On \(\alpha g\)-Separation Axioms

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Abstract

In this paper by using \(\alpha g\)-open sets we define almost \(\alpha g\)-normality and mild \(\alpha g\)-normality also we continue the study of further properties of \(\alpha g\)-normality. We show that these three axioms are regular open hereditary. We also define the class of almost \(\alpha g\)-irresolute mappings and show that \(\alpha g\)-normality is invariant under almost \(\alpha g\)-irresolute \(M-\alpha g\)-open continuous surjection.

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1. Introduction

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C.E. Aull studied some separation axioms between the \(T_1\) and \(T_2\) spaces, namely, \(S_1\) and \(S_2\). Next, in 1982, S.P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, sequentially s-compact notions. G.B. Navlagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P. Aruna Swathi Vysayanthi studied \(v\)-Normal Almost- \(v\)-Normal, Mildly- \(v\)-Normal, \(v\)-US, \(v\)-S\(_1\) and \(v\)-S\(_2\). Also we examine \(\alpha g\)-convergence, sequentially \(\alpha g\)-compact, sequentially \(\alpha g\)-continuous maps, and sequentially sub \(\alpha g\)-continuous maps in the context of these new concepts. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper \(X\) and \(Y\) denote Topological spaces on which no separation axioms are assumed explicitly stated.

2. Preliminaries

Definition 2.1: \(A \subseteq X\) is called
(i) g-closed if \(\text{cl } A \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
(ii) \(\alpha g\)-closed if \(\alpha g\text{-cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha g\)-open in \(X\).

Definition 2.2: A function \(f\) is said to be almost–pre–irresolute if for each \(x\) in \(X\) and each pre-neighborhood \(V\) of \(f(x),\) \(\text{pcl}(f^{-1}(V))\) is a pre-neighborhood of \(x\).

Definition 2.3: A space \(X\) is said to be
(i) \(T_1\) (\(T_2\)) if for any \(x \neq y\) in \(X\), there exist (disjoint) open sets \(U, V\) in \(X\) such that \(x \in U\) and \(y \in V\).
(ii) weakly Hausdorff if each point of \(X\) is the intersection of regular closed sets of \(X\).
(iii) normal [resp: mildly normal] if for any pair of disjoint [resp: regular-closed] closed sets \(F_1\) and \(F_2\), there exist disjoint open sets \(U\) and \(V\) such that \(F_1 \subseteq U\) and \(F_2 \subseteq V\).
(iv) almost normal if for each closed set \(A\) and each regular closed set \(B\) such that \(A \cap B = \emptyset\), there exist disjoint open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).
(v) weakly regular if for each pair consisting of a regular closed set \(A\) and a point \(x\) such that \(A \cap \{x\} = \emptyset\), there exist disjoint open sets \(U\) and \(V\) such that \(x \in U\) and \(A \subseteq V\).

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(vi) A subset \( A \) of a space \( X \) is \( S \)-closed relative to \( X \) if every cover of \( A \) by semiopen sets of \( X \) has a finite subfamily whose closures cover \( A \).

(vii) \( R_0 \) if for any point \( x \) and a closed set \( F \) with \( x \not\in F \) in \( X \), there exists an open set \( G \) containing \( F \) but not \( x \).

(viii) \( R_1 \) if for \( x, y \in X \) with \( \text{cl}\{x\} \neq \text{cl}\{y\} \), there exist disjoint open sets \( U \) and \( V \) such that \( \text{cl}\{x\} \subset U \) and \( \text{cl}\{y\} \subset V \).

(ix) US-space if every convergent sequence has exactly one limit point to which it converges.

(x) \( S \)-pre-US space if every pre-convergent sequence has exactly one limit point to which it converges.

(xi) \( S \)-pre-S, if it is \( S \)-pre-US and every sequence \( <x_n> \) pre-converges with subsequence of \( <x_n> \) pre-side points.

(xii) \( S \)-pre-S, if it is \( S \)-pre-US and every sequence \( <x_n> \) in \( X \) pre-converges which has no pre-side point.

(xiii) \( \alpha \) weakly countable compact if every infinite subset of \( X \) has a limit point in \( X \).

(xiv) Baire space if for any countable collection of closed sets with empty interior in \( X \), their union also has empty interior in \( X \).

\[ \text{Definition 2.4:} \text{ Let } A \subset X. \text{ Then a point } x \text{ is said to be a} \]

(i) limit point of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( x \neq y \).

(ii) \( \alpha \)-limit point of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( \text{cl}\{x\} \neq \text{cl}\{y\} \), or equivalently, such that they are topologically distinct.

(iii) \( \alpha \)-pre-\( \alpha \)-limit point of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( \text{pcl}\{x\} \neq \text{pcl}\{y\} \), or equivalently, such that they are topologically distinct.

\[ \text{Note 1:} \text{ Recall that two points are topologically distinguishable or distinct if there exists an open set containing one of the points but not the other; equivalently if they have disjoint closures. In fact, the } \alpha \text{-axiom is precisely to ensure that any two distinct points are topologically distinct.} \]

\[ \text{Example 1:} \text{ Let } X = \{a, b, c, d\} \text{ and } \tau = \{\{a\}, \{b, c\}, \{a, b, c\}, X, \phi\}. \text{ Then } b \text{ and } c \text{ are the limit points but not the } \alpha \text{-limit points of the set } \{b, c\}. \text{ Further } d \text{ is a } \alpha \text{-limit point of } \{b, c\}. \]

\[ \text{Example 2:} \text{ Let } X = (0, 1) \text{ and } \tau = \{\phi, X, \text{ and } U_n = (0, 1-\frac{1}{n}), n = 2, 3, 4, \ldots \}. \text{ Then every point of } X \text{ is a limit point of } X. \text{ Every point of } X-U_2 \text{ is a } \alpha \text{-limit point of } X, \text{ but no point of } U_2 \text{ is a } \alpha \text{-limit point of } X. \]

\[ \text{Definition 2.5:} \text{ A set } A \text{ together with all its } \alpha \text{-limit points will be denoted by } T_0\text{-cl} A. \]

\[ \text{Note 2:} \text{ i. Every } \alpha \text{-limit point of a set } A \text{ is a limit point of the set but the converse is not true in general.} \]

\[ ii. \text{ In } T_0\text{-space both are same.} \]

\[ \text{Note 3:} \text{ } R_\alpha \text{-axiom is weaker than } T_1\text{-axiom. It is independent of the } T_\alpha\text{-axiom. However } T_1 = R_\alpha + T_0 \]

\[ \text{Note 4:} \text{ Every countable compact space is weakly countable compact but converse is not true in general. However, a } T_1\text{-space is weakly countable compact iff it is countable compact.} \]

