On pg-Separation Axioms

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ABSTRACT

In this paper by using pg-open sets we define almost pg-normality and mild pg-normality also we continue the study of further properties of pg-normality. We show that these three axioms are regular open hereditary. We also define the class of almost pg-irresolute mappings and show that pg-normality is invariant under almost pg-irresolute M-pg-open continuous surjection.

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1. INTRODUCTION

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C.E. Aull studied some separation axioms between the T1 and T2 spaces, namely, S1 and S2. Next, in 1982, S.P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, sequentially s-compact notions. G.B. Navlagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P. Aruna Swathi Vyjayanthi studied v-Normal Almost-v-Normal, Mildly-v-Normal and v-US spaces. Inspired with these we introduce pg-Normal Almost-pg-Normal, Mildly-pg-Normal, pg-US, pg-S1 and pg-S2. Also we examine pg-convergence, sequentially pg-compact, sequentially pg-continuous maps, and sequentially sub pg-continuous maps in the context of these new concepts. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper X and Y denote Topological spaces on which no separation axioms are assumed explicitly stated.

2. PRELIMINARIES

Definition 2.1: A⊂ X is called
(i) g-closed if cl A⊆ U whenever A⊂ U and U is open in X.
(ii) pg-closed if pcl(A) ⊆ U whenever A⊂ U and U is preopen in X.

Definition 2.2: A space X is said to be
(i) T1 (T2) if for any x ≠ y in X, there exist (disjoint) open sets U; V in X such that x∈ U and y∈ V.
(ii) weakly Hausdorff if each point of X is the intersection of regular closed sets of X.
(iii) normal [resp: mildly normal] if for any pair of disjoint [resp: regular-closed]closed sets F1 and F2, there exist disjoint open sets U and V such that F1 ⊂ U and F2 ⊂ V.
(iv) almost normal if for each closed set A and each regular closed set B such that A∩B = ∅, there exist disjoint open sets U and V such that A⊆U and B⊂V.
(v) weakly regular if for each pair consisting of a regular closed set A and a point x such that A ∩ {x} = ∅, there exist disjoint open sets U and V such that x∈ U and A⊂V.
(vi) A subset A of a space X is S-closed relative to X if every cover of A by semiopen sets of X has a finite subfamily whose closures cover A.
Note 5:

open sets.

Lemma 3.02:

X. Every point of X such that point y of A

(i) If X is

(ii) If x is a pg-T₀–limit point of A then x is pg-limit point of A.

Lemma 3.02:

(i) If X is pg-T₀–space then every pg-T₀–limit point and every pg-limit point are equivalent.

(ii) If x is a pg-T₀–limit point of A then x is pg-limit point of A.

Theorem 3.03: For x ≠ y ∈ X,

(i) x is a pg-T₀–limit point of {y} iff x ∉ pgcl{y} and y ∈ pgcl{x}.

(ii) x is not a pg-T₀–limit point of {y} iff either x ∉ pgcl{y} or pgcl{x} = pgcl{y}.

(iii) x is not a pg-T₀–limit point of {y} iff either x ∉ pgcl{y} or y ∈ pgcl{x}.

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Corollary 3.04:
(i) If \(x\) is a \(pg-T_0\)-limit point of \(\{y\}\), then \(y\) cannot be a \(pg\)-limit point of \(\{x\}\).
(ii) If \(pgcl\{x\} = pgcl\{y\}\), then neither \(x\) is a \(pg-T_0\)-limit point of \(\{y\}\) nor \(y\) is a \(pg-T_0\)-limit point of \(\{x\}\).
(iii) If a singleton set \(A\) has no \(pg-T_0\)-limit point in \(X\), then \(pgcl\{A\} = pgcl\{x\}\) for all \(x \in pgcl\{A\}\).

Lemma 3.05:
In \(X\), if \(x\) is a \(pg\)-limit point of a set \(A\), then in each of the following cases \(x\) becomes \(pg-T_0\)-limit point of \(A\) \((\{x\} \neq A)\).

(i) \(pgcl\{x\} \neq pgcl\{y\}\) for \(y \in A\), \(x \neq y\).
(ii) \(pgcl\{x\} = \{x\}\)
(iii) \(X\) is a \(pg-T_0\)-space.
(iv) \(A \sim \{x\}\) is \(pg\)-open

4. \(pg-T_0\) AND \(pg-R_i\) AXIOMS, \(i = 0, 1\):

In view of Lemma 3.6(iii), \(pg-T_0\)-axiom implies the equivalence of the concept of limit point of a set with that of \(pg-T_0\)-limit point of the set. But for the converse, if \(x \in pgcl\{y\}\) then \(pgcl\{x\} \neq pgcl\{y\}\) in general, but if \(x\) is a \(pg-T_0\)-limit point of \(\{y\}\), then \(pgcl\{x\} = pgcl\{y\}\)

Lemma 4.01: In a space \(X\), a limit point \(x\) of \(\{y\}\) is a \(pg-T_0\)-limit point of \(\{y\}\) iff \(pgcl\{x\} \neq pgcl\{y\}\).

This lemma leads to characterize the equivalence of \(pg-T_0\)-limit point and \(pg\)-limit point of a set as the \(pg-T_0\)-axiom.

