{30}+y_{31}+.....)}{x+(y_{30}+y_{31}+....)}\\ .\end{array}\right]=0 \tag{A17}$$

$$(1-p)\left[\frac{d^2(y_{30}+py_{31}+....)}{dx^2}+\beta_1\phi_1{}^2(y_{10}+py_{11}+...)+\beta_2\phi_2{}^2(y_{20}+py_{21}+....)\right]$$
$$+p\left[\begin{array}{l}\frac{d^2(y_{30}+py_{31}+..)}{dx^2}+\beta_1\phi_1{}^2(y_{10}+y_{11}+......)+\beta_2\phi_2{}^2(y_{20}+y_{21}+.....)-\\ \frac{\alpha_1\beta_1\phi_1{}^2(y_{10}+y_{11}+...)(y_{30}+y_{31}+.....)}{x+(y_{30}+y_{31}+....)}+\frac{\alpha_2\beta_2\phi_2{}^2(y_{20}+y_{21}+...)(y_{30}+y_{31}+.....)}{x+(y_{30}+y_{31}+....)}\end{array}\right]=0 \tag{A18}$$

Comparing the coefficients of like powers of $p$ in equation (A16) we get

$$p^0:\frac{d^2y_{10}}{dx^2}-\phi_1{}^2y_{10}=0 \tag{A19}$$

$$p^1:\frac{d^2y_{11}}{dx^2}-\phi_1{}^2y_{11}-\frac{\phi_1{}^2\alpha_1y_{10}y_{30}}{(x+y_{30})}=0 \tag{A20}$$

Comparing the coefficients of like powers of $p$ in equation (A17) we obtain.

$$p^0:\frac{d^2y_{20}}{dx^2}-\phi_2{}^2y_{20}+\phi_1{}^2y_{10}D^{'}=0 \tag{A21}$$

$$p^1:\frac{d^2y_{21}}{dx^2}-\phi_2{}^2y_{21}+\phi_1{}^2y_{11}D^{'}-\frac{\phi_2{}^2y_2\alpha_{20}y_{30}}{(x+y_{30})}+\frac{\phi_1{}^2y_{10}D^{'}\alpha_1y_{30}}{(x+y_{30})} \tag{A22}$$

Comparing the coefficients of like powers of $p$ in equation (A18) we have

$$p^0:\frac{d^2y_{30}}{dx^2}+\beta_1\phi_1{}^2y_{10}+\beta_2\phi_2{}^2y_{20}=0 \tag{A23}$$

$$p^1:\left[\frac{d^2y_{31}}{dx^2}+\beta_1\phi_1{}^2y_{11}+\beta_2\phi_2{}^2y_{21}+\frac{\alpha_1\beta_1\phi_1{}^2y_{10}y_{30}}{(x+y_{30})}+\frac{\alpha_2\beta_2\phi_2{}^2y_{20}y_{30}}{(x+y_{30})}\right]=0 \tag{A24}$$

The initial approximations are as follows

$$x=1,\qquad \mathrm{y}_{10}(1)=1,\ \mathrm{y}_{20}(1)=0,\ \mathrm{y}_{30}(1)=0 \tag{A25}$$

$$\mathrm{y}_{1i}(1)=0,\ \mathrm{y}_{2i}(1)=0,\ \mathrm{y}_{3i}(1)=0,\ \mathrm{i}=1,2,3.... \tag{A26}$$

$$x = 0, \qquad y_{10}(0) = 0,\ y_{20}(0) = 0,\ y_{30}(0) = 0 \tag{A27}$$

$$y_{1i}(0) = 0,\ y_{2i}(1) = 0,\ y_{3i}(0) = 0,\ i = 1,2,3.. \tag{A28}$$

Solving the equations (A19) to (A24) and using the boundary conditions (A25) to (A28), we can obtain the following results:

$$y_{10} = \frac{\sinh(\phi_1 x)}{\sinh(\phi_1)} \tag{A29}$$

$$y_{11} = \frac{\phi_1 k \alpha_1}{\sinh^2(\phi_1)}\left[\sinh(\phi_1 x) - x\sinh(\phi_1)\right] \tag{A30}$$