3. \( \alpha \)-\( T_\alpha \) LIMIT POINT:

\[ \text{Definition 3.01:} \text{ In } X, \text{ a point } x \text{ is said to be a } \alpha \text{-}\( T_\alpha \)-limit point of } A \text{ if each } \alpha \text{-open set containing } x \text{ contains some point } y \text{ of } A \text{ such that } \text{agcl}\{x\} \neq \text{agcl}\{y\}, \text{ or equivalently; such that they are topologically distinct with respect to } \alpha \text{-open sets.} \]

\[ \text{Example 3:} \text{ regular open set } \Rightarrow \text{ open set } \Rightarrow \alpha \text{-open set } \Rightarrow \alpha \text{-open set we have} \]

\[ r-\alpha \text{-limit point } \Rightarrow \alpha \text{-limit point } \Rightarrow \alpha-\alpha \text{-limit point } \Rightarrow \alpha-\alpha \text{-limit point} \]

\[ \text{Definition 3.02:} \text{ A set } A \text{ together with all its } \alpha-\alpha \text{-limit points is denoted by } T_\alpha\text{-agcl}(A) \]

\[ \text{Lemma 3.01:} \text{ If } x \text{ is a } \alpha-\alpha \text{-limit point of a set } A \text{ then } x \text{ is } \alpha \text{-limit point of } A. \]

\[ \text{Lemma 3.02:} \text{ If } X \text{ is } \alpha-\alpha \text{-space then every } \alpha-\alpha \text{-limit point and every } \alpha \text{-limit point are equivalent.} \]

\[ \text{Corollary 3.03:} \text{ If } X \text{ is } r-\alpha \text{-space then every } \alpha-\alpha \text{-limit point and every } \alpha \text{-limit point are equivalent.} \]

\[ \text{Theorem 3.04:} \text{ For } x \neq y \in X, \]

\[ (i) \text{ } x \text{ is a } \alpha-\alpha \text{-limit point of } \{y\} \text{ iff } x \notin \text{agcl}\{y\} \text{ and } y \in \text{agcl}\{x\}, \]

\[ (ii) \text{ } x \text{ is not a } \alpha-\alpha \text{-limit point of } \{y\} \text{ iff either } x \notin \text{agcl}\{y\} \text{ or } \text{agcl}\{x\} = \text{agcl}\{y\}, \]

\[ (iii) \text{ } x \text{ is not a } \alpha-\alpha \text{-limit point of } \{y\} \text{ iff either } x \notin \text{agcl}\{y\} \text{ or } y \notin \text{agcl}\{x\}. \]
Corollary 3.05:
(i) If \( x \) is a \( \alpha-T_\vartheta \)-limit point of \( \{y\} \), then \( y \) cannot be a \( \alpha \)-limit point of \( \{x\} \).
(ii) If \( \text{agcl}\{x\} = \text{agcl}\{y\} \), then neither \( x \) is a \( \alpha-T_\vartheta \)-limit point of \( \{y\} \) nor \( y \) is a \( \alpha-T_\vartheta \)-limit point of \( \{x\} \).
(iii) If a singleton set \( A \) has no \( \alpha-T_\vartheta \)-limit point in \( X \), then \( \text{agcl}A = \text{agcl}\{x\} \) for all \( x \in \text{agcl}\{A\} \).

Lemma 3.06: In \( X \), if \( x \) is a \( \alpha \)-limit point of a set \( A \), then in each of the following cases \( x \) becomes \( \alpha-T_\vartheta \)-limit point of \( A \) \( (\{x\} \neq A) \).
(i) \( \text{agcl}\{x\} \neq \text{agcl}\{y\} \) for \( y \in A, x \neq y \).
(ii) \( \text{agcl}\{x\} = \{x\} \).
(iii) \( X \) is a \( \alpha-T_\vartheta \)-topological space.
(iv) \( A \sim \{x\} \) is \( \alpha \)-open

Corollary 3.07: In \( X \), if \( x \) is a limit point of a set \( A \), then in each of the following cases \( x \) becomes \( \alpha-T_\vartheta \)-limit point of \( A \) \( (\{x\} \neq A) \).
(i) \( \text{agcl}\{x\} \neq \text{agcl}\{y\} \) for \( y \in A, x \neq y \).
(ii) \( \text{agcl}\{x\} = \{x\} \).
(iii) \( X \) is a \( \alpha-T_\vartheta \)-topological space.
(iv) \( A \sim \{x\} \) is \( \alpha \)-open

4. \( \alpha-T_\vartheta \) AND \( \alpha-R_i \) AXIOMS, \( i = 0, 1 \):
In view of Lemma 3.6(iii), \( \alpha-T_\vartheta \)-axiom implies the equivalence of the concept of limit point of a set with that of \( \alpha-T_\vartheta \)-limit point of the set. But for the converse, if \( x \in \text{agcl}\{y\} \) then \( \text{agcl}\{x\} \neq \text{agcl}\{y\} \) in general, but if \( x \) is a \( \alpha-T_\vartheta \)-limit point of \( \{y\} \), then \( \text{agcl}\{x\} = \text{agcl}\{y\} \)

Lemma 4.01: In a space \( X \), a limit point \( x \) of \( \{y\} \) is a \( \alpha-T_\vartheta \)-limit point of \( \{y\} \) iff \( \text{agcl}\{x\} \neq \text{agcl}\{y\} \).