Theorem 4.02: The following conditions are equivalent:
(i) \(X\) is a \(pg-T_0\) space
(ii) Every \(pg\)-limit point of a set \(A\) is a \(pg-T_0\)-limit point of \(A\)
(iii) Every \(r\)-limit point of a singleton set \(\{x\}\) is a \(pg-T_0\)-limit point of \(\{x\}\)
(iv) For any \(x, y\) in \(X\), \(x \neq y\) if \(x \in pgcl\{y\}\), then \(x\) is a \(pg-T_0\)-limit point of \(\{y\}\)

Note 6: In a \(pg-T_0\)-space \(X\) if every point of \(X\) is a \(r\)-limit point of \(X\), then every point of \(X\) is \(pg-T_0\)-limit point of \(X\). But a space \(X\) in which each point is a \(pg-T_0\)-limit point of \(X\) is not necessarily a \(pg-T_0\)-space

Theorem 4.03: The following conditions are equivalent:
(i) \(X\) is a \(pg-R_0\) space
(ii) For any \(x, y\) in \(X\), if \(x \in pgcl\{y\}\), then \(x\) is not a \(pg-T_0\)-limit point of \(\{y\}\)
(iii) A point \(pg\)-closure set has no \(pg-T_0\)-limit point in \(X\)
(iv) A singleton set has no \(pg-T_0\)-limit point in \(X\).

Theorem 4.04: In a \(pg-R_0\) space \(X\), a point \(x\) is \(pg-T_0\)-limit point of \(A\) iff every \(pg\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct

Theorem 4.05: \(X\) is \(pg-R_0\) space iff a set \(A\) of the form \(A = \cup pgcl\{x_i, i = 1 to n\}\) a finite union of point closure sets has no \(pg-T_0\)-limit point.

If \(pg-R_0\) space is replaced by \(rR_0\) space in the above theorem, we have the following corollaries:

Corollary 4.06: The following conditions are equivalent:
(i) \(X\) is a \(r-R_0\) space
(ii) For any \(x, y\) in \(X\), if \(x \in pgcl\{y\}\), then \(x\) is not a \(pg-T_0\)-limit point of \(\{y\}\)
(iii) A point \(pg\)-closure set has no \(pg-T_0\)-limit point in \(X\)
(iv) A singleton set has no \(pg-T_0\)-limit point in \(X\).

Corollary 4.07: In an \(rR_0\)-space \(X\),
(i) If a point \(x\) is \(rT_0\)-limit point of a set then every \(pg\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct.
(ii) If a point \(x\) is \(pg-T_0\)-limit point of a set then every \(pg\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct.
(iii) If \(A = \cup pgcl\{x_i, i = 1 to n\}\) a finite union of point closure sets has no \(pg-T_0\)-limit point.
(iv) If \(X = \cup pgcl\{x_i, i = 1 to n\}\) then \(X\) has no \(pg-T_0\)-limit point.

Various characteristic properties of \(pg-T_0\)-limit points studied so far is enlisted in the following theorem.
Theorem 4.08: In a pg-R₀-space, we have the following:
(i) A singleton set has no pg-T₀-limit point in X.
(ii) A finite set has no pg-T₀-limit point in X.
(iii) A point pg-closure has no set pg-T₀-limit point in X.
(iv) A finite union point pg-closure sets have no set pg-T₀-limit point in X.
(v) For x, y ∈ X, x ∈ T₀–pgcl{y} iff x = y.
(vi) For any x, y ∈ X, x ≠ y iff neither x is pg-T₀-limit point of {y} nor y is pg-T₀-limit point of {x}.
(vii) For any x, y ∈ X, x ≠ y iff T₀–pgcl{x} ∩ T₀–pgcl{y} = φ.
(viii) Any point x ∈ X is a pg-T₀-limit point of a set A in X iff every pg-open set containing x contains infinitely many points of A with each which x is topologically distinct.

Theorem 4.09: X is pg-R_i iff for any pg-open set U in X and points x, y such that x ∈ X, y ∈ U, there exists a pg-open set V in X such that y ∈ V ⊂ U, x ∉ V.

Lemma 4.10: In pg-R₁ space X, if x is a pg-T₀-limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that V ⊂ U, x ∉ pgcl(V).

Lemma 4.11: In a pg-regular space X, if x is a pg-T₀-limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that pgcl(V) ⊂ U, x ∉ pgcl(V).

Corollary 4.12: In a regular space X,
(i) If x is a pg-T₀-limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that pgcl(V) ⊂ U, x ∉ pgcl(V).
(ii) If x is a T₀-limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that pgcl(V) ⊂ U, x ∉ pgcl(V).

Theorem 4.13: If X is a pg-compact pg-R₁-space, then X is a Baire Space.

Proof: Let {Aₙ} be a countable collection of pg-closed sets of X, each Aₙ having empty interior in X. Take A₁, since A₁ has empty interior, A₁ does not contain any pg-open set say U₀. Therefore we can choose a point y ∈ U₀ such that y ∉ A₁.

For X is pg-regular, and y ∈ (X-A₁) ∩ U₀, a pg-open set, we can find a pg-open set U₁ in X such that y ∈ U₁, pgcl(U₁) ⊂ (X-A₁) ∩ U₀. Hence U₁ is a non empty pg-open set in X such that pgcl(U₁) ⊂ U₀ and pgcl(U₁) ∩ A₁ = φ. Continuing this process, in general, for given non empty pg-open set Uₙ₋₁, we can choose a point of Uₙ₋₁ which is not in the pg-closed set Aₙ and a pg-open set Uₙ containing this point such that pgcl(Uₙ) ⊂ Uₙ₋₁ and pgcl(Uₙ) ∩ Aₙ = φ. Thus we get a sequence of nested non empty pg-closed sets which satisfies the finite intersection property. Therefore ∩ pgcl(Uₙ) ≠ φ.