$$y_{20} = \frac{\phi_1^{\,2} D^{'}}{(\phi_1^{\,2} - \phi_2^{\,2})}\left[\frac{\sinh(\phi_2 x)}{\sinh(\phi_2)} - \frac{\sinh(\phi_1 x)}{\sinh(\phi_1)}\right] \tag{A31}$$

$$y_{21} = \left[x - \frac{\sinh(\phi_2 x)}{\sinh(\phi_2)}\right]\left[\begin{array}{l}\dfrac{\phi_1^{\,3} D^{'} k\alpha_1(\phi_1 - \sinh(\phi_1))}{\phi_2^{\,2}\sinh^2(\phi_1)} + \dfrac{\phi_1^{\,3} D^{'} k\alpha_1}{\phi_2^{\,2}(k+1)\sinh(\phi_1)} \\ -\left(\dfrac{\phi_2}{\sinh(\phi_2)} - \dfrac{\phi_1}{\sinh(\phi_1)}\right)\left(\dfrac{\phi_1^{\,2} D^{'} k\alpha_2}{(k+1)(\phi_1^{\,2} - \phi_2^{\,2})}\right)\end{array}\right] \tag{A32}$$

$$y_{30} = (\beta_1 + \beta_2 D^{'} x) - \frac{\beta_1 \sinh(\phi_1 x)}{\sinh(\phi_1)} + \frac{\beta_2 D^{'}}{(\phi_1^{\,2} - \phi_2^{\,2})}\left[\frac{\phi^2 \sinh(\phi_1 x)}{\sinh(\phi_1)} - \frac{\phi_1^{\,2}\sinh(\phi_2 x)}{\sinh(\phi_2)}\right] \tag{A33}$$

$$y_{31} = l\left[\left(\frac{\sinh(\phi_1)}{\phi_1^{\,2}} - \frac{\sinh(\phi_1)}{6}\right)x - \frac{\sinh(\phi_1 x)}{\phi_1^{\,2}} + \frac{x^3\sinh(\phi_1 x)}{6}\right] + m\left[\left(\frac{1}{6} - \frac{1}{\phi_2^{\,2}}\right)x - \frac{x^3}{6} + \frac{\sinh(\phi_2 x)}{\phi_2^{\,2}\sinh(\phi_2)}\right] + n\left[\frac{x - x^3}{6}\right] \tag{A34}$$

Where

$$k = \beta_1 + \beta_2 D^{'} - \frac{\beta_1\phi_1}{\sinh(\phi_1)} + \frac{\beta_2 D^{'}}{(\phi_1^{\,2} - \phi_2^{\,2})}\left[\frac{\phi_1\phi_2^{\,2}}{\sinh(\phi_1)} - \frac{\phi_2\phi_1^{\,2}}{\sinh(\phi_2)}\right] \tag{A35}$$

$$l = \frac{\alpha_1\beta_1\phi_1^{\,3} k}{\sinh^2(\phi_1)} \tag{A36}$$

$$m = \beta_2 u\phi_2^{\,2}\left[\frac{\phi_1^{\,3} D^{'}\alpha_1(\phi_1 - \sinh(\phi_1)}{\phi_2^{\,2}\sinh^2(\phi_1)} + \frac{\phi_1^{\,3} D^{'}\alpha_1}{\phi_2^{\,2}(k+1)\sinh(\phi_1)} - \left(\frac{\phi_1^{\,2} D^{'}\alpha_2}{(\phi_1^{\,2} - \phi_2^{\,2})(k+1)}\right)\left(\frac{\phi_2}{\sinh(\phi_2)} - \frac{\phi_1}{\sinh(\phi_1)}\right)\right] \tag{A37}$$

$$n = \frac{k\left[\dfrac{\alpha_1\beta_1\phi_1^{\,3}}{\sinh(\phi_1)} + \left(\dfrac{\alpha_2\beta_2\phi_1^{\,2}\phi_2^{\,2} D^{'}}{(\phi_1^{\,2} - \phi_2^{\,2})}\right)\left(\dfrac{\phi_2}{\sinh(\phi_2)} - \dfrac{\phi_1}{\sinh(\phi_1)}\right)\right]}{\left(1 + \beta_1 + \beta_2 D^{'} + D^{'} - \dfrac{\beta_1\phi_1}{\sinh(\phi_1)}\right)} \tag{A38}$$