This lemma leads to characterize the equivalence of \( \alpha-T_\vartheta \)-limit point and \( \alpha \)-limit point of a set as the \( \alpha-T_\vartheta \)-axiom.

Theorem 4.02: The following conditions are equivalent:
(i) \( X \) is a \( \alpha-T_0 \) space
(ii) Every \( \alpha \)-limit point of a set \( A \) is a \( \alpha-T_\vartheta \)-limit point of \( A \)
(iii) Every \( r \)-limit point of a singleton set \( \{x\} \) is a \( \alpha-T_\vartheta \)-limit point of \( \{x\} \)
(iv) For any \( x, y \) in \( X \), \( x \neq y \) if \( x \in \text{agcl}\{y\} \), then \( x \) is a \( \alpha-T_\vartheta \)-limit point of \( \{y\} \)

Note 5: In a \( \alpha-T_\vartheta \)-space \( X \) if every point of \( X \) is a \( r \)-limit point of \( X \), then every point of \( X \) is \( \alpha-T_\vartheta \)-limit point of \( X \). But a space \( X \) in which each point is a \( \alpha-T_\vartheta \)-limit point of \( X \) is not necessarily a \( \alpha-T_\vartheta \)-space

Theorem 4.03: The following conditions are equivalent:
(i) \( X \) is a \( \alpha-R_0 \) space
(ii) For any \( x, y \) in \( X \), \( x \neq y \) if \( x \in \text{agcl}\{y\} \), then \( x \) is not a \( \alpha-T_\vartheta \)-limit point of \( \{y\} \)
(iii) A point \( \alpha \)-closure set has no \( \alpha-T_\vartheta \)-limit point in \( X \)
(iv) A singleton set has no \( \alpha-T_\vartheta \)-limit point in \( X \).

Since every \( r-R_0 \) space is \( \alpha-R_0 \) space, we have the following corollary

Corollary 4.04: The following conditions are equivalent:
(i) \( X \) is a \( r-R_0 \) space
(ii) For any \( x, y \) in \( X \), \( x \neq y \) if \( x \in \text{agcl}\{y\} \), then \( x \) is not a \( \alpha-T_\vartheta \)-limit point of \( \{y\} \)
(iii) A point \( \alpha \)-closure set has no \( \alpha-T_\vartheta \)-limit point in \( X \)
(iv) A singleton set has no \( \alpha-T_\vartheta \)-limit point in \( X \).

Theorem 4.05: In a \( \alpha-R_0 \) space \( X \), a point \( x \) is \( \alpha-T_\vartheta \)-limit point of \( A \) iff every \( \alpha \)-open set containing \( x \) contains infinitely many points of \( A \) with each of which \( x \) is topologically distinct

If \( \alpha-R_0 \) space is replaced by \( rR_\vartheta \) space in the above theorem, we have the following corollaries:

Corollary 4.06: In an \( rR_\vartheta \) space \( X \),
(i) If a point \( x \) is \( rT_\vartheta \)-limit point of a set then every \( \alpha \)-open set containing \( x \) contains infinitely many points of \( A \) with each of which \( x \) is topologically distinct.
(ii) If a point \( x \) is \( \alpha-T_\vartheta \)-limit point of a set then every \( \alpha \)-open set containing \( x \) contains infinitely many points of \( A \) with each of which \( x \) is topologically distinct.
Corollary 4.08: If X is rR₀ space and
(i) If \( A = \bigcup agcl\{x_i\} \) i = 1 to n, a finite union of point closure sets has no \( ag-T₀\)-limit point.
(ii) If \( X = \bigcup agcl\{x_i\} \) i = 1 to n, then X has no \( ag-T₀\)-limit point.

Theorem 4.09: The following conditions are equivalent:
(i) \( X \) is \( ag-R₀\)-space
(ii) For any \( x \) and a set in \( X \), \( x \) is a \( ag-T₀\)-limit point of \( A \) if every \( ag\)-open set containing \( x \) contains infinitely many points of \( A \) with each of which \( x \) is topologically distinct.

Various characteristic properties of \( ag-T₀\)-limit points studied so far is enlisted in the following theorem for a ready reference.

Theorem 4.10: In a \( ag-R₀\)-space, we have the following:
(i) A singleton set has no \( ag-T₀\)-limit point in \( X \).
(ii) A finite set has no \( ag-T₀\)-limit point in \( X \).
(iii) A point \( ag\)-closure has no set \( ag-T₀\)-limit point in \( X \).
(iv) A finite union point \( ag\)-closure sets have no set \( ag-T₀\)-limit point in \( X \).
(v) For \( x, y \in X, x \in agcl\{y\} \) if \( x = y \).
(vi) For \( x, y \in X, x \neq y \) if \( x \) is \( ag-T₀\)-limit point of \( \{y\} \) nor \( y \) is \( ag-T₀\)-limit point of \( \{x\} \).
(vii) For \( x, y \in X, x \neq y \) if \( \cap \{agcl\{y\} \cap \cap \{agcl\{x\}\} = \phi \).
(viii) Any point \( x \in X \) is a \( ag-T₀\)-limit point of a set \( A \) in \( X \) if every \( ag\)-open set containing \( x \) contains infinitely many points of \( A \) with each which \( x \) is topologically distinct.

Theorem 4.11: \( X \) is \( ag-R₁\) iff for any \( ag\)-open set \( U \) in \( X \) and points \( x, y \) such that \( x \in \text{cl}\{U \} \), \( y \in U \), there exists a \( ag\)-open set \( V \) in \( X \) such that \( y \in \text{cl}\{V \} \), \( x \notin V \).

Lemma 4.12: In \( ag-R₁\) space \( X \), if \( x \) is a \( ag-T₀\)-limit point of \( X \), then for any non empty \( ag\)-open set \( U \), there exists a non empty \( ag\)-open set \( V \) such that \( V \subseteq X \), \( x \notin agcl\{V \} \).