Then some x ∈ ∩ pgcl(Uₙ) which in turn implies that x ∈ Uₙ₋₁ as pgcl(Uₙ) ⊂ Uₙ₋₁ and x ∉ Aₙ for each n.

Corollary 4.14: If X is a compact pg-R₁-space, then X is a Baire Space.

Corollary 4.15: Let X be a pg-compact pg-R₁-space. If {Aₙ} is a countable collection of pg-closed sets in X, each Aₙ having non-empty pg-interior in X, then there is a point of X which is not in any of the Aₙ.

Corollary 4.16: Let X be a pg-compact R₁-space. If {Aₙ} is a countable collection of pg-closed sets in X, each Aₙ having non-empty pg-interior in X, then there is a point of X which is not in any of the Aₙ.

Theorem 4.17: Let X be a non empty compact pg-R₁-space. If every point of X is a pg-T₀-limit point of X then X is uncountable.

Proof: Since X is non empty and every point is a pg-T₀-limit point of X, X must be infinite. If X is countable, we construct a sequence of pg-open sets {Vₙ} in X as follows:

Let X = V₁, then for x₁ is a pg-T₀-limit point of X, we can choose a non empty pg-open set V₂ in X such that V₂ ⊂ V₁ and x₁ ∉ pgclV₂. Next for x₂ and non empty pg-open set V₃, we can choose a non empty pg-open set V₃ in X such that V₃ ⊂ V₂ and x₂ ∉ pgclV₃. Continuing this process for each xₙ and a non empty pg-open set Vₙ, we can choose a non empty pg-open set Vₙ₊₁ in X such that Vₙ₊₁ ⊂ Vₙ and xₙ ∉ pgclVₙ₊₁.

Now consider the nested sequence of pg-closed sets pgclV₁ ⊃ pgclV₂ ⊃ pgclV₃ ⊃ ⋯ ⋯ pgclVₙ ⊃ ⋯ Since X is pg-compact and {pgclVₙ} the sequence of pg-closed sets satisfies finite intersection property. By Cantors intersection theorem, there exists an x in X such that x ∈ pgclV₂. Further x ∈ X and x ∈ V₁, which is not equal to any of the points of X. Hence X is uncountable.
5. \textit{pg-}T_\sigma\textit{-IDENTIFICATION SPACES AND pg-SEPARATION AXIOMS}

\textbf{Definition 5.01:} Let \((X, \tau)\) be a topological space and let \(\mathcal{R}\) be the equivalence relation on \(X\) defined by \(x\mathcal{R}y\) iff 
\[\text{pgcl}\{x\} = \text{pgcl}\{y\}\]

\textbf{Problem 5.02:} show that \(x\mathcal{R}y\) iff \(\text{pgcl}\{x\} = \text{pgcl}\{y\}\) is an equivalence relation

\textbf{Definition 5.03:} The space \((X_0, Q(X_0))\) is called the pg-T_\sigma\textit{-identification space of }\((X, \tau)\), where \(X_0\) is the set of equivalence classes of \(\mathcal{R}\) and \(Q(X_0)\) is the decomposition topology on \(X_0\).

\textbf{Lemma 5.04:} If \(x \in X\) and \(A \subseteq X\), then \(x \in \text{pgcl} A\) iff every pg-open set containing \(x\) intersects \(A\).

\textbf{Theorem 5.05:} The natural map \(P_X : (X, \tau) \rightarrow (X_0, Q(X_0))\) is closed, and \(P_X^{-1}(O) = O\) for all \(O \in PO(X, \tau)\) and \((X_0, Q(X_0))\) is pg-T_\sigma\textit{.}

\textbf{Proof:} Let \(O \in PO(X, \tau)\) and \(C \subseteq P_X(O)\). Then there exists \(x \in O\) such that \(P_X(x) = C\). If \(y \in C\), then \(\text{pgcl}\{y\} = \text{pgcl}\{x\}\), which, by lemma, implies \(y \in O\). Since \(\tau \subseteq PO(X, \tau)\), then \(P_X^{-1}(P_X(U)) = U\) for all \(U \in \tau\), which implies \(P_X\) is closed and open.

Let \(G, H \subseteq X_0\) such that \(G \neq H\); let \(x \in G\) and \(y \in H\). Then \(\text{pgcl}\{x\} \neq \text{pgcl}\{y\}\), which implies \(x \notin \text{pgcl}\{y\}\) or \(y \notin \text{pgcl}\{x\}\), say \(x \notin \text{pgcl}\{y\}\). Since \(P_X\) is continuous and open, then \(G \in A = P_X\{X - \text{pgcl}\{y\}\} \notin PO(X_0, Q(X_0))\) and \(H \notin A\).