According to the HPM, we can conclude that

$$y_1 = \lim_{p\to 1} y_1(x) = y_{10} + y_{11} \tag{A39}$$

$$y_2 = \lim_{p\to 1} y_2(x) = y_{20} + y_{21} \tag{A40}$$

$$y_3 = \lim_{p\to 1} y_3(x) = y_{30} + y_{31} \tag{A41}$$

After putting equations (A29) and (A30) into equation (A39) and equations (A31) and (A32) into equation (A40) and equations (A33) and (A34) into equation (A41), we obtain the following solutions.

$$y_1(x) = \left[\frac{\sinh(\phi_1 x)}{\sinh(\phi_1)} + \frac{\phi_1 k\alpha_1}{\sinh^2(\phi_1)}\left[\sinh(\phi_1 x) - x\sinh(\phi_1)\right]\right] \tag{A42}$$

$$y_2(x) = \left[\begin{array}{l} \dfrac{\phi_1^{\,2} D'}{(\phi_1^{\,2} - \phi_2^{\,2})}\left[\dfrac{\sinh(\phi_2 x)}{\sinh(\phi_2)} - \dfrac{\sinh(\phi_1 x)}{\sinh(\phi_1)}\right] + \left[x - \dfrac{\sinh(\phi_2 x)}{\sinh(\phi_2)}\right] \\ \left[\begin{array}{l} \dfrac{\phi_1^{\,3} D' k\alpha_1(\phi_1 - \sinh(\phi_1))}{\phi_2^{\,2}\sinh^2(\phi_1)} + \dfrac{\phi_1^{\,3} D' k\alpha_1}{\phi_2^{\,2}(k+1)\sinh(\phi_1)} \\ -\left(\dfrac{\phi_2}{\sinh(\phi_2)} - \dfrac{\phi_1}{\sinh(\phi_1)}\right)\left(\dfrac{\phi_1^{\,2} D' k\alpha_2}{(k+1)(\phi_1^{\,2} - \phi_2^{\,2})}\right) \end{array}\right] \end{array}\right] \tag{A43}$$

$$\begin{aligned} y_3(x) = {} & (\beta_1 + \beta_2 D')x - \frac{\beta_1 \sinh(\phi_1 x)}{\sinh(\phi_1)} + \frac{\beta_2 D'}{(\phi_1^{\,2} - \phi_2^{\,2})}\left[\frac{\phi^2 \sinh(\phi_1 x)}{\sinh(\phi_1)} - \frac{\phi_1^{\,2}\sinh(\phi_2 x)}{\sinh(\phi_2)}\right] \\ & + l\left[\left(\frac{\sinh(\phi_1)}{\phi_1^{\,2}} - \frac{\sinh(\phi_1)}{6}\right)x - \frac{\sinh(\phi_1 x)}{\phi_1^{\,2}} + \frac{x^3\sinh(\phi_1 x)}{6}\right] \\ & + m\left[\left(\frac{1}{6} - \frac{1}{\phi_2^{\,2}}\right)x - \frac{x^3}{6} + \frac{\sinh(\phi_2 x)}{\phi_2^{\,2}\sinh(\phi_2)}\right] + n\left[\frac{x - x^3}{6}\right] \end{aligned} \tag{A44}$$

**Appendix B. Nomenclature**

| Symbol | Meaning |
|---|---|
| $A$ | Dimensionless concentration of the component *A*. |
| $B$ | Dimensionless concentration of the component $B$ |
| $x$ | Dimensionless radius |
| $\alpha_1$ | Dimensionless activation energy of the component $A$ |
| $\alpha_2$ | Dimensionless activation energy of the component $B$ |
| $\beta_1$ | Dimensionless degree of thermicity of the component $A$ |
| $\beta_2$ | Dimensionless degree of thermicity of the component $B$ |
| $\phi_1$ | Dimensionless Thiele modulus of the component $A$. |
| $\phi_2$ | Dimensionless Thiele modulus of the component $B$. |
| $D_A$ | Effective diffusivity of the component $A$ |
| $D_B$ | Effective diffusivity of the component $B$ |
| $D'$ | $\dfrac{D_A}{D_B}$ |