Lemma 4.13: In a \( ag\)-regular space \( X \), if \( x \) is a \( ag-T₀\)-limit point of \( X \), then for any non empty \( ag\)-open set \( U \), there exists a non empty \( ag\)-open set \( V \) such that \( agcl\{V \} \subseteq U \), \( x \notin agcl\{V \} \).

Corollary 4.14: In a regular space \( X \),
(i) If \( x \) is a \( ag-T₀\)-limit point of \( X \), then for any non empty \( ag\)-open set \( U \), there exists a non empty \( ag\)-open set \( V \) such that \( agcl\{V \} \subseteq U \), \( x \notin agcl\{V \} \).
(ii) If \( x \) is a \( T₀\)-limit point of \( X \), then for any non empty \( ag\)-open set \( U \), there exists a non empty \( ag\)-open set \( V \) such that \( agcl\{V \} \subseteq U \), \( x \notin agcl\{V \} \).

Theorem 4.15: If \( X \) is a \( ag\)-compact \( ag-R₁\)-space, then \( X \) is a Baire Space.

Proof: Let \( \{A_n\} \) be a countable collection of \( ag\)-closed sets of \( X \), each \( A_n \) having empty interior in \( X \). Take \( A_1 \), since \( A_1 \) has empty interior, \( A_1 \) does not contain any \( ag\)-open set say \( U_0 \). Therefore we can choose a point \( y \in U_0 \) such that \( y \notin A_1 \). For \( X \) is \( ag\)-regular, and \( y \in (X-A_1) \cap U_0 \), a \( ag\)-open set, we can find a \( ag\)-open set \( U_1 \) in \( X \) such that \( y \in U_1 \), \( agcl\{U_1\} \subseteq (X-A_1) \cap U_0 \). Hence \( U_1 \) is a non empty \( ag\)-open set in \( X \) such that \( agcl\{U_1\} \subseteq U_0 \) and \( vcl(U_1) \cap A_1 = \phi \). Continuing this process, in general, for given non empty \( ag\)-open set \( U_{n-1} \), we can choose a point of \( U_{n-1} \), which is not in the \( ag\)-closed set \( A_n \) and a \( ag\)-open set \( U_n \) containing this point such that \( agcl\{U_n\} \subseteq U_{n-1} \) and \( agcl\{U_n\} \cap A_n = \phi \). Thus we get a sequence of nested non empty \( ag\)-closed sets which satisfies the finite intersection property. Therefore \( \bigcap agcl\{U_n\} = \phi \). Then some \( x \in \bigcap agcl\{U_n\} \) which in turn implies that \( x \in \bigcap U_{n-1} \) as \( agcl\{U_n\} \subseteq U_{n-1} \) and \( x \notin A_n \) for each \( n \).

Corollary 4.16: If \( X \) is a compact \( ag-R₁\)-space, then \( X \) is a Baire Space.

Corollary 4.17: Let \( X \) be a \( ag\)-compact \( ag-R₁\)-space. If \( \{A_n\} \) is a countable collection of \( ag\)-closed sets in \( X \), each \( A_n \) having non-empty \( ag\)-interior in \( X \), then there is a point of \( X \) which is not in any of the \( A_n \).

Corollary 4.18: Let \( X \) be a \( ag\)-compact \( R₁\)-space. If \( \{A_n\} \) is a countable collection of \( ag\)-closed sets in \( X \), each \( A_n \) having non-empty \( ag\)-interior in \( X \), then there is a point of \( X \) which is not in any of the \( A_n \).

Theorem 4.19: Let \( X \) be a non empty compact \( ag-R₁\)-space. If every point of \( X \) is a \( ag-T₀\)-limit point of \( X \) then \( X \) is uncountable.
Proof: Since X is non empty and every point is a $T_{\emptyset}$-limit point of X, X must be infinite. If X is countable, we construct a sequence of $\alpha$-open sets $\{V_n\}$ in X as follows:

Let $X = V_1$, then for $x_1$ is a $T_{\emptyset}$-limit point of X, we can choose a non empty $\alpha$-open set $V_2$ in X such that $V_2 \subset V_1$ and $x_1 \not\in \text{agcl} V_2$. Next for $x_2$ and non empty $\alpha$-open set $V_3$, we can choose a non empty $\alpha$-open set $V_3$ in X such that $V_3 \subset V_2$ and $x_2 \not\in \text{agcl} V_3$. Continuing this process for each $x_n$ and a non empty $\alpha$-open set $V_n$, we can choose a non empty $\alpha$-open set $V_{n+1}$ in X such that $V_{n+1} \subset V_n$ and $x_n \not\in \text{agcl} V_{n+1}$.

Now consider the nested sequence of $\alpha$-closed sets $\text{agcl}V_1 \supset \text{agcl}V_2 \supset \text{agcl}V_3 \supset \ldots \supset \text{agcl}V_n \supset \ldots$ Since X is $\alpha$-compact and $\text{agcl}V_n$ the sequence of $\alpha$-closed sets satisfies finite intersection property. By Cantors intersection theorem, there exists an $x$ in X such that $x \in \text{agcl}V_n$. Further $x \in X$ and $x \in V_n$, which is not equal to any of the points of X. Hence X is uncountable.

**Corollary 4.20:** Let X be a non empty $\alpha$-compact $\alpha$-$R_i$-space. If every point of X is a $T_{\emptyset}$-limit point of X then X is uncountable

5. $\alpha$ – $T_{\emptyset}$ IDENTIFICATION SPACES AND $\alpha$ – SEPARATION AXIOMS

**Definition 5.01:** Let $(X, \tau)$ be a topological space and let $\mathfrak{R}$ be the equivalence relation on X defined by $x \mathfrak{R} y$ if $x \in \text{agcl} \{x\} = \text{agcl} \{y\}$

**Problem 5.02:** show that $x \mathfrak{R} y$ if $x \in \text{agcl} \{x\} = \text{agcl} \{y\}$ is an equivalence relation

**Definition 5.03:** The space $(X_0, \text{Q}(X_0))$ is called the $\alpha$-$T_{\emptyset}$-identification space of $(X, \tau)$, where $X_0$ is the set of equivalence classes of $\mathfrak{R}$ and $\text{Q}(X_0)$ is the decomposition topology on $X_0$.