\textbf{Theorem 5.06:} The following are equivalent:

(i) \(X\) is pg-R_0

(ii) \(X_0 = \{\text{pgcl}\{x\} : x \in X\}\) and \((X_0, Q(X_0))\) is pg-T_\sigma\textit{.}

\textbf{Proof:} (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) Let \(C \subseteq X_0\) and \(x \in C\). If \(y \in C\), then \(\text{pgcl}\{y\} = \text{pgcl}\{x\}\), which implies \(C = \text{pgcl}\{x\}\). If \(y \notin \text{pgcl}\{x\}\), then \(x \notin \text{pgcl}\{y\}\), since, otherwise, \(x \in X - \text{pgcl}\{y\} \subseteq PO(X, \tau)\) which implies \(\text{pgcl}\{x\} \subseteq X - \text{pgcl}\{y\}\), which is a contradiction. Thus, if \(y \notin \text{pgcl}\{x\}\), then \(x \notin \text{pgcl}\{y\}\), which implies \(\text{pgcl}\{y\} = \text{pgcl}\{x\}\) and \(y \in C\). Hence \(X_0 = \{\text{pgcl}\{x\} : x \in X\}\)

(ii) \(\Rightarrow\) (iii) Let \(A \neq B \in X_0\). Then there exists \(x, y \in X\) such that \(A = \text{pgcl}\{x\}\); \(B = \text{pgcl}\{y\}\), and \(\text{pgcl}\{x\} \cap \text{pgcl}\{y\} = \emptyset\). Then \(A \cap = P_X\{X - \text{pgcl}\{y\}\} \subseteq PO(X_0, Q(X_0))\) and \(B \notin A\). Thus \((X_0, Q(X_0))\) is pg-T_\sigma\textit{.}

(iii) \(\Rightarrow\) (i) Let \(x \in U \subseteq aGO(X)\). Let \(y \notin U\) and \(C_x, C_y \subseteq X_0\) containing \(x\) and \(y\) respectively. Then \(x \notin \text{pgcl}\{y\}\), which implies \(C_x \neq C_y\) and there exists pg-open set \(A\) such that \(C_x \subseteq A\) and \(C_y \subseteq A\). Since \(P_X\) is continuous and open, then \(y \in B = P_X^{-1}(A) \subseteq PGO(X)\) and \(x \in B\), which implies \(y \notin \text{pgcl}\{x\}\). Thus \(\text{pgcl}\{x\} \subseteq U\). This is true for all \(\text{pgcl}\{x\}\) implies \(\cap \text{pgcl}\{x\} \subseteq U\). Hence \(X\) is pg-R_0.

\textbf{Theorem 5.07:} \((X, \tau)\) is pg-R_\sigma\textit{ iff }\((X_0, Q(X_0))\) is pg-T_2\textit{.}

The proof is straightforward from theorems 5.05 and 5.06 and is omitted.

\textbf{Theorem 5.08:} \(X\) is pg-T_\sigma\textit{; }i = 0, 1, 2. iff there exists a pg-continuous, almost–open, 1–1 function from \((X, \tau)\) into a pg-T_\sigma\textit{-space; }i = 0, 1, 2. respectively.

\textbf{Theorem 5.09:} If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is pg-continuous, pg-open, and \(x, y \in X\) such that \(\text{pgcl}\{x\} = \text{pgcl}\{y\}\), then \(\text{pgcl}\{f(x)\} = \text{pgcl}\{f(y)\}\).

\textbf{Theorem 5.10:} The following are equivalent:

(i) \((X, \tau)\) is pg-T_\sigma\textit{.}

(ii) Elements of \(X_0\) are singleton sets and

(iii) There exists a pg-continuous, pg-open, 1–1 function \(f : (X, \tau) \rightarrow (Y, \sigma)\), where \((Y, \sigma)\) is pg-T_\sigma\textit{.}

\textbf{Proof:} (i) is equivalent to (ii) and (i) \(\Rightarrow\) (iii) are straightforward and is omitted.

(iii) \(\Rightarrow\) (i) Let \(x, y \in X\) such that \(f(x) \neq f(y)\), which implies \(\text{pgcl}\{f(x)\} \neq \text{pgcl}\{f(y)\}\). Then by theorem 5.09, \(\text{pgcl}\{x\} \neq \text{pgcl}\{y\}\). Hence \((X, \tau)\) is pg-T_\sigma\textit{.}
Corollary 5.11: A space \((X, \tau)\) is pg-T\(_i\) ; \(i = 1,2\) iff \((X, \tau)\) is pg-T\(_{i-1}\) ; \(i = 1,2\), respectively, and there exists a pg-continuous, pg-open, \(i-1\) function \(f: (X, \tau)\) into a pg-T\(_0\) space.

Definition 5.04: \(f: X \rightarrow Y\) is point–pg-closure \(i-1\) iff for \(x, y \in X\) such that pgcl\([x]\) \(\neq\) pgcl\([y]\), pgcl\([f(x)]\) \(\neq\) pgcl\([f(y)]\).

Theorem 6.12: (i) \(f: (X, \tau) \rightarrow (Y, \sigma)\) is point–pg-closure \(i-1\) and \((X, \tau)\) is pg-T\(_0\), then \(f\) is \(i-1\)
(ii) \(f: (X, \tau) \rightarrow (Y, \sigma)\), where \((X, \tau)\) and \((Y, \sigma)\) are pg-T\(_0\) then \(f\) is point–pg-closure \(i-1\) iff \(f\) is \(i-1\)

The following result can be obtained by combining results for pg-T\(_0\)–identification spaces, pg-induced functions and pg-T\(_i\) spaces; \(i = 1,2\).

Theorem 6.13: \(X\) is pg-R\(_i\) ; \(i = 0,1\) iff there exists a pg-continuous, almost–open point–pg-closure \(i-1\) function \(f: (X, \tau)\) into a pg-R\(_i\) space; \(i = 0,1\) respectively.