**Figure: 1(a)**

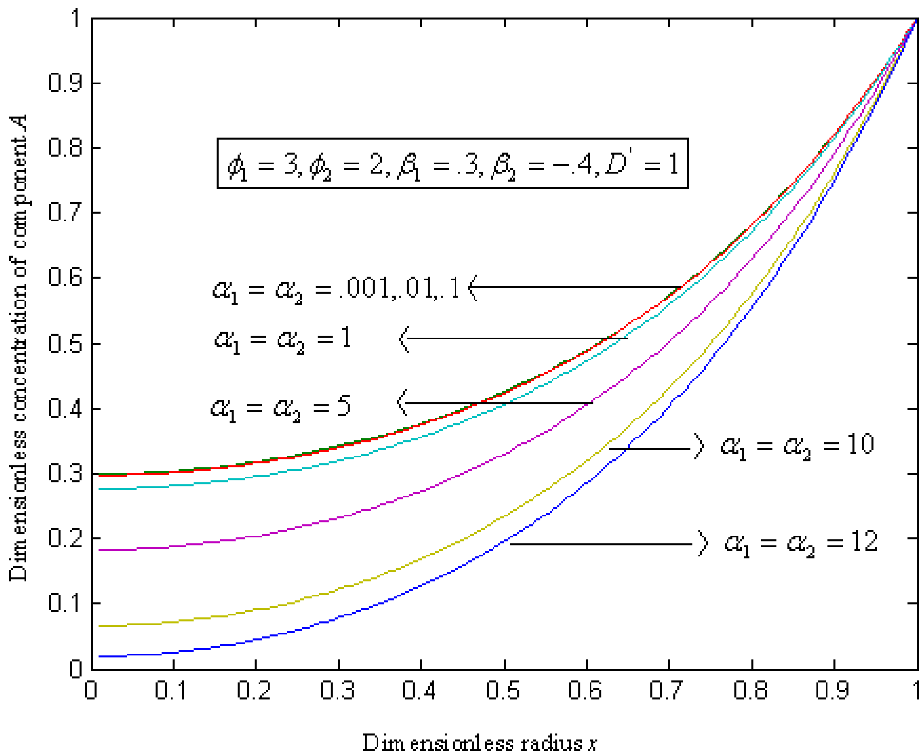


**Figure: 1(b)**

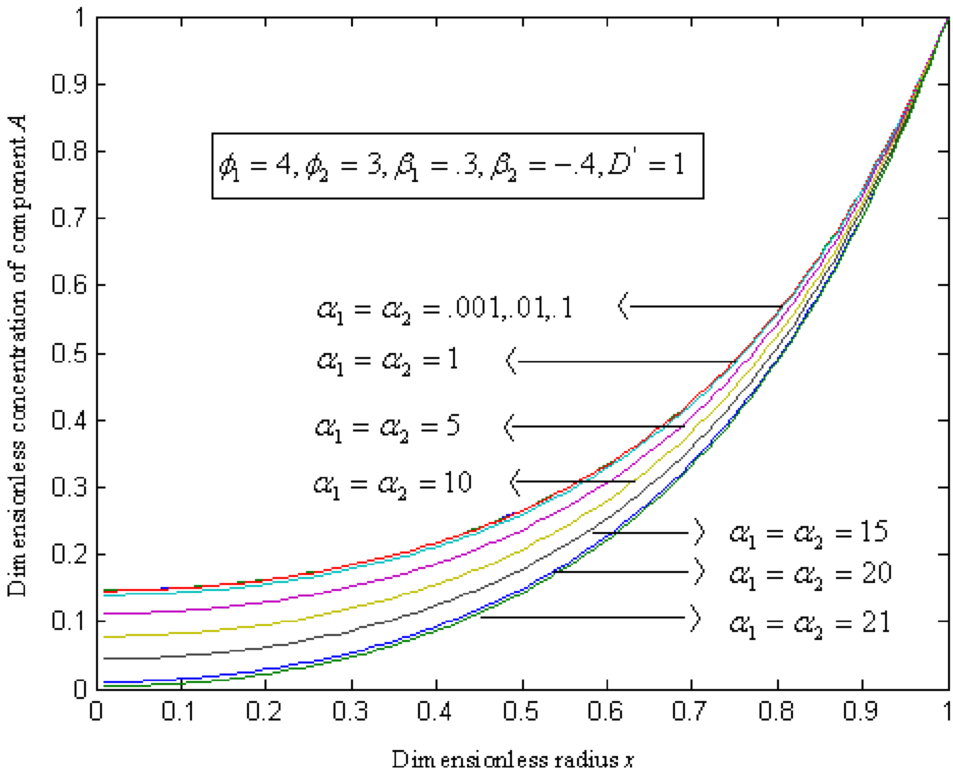


**Figure 1:** Influence of dimensionless activation energies $\alpha_1,\ \alpha_2$ on the dimensionless concentration of component $A$ obtained from the equation (7). The curve is plotted for some fixed values of $\phi_1,\ \phi_2,\ \beta_1,\ \beta_2$ and $D'$,

(a) $\phi_1 = 3,\ \phi_2 = 2,\ \beta_1 = .3,\ \beta_2 = -.4,\ D' = 1$ ,

(b) $\phi_1 = 4,\ \phi_2 = 3,\ \beta_1 = .3,\ \beta_2 = -.4,\ D' = 1$

 