Let $P_X: (X, \tau) \to (X_0, \text{Q}(X_0))$ denote the natural map

**Lemma 5.04:** If $x \in X$ and $A \subset X$, then $x \in \text{agcl} A$ iff every $\alpha$-open set containing $x$ intersects $A$.

**Theorem 5.05:** The natural map $P_X: (X, \tau) \to (X_0, \text{Q}(X_0))$ is closed, open and $P_X^{-1}(P_X(O)) = O$ for all $O \in \text{PO}(X, \tau)$ and $(X_0, \text{Q}(X_0))$ is $\alpha$-$T_{\emptyset}$

**Proof:** Let $O \in \text{PO}(X, \tau)$ and let $C \subset P_X(O)$. Then there exists $x \in O$ such that $P_X(x) = C$. If $y \in C$, then $\text{agcl} \{y\} = \text{agcl} \{x\}$, which, by lemma, implies $y \in O$. Since $\tau \subset \text{PO}(X, \tau)$, then $P_X^{-1}(P_X(U)) = U$ for all $U \in \tau$, which implies $P_X$ is closed and open.

Let $G, H \in X$ such that $G \neq H$; let $x \in G$ and $y \in H$. Then $\text{agcl} \{x\} \neq \text{agcl} \{y\}$, which implies $x \in \text{agcl} \{y\}$ or $y \in \text{agcl} \{x\}$, say $x \in \text{agcl} \{y\}$. Since $P_X$ is continuous and open, then $G \cap A = P_X(X \cap \text{agcl} \{y\}) \in \text{PO}(X_0, \text{Q}(X_0))$ and $H \neq A$

**Theorem 5.06:** The following are equivalent:

(i) $X$ is $\alpha$-$R_0$ if (ii) $X_0 = \{\text{agcl} \{x\}; x \in X\}$ and (iii) $(X_0, \text{Q}(X_0))$ is $\alpha$-$T_1$

**Proof:**

(i) $\Rightarrow$ (ii) Let $C \subset X_0$ and let $x \in C$. If $y \in C$, then $y \in \text{agcl} \{y\} = \text{agcl} \{x\}$, which implies $C \subset \text{agcl} \{x\}$. If $y \in \text{agcl} \{x\}$, then $x \in \text{agcl} \{y\}$, since, otherwise, $x \not\in \text{agcl} \{y\} \in \text{PO}(X, \tau)$ which implies $\text{agcl} \{x\} \subset X \cap \text{agcl} \{y\}$, which is a contradiction. Thus, if $y \in \text{agcl} \{x\}$, then $x \in \text{agcl} \{y\}$, which implies $\text{agcl} \{y\} = \text{agcl} \{x\}$ and $y \in C$. Hence $X_0 = \{\text{agcl} \{x\}; x \in X\}$

(ii) $\Rightarrow$ (iii) Let $A \neq B \in X_0$. Then there exists $x, y \in X$ such that $A = \text{agcl} \{x\}; B = \text{agcl} \{y\}$, and $\text{agcl} \{x\} \cap \text{agcl} \{y\} = \phi$. Then $A \subset C = P_X(X \cap \text{agcl} \{y\}) \in \text{PO}(X_0, \text{Q}(X_0))$ and $B \neq C$. Thus $(X_0, \text{Q}(X_0))$ is $\alpha$-$T_1$

(iii) $\Rightarrow$ (i) Let $x \in U \in \text{agO}(X)$. Let $y \in U$ and $C_x \subset X_0$ containing $x$ and $y$ respectively. Then $x \not\in \text{agcl} \{y\}$, which implies $C_y \subset C_x$ and there exists $\alpha$-open set $A$ such that $C_x \subset A$ and $C_y \not\in A$. Since $P_X$ is continuous and open, then $y \in B = P_X^{-1}(A) \subset x \in \text{ag} \text{O}(X)$ and $x \not\in B$, which implies $y \not\in \text{agcl} \{x\}$. Thus $\text{agcl} \{x\} \subset U$. This is true for all $\text{agcl} \{x\}$ implies $\cap \text{agcl} \{x\} \subset U$. Hence $X$ is $\alpha$-$R_0$

**Theorem 5.07:** $(X, \tau)$ is $\alpha$-$R_1$ if $(X_0, \text{Q}(X_0))$ is $\alpha$-$T_2$

The proof is straight forward from using theorems 5.05 and 5.06 and is omitted

**Theorem 5.08:** $X$ is $\alpha$-$T_2$; $i = 0, 1, 2$ iff there exists a $\alpha$-continuous, almost–open, 1–1 function from $(X, \tau)$ into a $\alpha$-$T_i$ space; $i = 0, 1, 2$, respectively.

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Proof: If $X$ is $\alpha$-$T_i$; $i = 0,1,2$, then the identity function on $X$ satisfies the desired properties. The converse is (ii) part of Theorem 2.13.

The following example shows that if $f: (X, \tau) \rightarrow (Y, \sigma)$ is continuous, $\alpha$-open, bijective, $A \in P_O(Y, \sigma)$, and $(Y, \sigma)$ $\alpha$-$T_i$; $i = 0,1,2$, then $f^{-1}(A)$ need not be $\alpha$-open and $(X, \tau)$ need not be $\alpha$-$T_i$; $i = 0,1,2$

**Theorem 5.09:** If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\alpha$-continuous, $\alpha$-open, and $x, y \in X$ such that $\alpha cl\{x\} = \alpha cl\{y\}$, then $\alpha cl\{f(x)\} = \alpha cl\{f(y)\}$.