6. pg-Normal: Almost pg-normal and Mildly pg-normal spaces

Definition 6.1: A space \(X\) is said to be pg-normal if for any pair of disjoint closed sets \(F_1\) and \(F_2\), there exist disjoint pg-open sets \(U\) and \(V\) such that \(F_1 \subset U\) and \(F_2 \subset V\).

Example 4: Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b, c\}, X\}\). Then \(X\) is pg-normal.

Example 5: Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}\). Then \(X\) is not pg-normal and is not normal.

Example 6: Let \(X = \{a, b, c, d\}\) with \(\tau = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}\) is pg-normal, normal and almost normal.

We have the following characterization of pg-normality.

Theorem 6.1: For a space \(X\) the following are equivalent:
(i) \(X\) is pg-normal.
(ii) For every pair of open sets \(U\) and \(V\) whose union is \(X\), there exist pg-closed sets \(A\) and \(B\) such that \(A \subset U\) and \(B \subset V\).
(iii) For every closed set \(F\) and every open set \(G\) containing \(F\), there exists a pg-open set \(U\) such that \(F \subset U \subset pgcl(U) \subset G\).

Proof: (i)\(\Rightarrow\)(ii): Let \(U\) and \(V\) be a pair of open sets in a pg-normal space \(X\) such that \(X = U \cup V\). Then \(X-U\), \(X-V\) are disjoint closed sets. Since \(X\) is pg-normal there exist disjoint pg-open sets \(U_i\) and \(V_i\) such that \(X-U \subset U_i\) and \(X-V \subset V_i\).

Let \(A = X-U_i\), \(B = X-V_i\). Then \(A\) and \(B\) are pg-closed sets such that \(A \subset U\), \(B \subset V\) and \(A \cup B = X\).

(b) \(\Rightarrow\)(c): Let \(F\) be a closed set and \(G\) be an open set containing \(F\). Then \(X-F\) and \(G\) are open sets whose union is \(X\). Then by (b), there exist pg-closed sets \(W_1\) and \(W_2\) such that \(W_1 \subset X-F\) and \(W_2 \subset G\) and \(W_1 \cup W_2 = X\). Then \(F \subset X-W_1\), \(X-G \subset X-W_2\), and \((X-W_1) \cap (X-W_2) = \emptyset\). Let \(U = X-W_1\) and \(V = X-W_2\). Then \(U\) and \(V\) are disjoint pg-open sets such that \(F \subset U \cap V \subset G\). As \(X-V\) is pg-closed set, we have pgcl\((U) \subset X-V\) and pgcl\((U) \subset G\).

(c) \(\Rightarrow\)(a): Let \(F_1\) and \(F_2\) be any two disjoint closed sets of \(X\). Put \(G = X-F_2\), then \(F_1 \cap G = \emptyset\). \(F_1 \subset G\) where \(G\) is an open set. Then by (c), there exists a pg-open set \(U\) of \(X\) such that \(F_1 \subset U \subset pgcl(U) \subset G\). It follows that \(F_2 \subset X-pgcl(U) = V\), say, then \(V\) is pg-open and \(U \cap V = \emptyset\). Hence \(F_1\) and \(F_2\) are separated by pg-open sets \(U\) and \(V\). Therefore \(X\) is pg-normal.

Theorem 6.2: A regular open subspace of a pg-normal space is pg-normal.

Example 7: Let \(X = \{a, b, c, d\}\) with \(\tau = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}\) is pg-normal and pg-regular.

However we observe that every pg-normal pg-R\(_0\) space is pg-regular.

Definition 6.2: A function \(f: X \rightarrow Y\) is said to be almost –pg-irresolute if for each \(x\) in \(X\) and each pg-neighborhood \(V\) of \(f(x)\), pgcl\((f^{-1}(V))\) is a pg-neighborhood of \(x\).

Clearly every pg-irresolute map is almost pg-irresolute.

The Proof of the following lemma is straightforward and hence omitted.
Lemma 6.1: \( f \) is almost pg-irresolute iff \( f^{-1}(V) \subseteq pg-int(pgcl(f^{-1}(V))) \) for every \( V \in PGO(Y) \).

Lemma 6.2: \( f \) is almost pg-irresolute iff \( f(pgcl(U)) \subseteq pgcl(f(U)) \) for every \( U \in PGO(X) \).

Proof: Let \( U \in PGO(X) \). Suppose \( y \notin pgcl(f(U)) \). Then there exists \( V \in PGO(Y) \) such that \( V \cap (f(U)) = \emptyset \). Hence \( f^{-1}(V) \cap U = \emptyset \). Since \( U \in PGO(X) \), we have \( pg-int(pgcl(f^{-1}(V))) = pgcl(f(U)) \). Then by lemma 6.1, \( f^{-1}(V) \cap pgcl(U) = \emptyset \) and hence \( V \cap f(pgcl(U)) = \emptyset \). This implies that \( y \notin f(pgcl(U)) \).

Conversely, if \( V \in PGO(Y) \), then \( W = X - pgcl(f^{-1}(V)) \). By hypothesis, \( f(pgcl(W)) \subseteq pgcl(f(W)) \) and hence \( X - pg-int(pgcl(f^{-1}(V))) = pgcl(W) \). Then \( f^{-1}(V) \subseteq pgcl(f(U)) \). Since \( f \) is M-pg-open almost pg-irresolute surjection, we obtain \( A \subseteq f(U) \subseteq pgcl(f(U)) \). Then again by Theorem 6.1, the space \( Y \) is pg-normal.