**Figure: 2(a)**

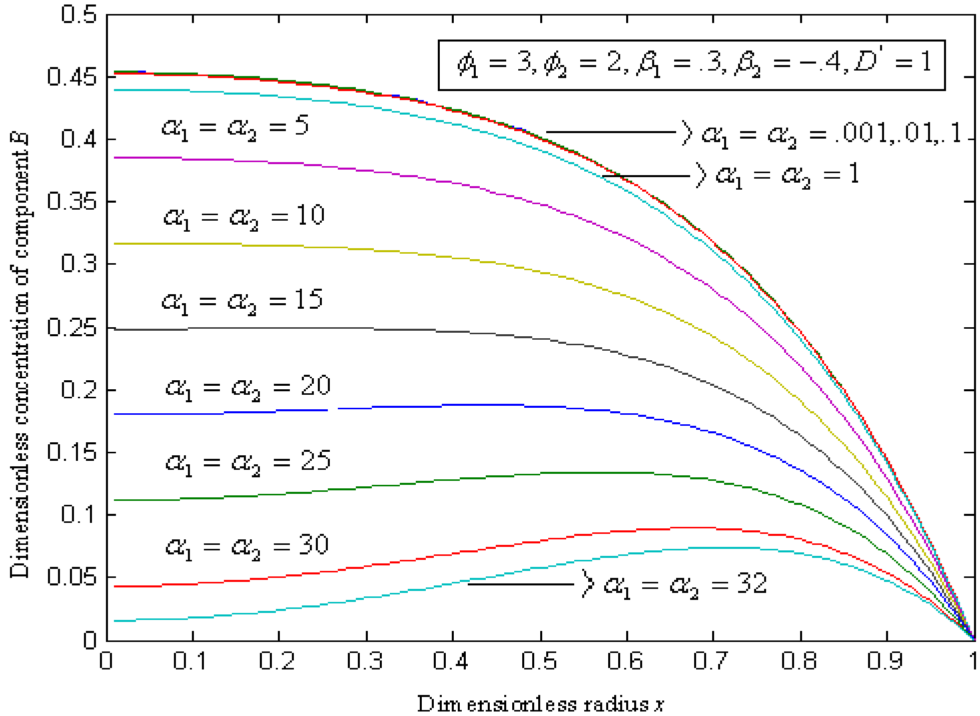


**Figure: 2(b)**

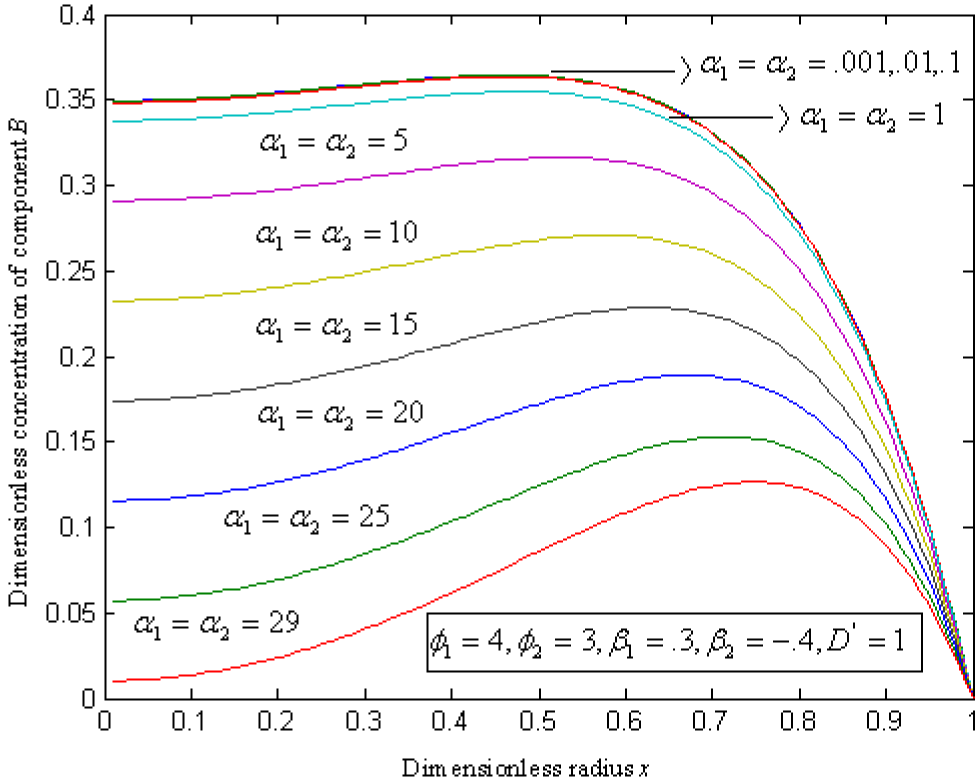


**Figure 2:** Influence of dimensionless activation energies $\alpha_1, \alpha_2$ on the dimensionless concentration of component $B$ obtained from the equation (8). The curve is plotted for some fixed values of $\phi_1, \phi_2, \beta_1, \beta_2$ and $D'$,