**Theorem 5.10:** The following are equivalent

(i) $(X, \tau)$ is $\alpha$-$T_0$

(ii) Elements of $X_0$ are singleton sets and

(iii) There exists a $\alpha$-continuous, $\alpha$-open, 1–1 function $f: (X, \tau) \rightarrow (Y, \sigma)$, where $(Y, \sigma)$ is $\alpha$-$T_0$

**Proof:** (i) is equivalent to (ii) and (i) $\Rightarrow$ (iii) are straight forward and is omitted.

(iii) $\Rightarrow$ (i) Let $x, y \in X$ such that $f(x) \neq f(y)$, which implies $\alpha cl\{f(x)\} \neq \alpha cl\{f(y)\}$. Then by theorem 5.09, $\alpha cl\{x\} \neq \alpha cl\{y\}$. Hence $(X, \tau)$ is $\alpha$-$T_0$

**Corollary 5.11:** A space $(X, \tau)$ is $\alpha$-$T_i$; $i = 1,2$ iff $(X, \tau)$ is $\alpha$-$T_{i-1}$; $i = 1,2$, respectively, and there exists a $\alpha$-continuous, $\alpha$-open, 1–1 function $f: (X, \tau) \rightarrow$ into a $\alpha$-$T_0$ space.

**Definition 5.04:** $f: X \rightarrow Y$ is point–$\alpha$-closure 1–1 iff for $x, y \in X$ such that $\alpha cl\{x\} \neq \alpha cl\{y\}, \alpha cl\{f(x)\} \neq \alpha cl\{f(y)\}$.

**Theorem 5.12:**

(i) If $f: (X, \tau) \rightarrow (Y, \sigma)$ is point–$\alpha$-closure 1–1 and $(X, \tau)$ is $\alpha$-$T_0$, then $f$ is 1–1

(ii) If $f: (X, \tau) \rightarrow (Y, \sigma)$, where $(X, \tau)$ and $(Y, \sigma)$ are $\alpha$-$T_0$ then $f$ is point–$\alpha$-closure 1–1 iff $f$ is 1–1

**Proof:** omitted

The following result can be obtained by combining results for $\alpha$-$T_0$-identification spaces, $\alpha$-induced functions and $\alpha$-$T_i$ spaces; $i = 1,2$.

**Theorem 5.13:** $X$ is $\alpha$-$\sigma_i$; $i = 0,1$ iff there exists a $\alpha$-continuous, $\alpha$-open, almost–point–$\alpha$-closure 1–1 function $f$: $(X, \tau)$ into a $\alpha$-$\sigma_i$ space; $i = 0,1$, respectively.

6. $\alpha$-Normal; Almost $\alpha$-normal and Mildly $\alpha$-normal spaces

**Definition 6.1:** A space $X$ is said to be $\alpha$-normal if for any pair of disjoint closed sets $F_1$ and $F_2$, there exist disjoint $\alpha$-open sets $U$ and $V$ such that $F_1 \subset U$ and $F_2 \subset V$.

**Example 4:** Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then $X$ is $\alpha$-normal.

**Example 5:** Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Then $X$ is not $\alpha$-normal and is not normal.

We have the following characterization of $\alpha$-normality.

**Theorem 6.1:** For a space $X$ the following are equivalent:

(i) $X$ is $\alpha$-normal.

(ii) For every pair of open sets $U$ and $V$ whose union is $X$, there exist $\alpha$-closed sets $A$ and $B$ such that $A \subset U$, $B \subset V$ and $A \cap B = X$.

(iii) For every closed set $F$ and every open set $G$ containing $F$, there exists a $\alpha$-open set $U$ such that $F \subset U \subset \alpha cl\{U\} \subset G$.

**Proof:** (i)$\Rightarrow$(ii): Let $U$ and $V$ be a pair of open sets in a $\alpha$-normal space $X$ such that $X = U \cup V$. Then $X = U \cup V$ are disjoint closed sets. Since $X$ is $\alpha$-normal there exist disjoint $\alpha$-open sets $U_1$ and $V_1$, such that $X = U_1 \cup V_1$ and $X = U_1 \cup V_1$. Let $A = X - U_1$, $B = X - V_1$. Then $A$ and $B$ are $\alpha$-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(b) $\Rightarrow$(c): Let $F$ be a closed set and $G$ be an open set containing $F$. Then $X - F$ and $G$ are open sets whose union is $X$.  

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Then by (b), there exist $\alpha$-closed sets $W_1$ and $W_2$ such that $W_1 \subset X-F$ and $W_2 \subset G$ and $W_1 \cup W_2 = X$. Then $F \subset X-W_1$, $X-G \subset X-W_2$ and $(X-W_1) \cap (X-W_2) = \phi$. Let $U = X-W_1$ and $V = X-W_2$. Then $U$ and $V$ are disjoint $\alpha$-open sets such that $F \subset U \subset X-V \subset G$. As $X-V$ is $\alpha$-closed set, we have $agcl(U) \subset X-V$ and $F \subset U \subset agcl(U) \subset G$.

(c) $\Rightarrow$ (a): Let $F_1$ and $F_2$ be any two disjoint closed sets of $X$. Put $G = X-F_2$, then $F_1 \cap G = \phi$. Hence $F_1 \subset G \subset agcl(U) \subset G$. It follows that $F_2 \subset X-\text{agcl}(U) = V$, say, then $V$ is $\alpha$-open and $U \cap V = \phi$. Hence $F_1$ and $F_2$ are separated by $\alpha$-open sets $U$ and $V$. Therefore $X$ is $\alpha$-normal.

Theorem 6.2: A regular open subspace of a $\alpha$-normal space is $\alpha$-normal.

Proof: Let $Y$ be a regular open subspace of a $\alpha$-normal space $X$. Let $A$ and $B$ be disjoint closed subsets of $Y$. As $Y$ is regular open, $A$, $B$ are closed sets of $X$. By $\alpha$-normality of $X$, there exist disjoint $\alpha$-open sets $U$ and $V$ in $X$ such that $A \subset U$ and $B \subset V$, $U \cap Y$ and $V \cap Y$ are $\alpha$-open in $Y$ such that $A \subset U \cap Y$ and $B \subset V \cap Y$. Hence $Y$ is $\alpha$-normal.

Example 6: Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ is $\alpha$-normal and $\alpha$-regular.