Lemma 6.3: A mapping \( f \) is M-pg-closed if and only if for each subset \( B \) in \( Y \) and for each pg-open set \( U \) in \( X \) containing \( f^{-1}(B) \), there exists a pg-open set \( V \) containing \( B \) such that \( f^{-1}(V) \subseteq U \).

Theorem 6.4: If \( f \) is an M-pg-open continuous almost pg-irresolute function from a pg-normal space \( X \) onto a space \( Y \), then \( Y \) is pg-normal.

Proof: Let \( A \) be a closed subset of \( Y \) and \( B \) be an open set containing \( A \). Then by continuity of \( f \), \( f^{-1}(A) \) is closed and \( f^{-1}(B) \) is an open set of \( X \) such that \( f^{-1}(A) \subseteq f^{-1}(B) \). As \( X \) is pg-normal, there exists a pg-open set \( U \) in \( X \) such that \( f^{-1}(A) \subseteq U \subseteq pgcl(U) \subseteq f^{-1}(B) \). Then \( f^{-1}(A) \subseteq f(U) \subseteq pgcl(f(U)) \subseteq f(f^{-1}(B)) \). Since \( f \) is M-pg-open almost pg-irresolute, we obtain \( A \subseteq f(U) \subseteq pgcl(f(U)) \). Then again by Theorem 6.1, the space \( Y \) is pg-normal.

Theorem 6.6: For a space \( X \) we have the following: (a) If \( X \) is normal then for any disjoint closed sets \( A \) and \( B \), there exist disjoint pg-open sets \( U \) and \( V \) such that \( A \cup U \) and \( B \subseteq V \); (b) If \( X \) is normal then for any closed set \( A \) and any open set \( V \) containing \( A \), there exists an pg-open set \( U \) of \( X \) such that \( A \subseteq pgcl(U) \).
exist disjoint pg-open sets \( U_1 \) and \( V_1 \) such that \( X - U \subset U_1 \) and \( X - V \subset V_1 \). Let \( G = X - U_1 \) and \( H = X - V_1 \). Then \( G \) and \( H \) are pg-closed sets such that \( G \subset U, H \subset V \) and \( G \cup H = X \).

(b) \( \Rightarrow \) (c) and (c) \( \Rightarrow \) (a) are obvious.
One can prove that almost pg-normality is also regular open hereditary.

Almost pg-normality does not imply almost pg-regularity in general. However, we observe that every almost pg-normal pg-R0 space is almost pg-regular.

**Theorem 6.8:** Every almost regular, pg-compact space \( X \) is almost pg-normal.

Recall that a function \( f : X \to Y \) is called rc-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost pg-normality in the following.

**Theorem 6.9:** If \( f \) is continuous M-pg-open rc-continuous and almost pg-irresolute surjection from an almost pg-normal space \( X \) onto a space \( Y \), then \( Y \) is almost pg-normal.

We have the following characterization of mild pg-normality.

**Theorem 6.10:** For a space \( X \) the following are equivalent.
(i) \( X \) is mildly pg-normal.
(ii) For every pair of regular open sets \( U \) and \( V \) whose union is \( X \), there exist pg-closed sets \( G \) and \( H \) such that \( G \subset U, H \subset V \) and \( G \cup H = X \).
(iii) For any regular closed set \( A \) and every regular open set \( B \) containing \( A \), there exists a pg-open set \( U \) such that \( A \subset U \subset pgcl(U) \subset B \).
(iv) For every pair of disjoint regular closed sets, there exist pg-open sets \( U \) and \( V \) such that \( A \subset U, B \subset V \) and \( pgcl(U) \cap pgcl(V) = \emptyset \).

This theorem may be proved by using the arguments similar to those of Theorem 6.7.

Also, we observe that mild pg-normality is regular open hereditary.

**Definition 6.4:** A space \( X \) is weakly pg-regular if for each point \( x \) and a regular open set \( U \) containing \( \{x\} \), there is a pg-open set \( V \) such that \( x \in V \subset clV \subset U \).

**Example 9:** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{b, c\}, \{b\}, \{a, b\}, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}, X\} \). Then \( X \) is not weakly pg-regular.

**Theorem 6.11:** If \( f : X \to Y \) is an M-pg-open rc-continuous and almost pg-irresolute function from a mildly pg-normal space \( X \) onto a space \( Y \), then \( Y \) is mildly pg-normal.

Proof: Let \( A \) be a regular closed set and \( B \) be a regular open set containing \( A \). Then by rc-continuity of \( f \), \( f^{-1}(A) \) is a regular closed set contained in the regular open set \( f^{-1}(B) \). Since \( X \) is mildly pg-normal, there exists a pg-open set \( V \) such that \( f^{-1}(A) \subset pgcl(V) \subset f^{-1}(B) \) by Theorem 6.10. As \( f \) is M-pg-open and almost pg-irresolute surjection, it follows that \( f(V) \in PGO(Y) \) and \( A \subset f(V) \subset pgcl(f(V)) \subset B \). Hence \( Y \) is mildly pg-normal.

**Theorem 6.12:** If \( f : X \to Y \) is rc-continuous, M-pg-closed map from a mildly pg-normal space \( X \) onto a space \( Y \), then \( Y \) is mildly pg-normal.

7. pg-US Spaces:

**Definition 7.1:** A sequence \( \langle x_n \rangle \) is said to be pg-converges to a point \( x \) of \( X \), written as \( \langle x_n \rangle \to_{pg}^x \) if \( \langle x_n \rangle \) is eventually in every pg-open set containing \( x \).
Clearly, if a sequence \(<x_n>\) \(r\)-converges to a point \(x\) of \(X\), then \(<x_n>\) \(pg\)-converges to \(x\).