(a) $\phi_1 = 3,\ \phi_2 = 2,\ \beta_1 = .3,\ \beta_2 = -.4,\ D' = 1$ ,

(b) $\phi_1 = 4,\ \phi_2 = 3,\ \beta_1 = .3,\ \beta_2 = -.4,\ D' = 1$

**Figure: 3(a)**

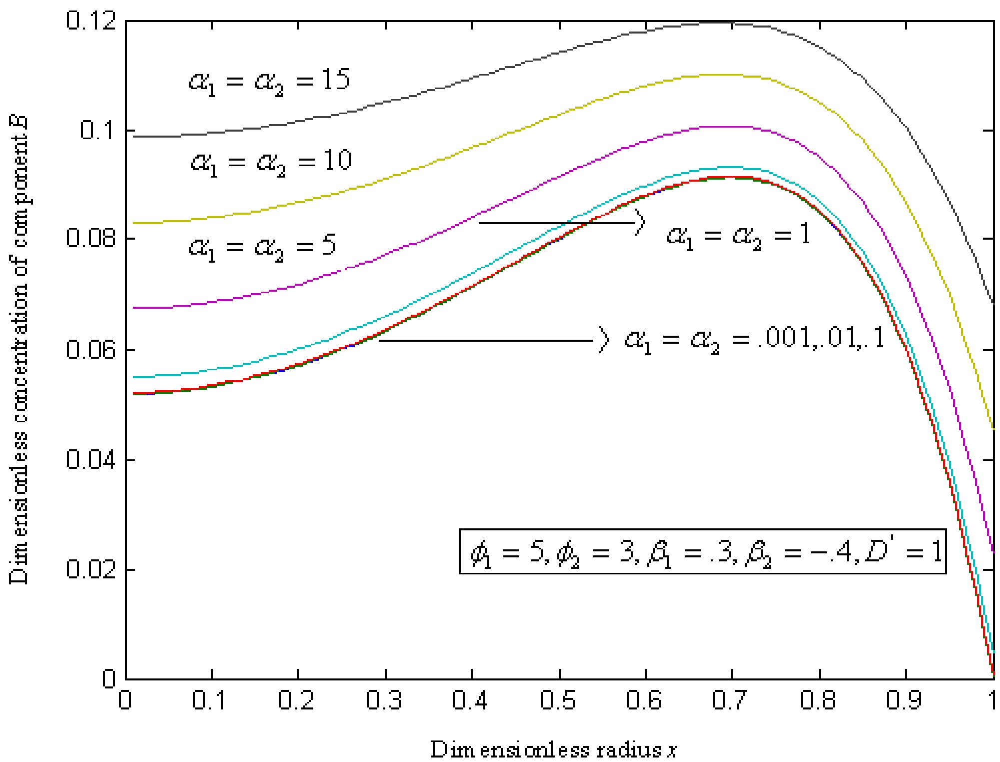


**Figure: 3(b)**

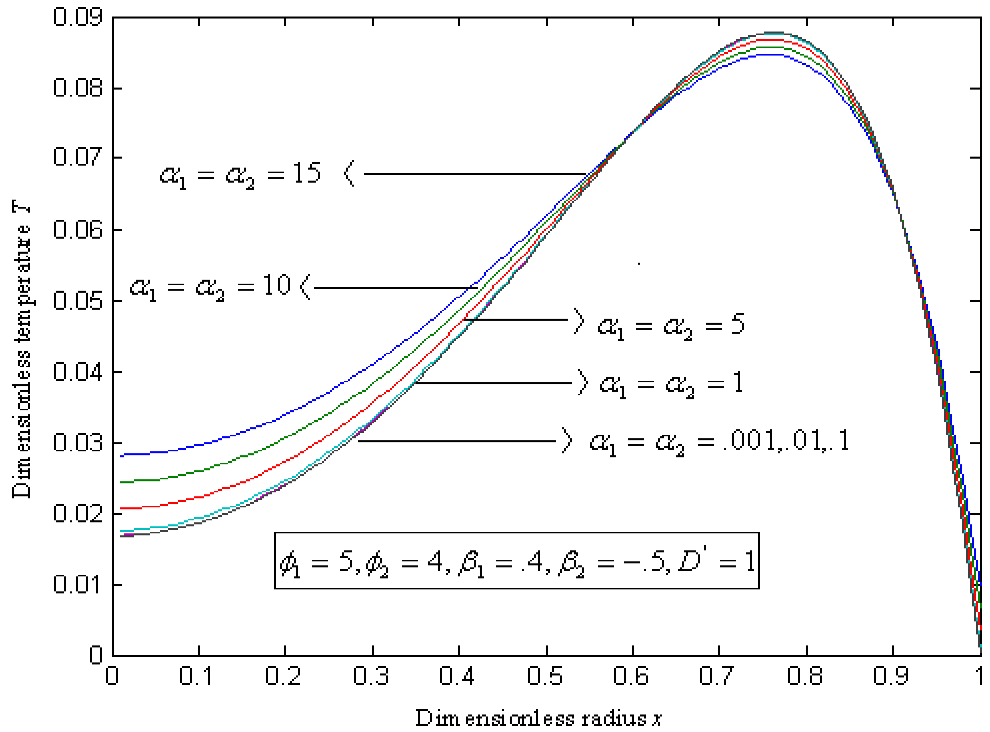


**Figure 3:** Influence of dimensionless activation energies $\alpha_1,\ \alpha_2$ on the dimensionless temperature $T$ obtained from the equation (9). The curve is plotted for some fixed values of $\phi_1,\ \phi_2,\ \beta_1,\ \beta_2$ and $D'$,