However we observe that every $\alpha$-normal $\alpha$-$R_0$ space is $\alpha$-regular.

Now, we define the following.

Definition 6.2: A function $f: X \to Y$ is said to be almost-$\alpha$-irresolute if for each $x$ in $X$ and each $\alpha$-neighborhood $V$ of $f(x)$, $agcl(f^{-1}(V))$ is a $\alpha$-neighborhood of $x$.

Clearly every $\alpha$-irresolute map is almost $\alpha$-irresolute.

The Proof of the following lemma is straightforward and hence omitted.

Lemma 6.1: $f$ is almost $\alpha$-irresolute iff $f^{-1}(V) \subset agcl(\text{int}(agcl(f^{-1}(V))))$ for every $V \in agO(Y)$.

Now we prove the following.

Lemma 6.2: $f$ is almost $\alpha$-irresolute iff $f(\text{agcl}(U)) \subset \text{agcl}(f(U))$ for every $U \in ag O(X)$.

Proof: Let $U \in ag O(X)$. Suppose $y \notin \text{agcl}(f(U))$. Then there exists $V \in agO(Y)$ such that $V \cap f(U) = \phi$. Hence $f^{-1}(V) \cap U = \phi$. Since $U \in agO(X)$, we have $\alpha$-int$(agcl(f^{-1}(V))) \cap \text{agcl}(U) = \phi$. Then by lemma 6.1, $f^{-1}(V) \cap \text{agcl}(U) = \phi$ and hence $V \cap f(\text{agcl(U)}) = \phi$. This implies that $y \notin f(\text{agcl}(U))$.

Conversely, if $V \in ag O(Y)$, then $W = X- agcl(f^{-1}(V)) \in ag O(X)$. By hypothesis, $f(\text{agcl}(W)) \subset \text{agcl} (f(W))$ and hence $X- \alpha$-int$(agcl(f^{-1}(V))) = \text{agcl}(W)-f^{-1}(agcl(f(W))) \subset \text{agcl}(f(X-f^{-1}(V))) \subset f^{-1}(\text{agcl}(Y-V)) = f^{-1}(Y-V) = X-f^{-1}(V)$. Therefore, $f^{-1}(V) \subset \alpha$-int$(agcl(f^{-1}(V)))$. By lemma 6.1, $f$ is almost $\alpha$-irresolute.

Now we prove the following result on the invariance of $\alpha$-normality.

Theorem 6.3: If $f$ is an $M$-$\alpha$-open continuous almost $\alpha$-irresolute function from an $\alpha$-normal space $X$ onto a space $Y$, then $Y$ is $\alpha$-normal.

Proof: Let $A$ be a closed subset of $Y$ and $B$ be an open set containing $A$. Then by continuity of $f$, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of $X$ such that $f^{-1}(A) \subset f^{-1}(B)$. As $X$ is $\alpha$-normal, there exists a $\alpha$-open set $U$ in $X$ such that $f^{-1}(A) \subset U \subset agcl(U) \subset f^{-1}(B)$. Then $f(U) \subset \text{agcl}(U) \subset f(U)$. Since $f$ is $M$-$\alpha$-open almost $\alpha$-irresolute surjection, we obtain $A \subset f(U) \subset agcl(f(U)) \subset B$. Then again by theorem 6.1 the space $Y$ is $\alpha$-normal.

Lemma 6.3: A mapping $f$ is $M$-$\alpha$-closed if and only if for each subset $B$ in $Y$ and for each $\alpha$-open set $U$ in $X$ containing $f^{-1}(B)$, there exists a $\alpha$-open set $V$ containing $B$ such that $f^{-1}(V) \subset U$.

Now we prove the following:

Theorem 6.4: If $f$ is an $M$-$\alpha$-closed continuous function from a $\alpha$-normal space onto a space $Y$, then $Y$ is $\alpha$-normal.

Proof of the theorem is routine and hence omitted.

Now in view of lemma 2.2 [9] and lemma 6.3, we prove that the following result.
Theorem 6.5: If \( f \) is an M-\( \alpha g \)-closed map from a weakly Hausdorff \( \alpha g \)-normal space \( X \) onto a space \( Y \) such that \( f^{-1}(y) \) is \( S \)-closed relative to \( X \) for each \( y \in Y \), then \( Y \) is \( \alpha g \)-T2.

Proof: Let \( y_1 \) and \( y_2 \) be any two distinct points of \( Y \). Since \( X \) is weakly Hausdorff, \( f^{-1}(y_1) \) and \( f^{-1}(y_2) \) are disjoint closed subsets of \( X \) by lemma 2.2 [9]. As \( X \) is \( \alpha g \)-normal, there exist disjoint \( \alpha g \)-open sets \( V_1 \) and \( V_2 \) such that \( f^{-1}(y_i) \subset V_i \), for \( i = 1, 2 \). Then it follows that \( U_1 \cap U_2 = \phi \). Hence \( Y \) is \( \alpha g \)-T2.

Theorem 6.6: For a space \( X \) we have the following:

(a) If \( X \) is normal then for any disjoint closed sets \( A \) and \( B \), there exist disjoint \( \alpha g \)-open sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \);

(b) If \( X \) is normal then for any closed set \( A \) and any open set \( V \) containing \( A \), there exists an \( \alpha g \)-open set \( U \) of \( X \) such that \( A \subset U \subset \alpha g \text{cl}(U) \subset V \).

Definition 6.2: \( X \) is said to be almost \( \alpha g \)-normal if for each closed set \( A \) and each regular closed set \( B \) such that \( A \cap B = \phi \), there exist disjoint \( \alpha g \)-open sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

Clearly, every \( \alpha g \)-normal space is almost \( \alpha g \)-normal, but not conversely in general.

Example 7: Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{a, b\}, \{b, c\}, X\} \). Then \( X \) is almost \( \alpha g \)-normal and not \( \alpha g \)-normal.

Now, we have characterization of almost \( \alpha g \)-normality in the following.