**Definition 7.2:** \(X\) is said to be \(pg\)-US if every sequence \(<x_n>\) in \(X\) \(pg\)-converges to a unique point.

**Definition 7.3:** A set \(F\) is sequentially \(pg\)-closed if every sequence in \(F\) \(pg\)-converges to a point in \(F\).

**Definition 7.4:** A subset \(G\) of a space \(X\) is said to be sequentially \(pg\)-compact if every sequence in \(G\) has a subsequence which \(pg\)-converges to a point in \(G\).

**Definition 7.5:** A point \(y\) is a \(pg\)-cluster point of sequence \(<x_n>\) if \(<x_n>\) is frequently in every \(pg\)-open set containing \(x\). The set of all \(pg\)-cluster points of \(<x_n>\) will be denoted by \(pg\-cl(x_n)\).

**Definition 7.6:** A point \(y\) is \(pg\)-side point of a sequence \(<x_n>\) if \(y\) is a \(pg\)-cluster point of \(<x_n>\) but no subsequence of \(<x_n>\) \(pg\)-converges to \(y\).

**Definition 7.7:** A space \(X\) is said to be
(i) \(pg\)-S1 if it is \(pg\)-US and every sequence \(<x_n>\) \(pg\)-converges with subsequence of \(<x_n>\) \(pg\)-side points.
(ii) \(pg\)-S2 if it is \(pg\)-US and every sequence \(<x_n>\) in \(X\) \(pg\)-converges which has no \(pg\)-side point.

Using sequentially continuous functions, we define sequentially \(pg\)-continuous functions.

**Definition 7.8:** A function \(f\) is said to be sequentially \(pg\)-continuous at \(x\) in \(X\) if \(f(x_n)\) \(\rightarrow^{pg} f(x)\) whenever \(x_n\) \(\rightarrow^{pg} x\). If \(f\) is sequentially \(pg\)-continuous at all \(x \in X\), then \(f\) is said to be sequentially \(pg\)-continuous.

**Theorem 7.1:** We have the following:
(i) Every \(pg\)-T2 space is \(pg\)-US.
(ii) Every \(pg\)-US space is \(pg\)-T1.
(iii) \(X\) is \(pg\)-US if the diagonal set is a sequentially \(pg\)-closed subset of \(X \times X\).
(iv) \(X\) is \(pg\)-T2 if it is both \(pg\)-R1 and \(pg\)-US.
(v) Every regular open subset of a \(pg\)-US space is \(pg\)-US.
(vi) Product of arbitrary family of \(pg\)-US spaces is \(pg\)-US.
(vii) Every \(pg\)-S2 space is \(pg\)-S1 and Every \(pg\)-S1 space is \(pg\)-US.

**Theorem 7.2:** In a \(pg\)-US space every sequentially \(pg\)-compact set is sequentially \(pg\)-closed.

**Proof:** Let \(X\) be \(pg\)-US space. Let \(Y\) be a sequentially \(pg\)-compact subset of \(X\). Let \(<x_n>\) be a sequence in \(Y\). Suppose that \(<x_n>\) \(pg\)-converges to a point in \(X-Y\). Let \(<x_{np}>\) be a subsequence of \(<x_n>\) that \(pg\)-converges to a point \(y\) in \(Y\) since \(Y\) is sequentially \(pg\)-compact. Also, let a subsequence \(<x_{np}>\) of \(<x_{np}>\) \(pg\)-converge to \(x\) in \(X-Y\). Since \(<x_{np}>\) is a sequence in the \(pg\)-US space \(X, x = y\). Thus, \(Y\) is sequentially \(pg\)-closed set.

**Theorem 7.3:** Let \(f\) and \(g\) be two sequentially \(pg\)-continuous functions. If \(Y\) is \(pg\)-US, then the set \(A = \{x | f(x) = g(x)\}\) is sequentially \(pg\)-closed.

**Proof:** Let \(Y\) be \(pg\)-US and suppose that there is a sequence \(<x_n>\) in \(A\) \(pg\)-converging to \(x \in X\). Since \(f\) and \(g\) are sequentially \(pg\)-continuous functions, \(f(x_n)\) \(\rightarrow^{pg} f(x)\) and \(g(x_n)\) \(\rightarrow^{pg} g(x)\). Hence \(f(x) = g(x)\) and \(x \in A\). Therefore, \(A\) is sequentially \(pg\)-closed.

8. SEQUENTIALLY sub-\(pg\)-CONTINUITY:

In this section we introduce and study the concepts of sequentially sub-\(pg\)-continuity, sequentially nearly \(pg\)-continuity and sequentially \(pg\)-compact preserving functions and study their relations and the property of \(pg\)-US spaces.

**Definition 8.1:** A function \(f\) is said to be
(i) sequentially nearly \(pg\)-continuous if for each point \(x \in X\) and each sequence \(<x_n>\) \(\rightarrow^{pg} x\) in \(X\), there exists a subsequence \(<x_{nak}>\) of \(<x_n>\) such that \(f(x_{nak})\) \(\rightarrow^{pg} f(x)\).
(ii) sequentially sub-\(pg\)-continuous if for each point \(x \in X\) and each sequence \(<x_n>\) \(\rightarrow^{pg} x\) in \(X\), there exists a subsequence \(<x_{nak}>\) of \(<x_n>\) and a point \(y \in Y\) such that \(f(x_{nak})\) \(\rightarrow^{pg} y\).
(iii) sequentially \(pg\)-compact preserving if \(f(K)\) is sequentially \(pg\)-compact in \(Y\) for every sequentially \(pg\)-compact set \(K\) of \(X\).