(a) $\phi_1 = 5,\ \phi_2 = 3,\ \beta_1 = .3,\ \beta_2 = -.4,\ D' = 1$ ,

(b) $\phi_1 = 5,\ \phi_2 = 4,\ \beta_1 = .4,\ \beta_2 = -.5,\ D' = 1$

**Figure: 4**

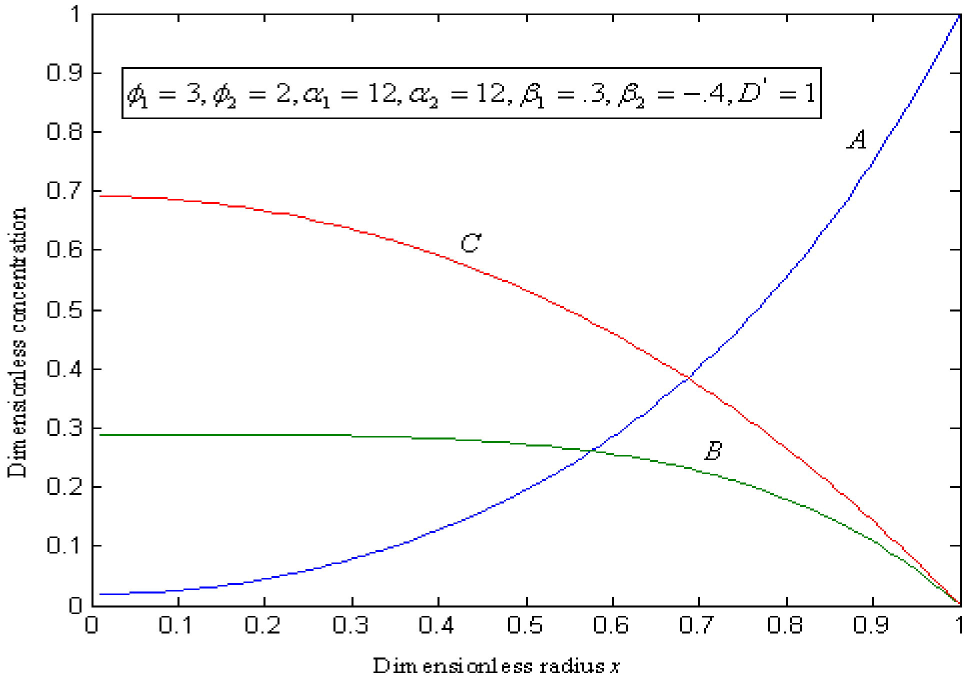


**Figure 4:** Influence of dimensionless activation energies $\alpha_1$, $\alpha_2$, Thiele modulus $\phi_1$, $\phi_2$ and dimensionless degree of thermicities $\beta_1$, $\beta_2$ and $D' = 1$ of the dimensionless concentrations $A$, $B$ & $C$ obtained from the equations (6), (7) and (8).

**Figure: 5**

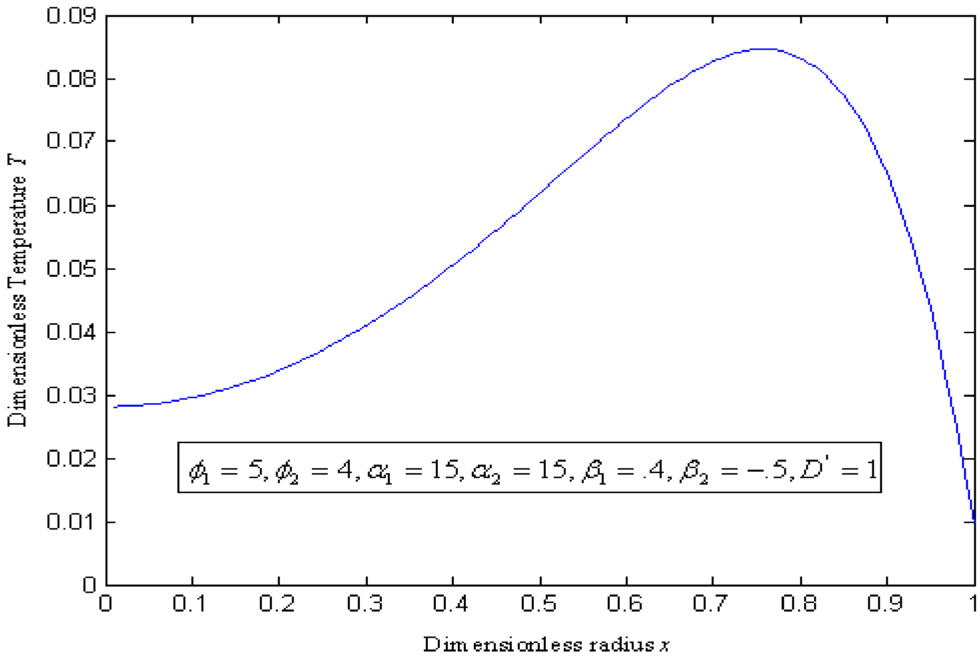


**Figure 5:** Influence of the dimensionless activation energies $\alpha_1$, $\alpha_2$, Thiele modulus $\phi_1$, $\phi_2$, dimensionless degree of thermicities $\beta_1$, $\beta_2$ and $D' = 1$, of dimensionless temperature $T$ obtained from equation (8).

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