**Lemma 8.1:** Every function \(f\) is sequentially sub-\(pg\)-continuous if \(Y\) is a sequentially \(pg\)-compact.
Proof: Let \( \langle x_n \rangle \rightarrow^p x \) in X. Since Y is sequentially pg-compact, there exists a subsequence \( \langle f(x_{n_k}) \rangle \) of \( \langle f(x_n) \rangle \) pg-converging to a point \( y \in Y \). Hence \( f \) is sequentially sub-pg-continuous.

**Theorem 8.1:** Every sequentially nearly pg-continuous function is sequentially pg-compact preserving.

**Proof:** Assume \( f \) is sequentially nearly pg-continuous and \( K \) any sequentially pg-compact subset of X. Let \( \langle y_n \rangle \) be any sequence in \( f(K) \). Then for each positive integer \( n \), there exists a point \( x_n \in K \) such that \( f(x_n) = y_n \). Since \( \langle x_n \rangle \) is a sequence in the sequentially pg-compact set \( K \), there exists a subsequence \( \langle x_{n_k} \rangle \) of \( \langle x_n \rangle \) pg-converging to a point \( x \in K \). By hypothesis, \( f \) is sequentially nearly pg-continuous and hence there exists a subsequence \( \langle x_{n_{k_l}} \rangle \) of \( \langle x_{n_k} \rangle \) such that \( f(x_{n_{k_l}}) \rightarrow^p f(x) \). Thus, there exists a subsequence \( \langle y_{j_l} \rangle \) of \( \langle y_{n_l} \rangle \) pg-converging to \( f(x) \in f(K) \). This shows that \( f(K) \) is sequentially pg-compact set in Y.

**Theorem 8.2:** Every sequentially pre-continuous function is sequentially pg-continuous.

**Proof:** Let \( f \) be a sequentially pre-continuous and \( \langle x_n \rangle \rightarrow^p x \in X \). Then \( \langle x_n \rangle \rightarrow^p x \). Since \( f \) is sequentially pre-continuous, \( f(\langle x_n \rangle) \rightarrow^p f(x) \). But we know that \( \langle x_n \rangle \rightarrow^p x \) implies \( \langle x_n \rangle \rightarrow^p x \) and hence \( f(\langle x_n \rangle) \rightarrow^p f(x) \) implies \( f \) is sequentially pg-continuous.

**Theorem 8.3:** Every sequentially pg-compact preserving function is sequentially sub-pg-continuous.

**Proof:** Suppose \( f \) is a sequentially pg-compact preserving function. Let \( x \) be any point of X and \( \langle x_n \rangle \) any sequence in X pg-converging to \( x \). We shall denote the set \( \{x_n \mid n = 1, 2, 3, \ldots\} \) by \( A \) and \( K = A \cup \{x\} \). Then \( K \) is sequentially pg-compact since \( \langle x_n \rangle \rightarrow^p x \). By hypothesis, \( f \) is sequentially pg-compact preserving and hence \( f(K) \) is a sequentially pg-compact set of Y. Since \( \langle f(x_n) \rangle \) is a sequence in \( f(K) \), there exists a subsequence \( \langle f(x_{n_k}) \rangle \) of \( \langle f(x_n) \rangle \) pg-converging to a point \( y \in f(K) \). This implies that \( f \) is sequentially sub-pg-continuous.

**Theorem 8.4:** A function \( f: X \rightarrow Y \) is sequentially pg-compact preserving iff \( f_K: K \rightarrow f(K) \) is sequentially sub-pg-continuous for each sequentially pg-compact subset \( K \) of X.

**Proof:** Suppose \( f \) is a sequentially pg-compact preserving function. Then \( f(K) \) is sequentially pg-compact set in Y for each sequentially pg-compact set \( K \) of X. Therefore, by Lemma 8.1 above, \( f_K: K \rightarrow f(K) \) is sequentially pg-continuous function.

Conversely, let \( K \) be any sequentially pg-compact set of X. Let \( \langle y_n \rangle \) be any sequence in \( f(K) \). Then for each positive integer \( n \), there exists a point \( x_n \in K \) such that \( f(x_n) = y_n \). Since \( \langle x_n \rangle \) is a sequence in the sequentially pg-compact set \( K \), there exists a subsequence \( \langle x_{n_{k_l}} \rangle \) of \( \langle x_n \rangle \) pg-converging to a point \( x \in K \). By hypothesis, \( f_K: K \rightarrow f(K) \) is sequentially sub-pg-continuous and hence there exists a subsequence \( \langle y_{n_{k_l}} \rangle \) of \( \langle y_n \rangle \) pg-converging to a point \( y \in f(K) \). This implies that \( f(K) \) is sequentially pg-compact set in Y. Thus, \( f \) is sequentially pg-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sub-pg-continuous function to be sequentially pg-compact preserving.

**Corollary 8.1:** If \( f \) is sequentially sub-pg-continuous and \( f(K) \) is sequentially pg-closed set in Y for each sequentially pg-compact set \( K \) of X, then \( f \) is sequentially pg-compact preserving function.

**REFERENCES**


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