Theorem 6.7: For a space \( X \) the following statements are equivalent:

(i) \( X \) is almost \( \alpha g \)-normal

(ii) For every pair of sets \( U \) and \( V \), one of which is open and the other is regular open whose union is \( X \), there exist \( \alpha g \)-closed sets \( G \) and \( H \) such that \( G \subset U \) and \( H \subset \alpha g \text{cl}(V) \subset B \).

(iii) For every closed set \( A \) and every regular open set \( B \) containing \( A \), there is a \( \alpha g \)-open set \( V \) such that \( A \subset V \subset \alpha g \text{cl}(V) \subset B \).

Proof:

(a) \( \Rightarrow \) (b) Let \( U \) be an open set and \( V \) be a regular open set in an almost \( \alpha g \)-normal space \( X \) such that \( U \cup V = X \). Then \( (X-U) \) is closed set and \( (X-V) \) is regular closed set with \( (X-U) \cap (X-V) = \phi \). By almost \( \alpha g \)-normality of \( X \), there exist disjoint \( \alpha g \)-open sets \( U_1 \) and \( V_1 \) such that \( X-U \subset U_1 \) and \( X-V \subset V_1 \). Let \( G = X - U_1 \) and \( H = X - V_1 \). Then \( G \) and \( H \) are \( \alpha g \)-closed sets such that \( G \subset U \), \( H \subset V \), and \( G \cap H = X \).

(b) \( \Rightarrow \) (c) and (c) \( \Rightarrow \) (a) are obvious.

One can prove that almost \( \alpha g \)-normality is also regular open hereditary.

Almost \( \alpha g \)-normality does not imply almost \( \alpha g \)-regularity in general. However, we observe that every almost \( \alpha g \)-normal \( \alpha g \)-R0 space is almost \( \alpha g \)-regular.

Next, we prove the following.

Theorem 6.8: Every almost regular, \( \nu \)-compact space \( X \) is almost \( \alpha g \)-normal.

Recall that a function \( f \): \( X \to Y \) is called rc-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost \( \alpha g \)-normality in the following.

Theorem 6.9: If \( f \) is continuous M-\( \alpha g \)-open rc-continuous and almost \( \alpha g \)-irresolute surjection from an almost \( \alpha g \)-normal space \( X \) onto a space \( Y \), then \( Y \) is almost \( \alpha g \)-normal.

Definition 6.3: A space \( X \) is said to be mildly \( \alpha g \)-normal if for every pair of disjoint regular closed sets \( F_1 \) and \( F_2 \) of \( X \), there exist disjoint \( \alpha g \)-open sets \( U \) and \( V \) such that \( F_1 \subset U \) and \( F_2 \subset V \).

Example 8: Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\} \). Then \( X \) is mildly \( \alpha g \)-normal.

We have the following characterization of mild \( \alpha g \)-normality.
Theorem 6.10: For a space $X$ the following are equivalent.
(i) $X$ is mildly $\alpha g$-normal.
(ii) For every pair of regular open sets $U$ and $V$ whose union is $X$, there exist $\alpha g$-closed sets $G$ and $H$ such that $G \subseteq U$, $H \subseteq V$ and $G \cup H = X$.
(iii) For any regular closed set $A$ and every regular open set $B$ containing $A$, there exists a $\alpha g$-open set $U$ such that $A \subseteq U \subseteq \text{agcl}(U) \subseteq B$.
(iv) For every pair of disjoint regular closed sets, there exist $\alpha g$-open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $\text{agcl}(U) \cap \text{agcl}(V) = \emptyset$.

This theorem may be proved by using the arguments similar to those of Theorem 6.7.

Also, we observe that mild $\alpha g$-normality is regular open hereditary.

We define the following

Definition 6.4: A space $X$ is weakly $\alpha g$-regular if for each point $x$ and a regular open set $U$ containing $\{x\}$, there is a $\alpha g$-open set $V$ such that $x \in V \subseteq \text{cl}V \subseteq U$.

Example 9: Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then $X$ is weakly $\alpha g$-regular.

Example 10: Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $X$ is not weakly $\alpha g$-regular.

Theorem 6.11: If $f : X \to Y$ is an $M$-$\alpha g$-open re-continuous and almost $\alpha g$-irresolute function from a mildly $\alpha g$-normal space $X$ onto a space $Y$, then $Y$ is mildly $\alpha g$-normal.

Proof: Let $A$ be a regular closed set and $B$ be a regular open set containing $A$. Then by rc-continuity of $f$, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$ since $X$ is mildly $\alpha g$-normal, there exists a $\alpha g$-open set $V$ such that $f^{-1}(A) \subseteq V \subseteq \text{agcl}(V) \subseteq f^{-1}(B)$ by Theorem 6.10. As $f$ is $M$-$\alpha g$-open and almost $\alpha g$-irresolute surjection, it follows that $f(V) \subseteq \alpha g f(U)$ and $\alpha g f(U) \subseteq \text{agcl}(f(U)) \subseteq B$. Hence $Y$ is mildly $\alpha g$-normal.

Theorem 6.12: If $f : X \to Y$ is rc-continuous, $M$-$\alpha g$-closed map from a mildly $\alpha g$-normal space $X$ onto a space $Y$, then $Y$ is mildly $\alpha g$-normal.

7. $\alpha g$-US spaces:

Definition 7.1: A sequence $<x_n>$ is said to be $\alpha g$-converges to a point $x$ of $X$, written as $<x_n> \to^g x$ if $<x_n>$ is eventually in every $\alpha g$-open set containing $x$.

Clearly, if a sequence $<x_n>$ $g$-converges to a point $x$, then $<x_n>$ $\alpha g$-converges to $x$.

Definition 7.2: $X$ is said to be $\alpha g$-US if every sequence $<x_n>$ in $X$ $\alpha g$-converges to a unique point.

Theorem 7.1: Every $\alpha g$-US space is $\alpha g$-$T_1$.

Proof: Let $X$ be $\alpha g$-US space. Let $x$ and $y$ be two distinct points of $X$. Consider the sequence $<x_n>$ where $x_n = x$ for every $n$. Clearly, $<x_n> \to^g x$. Also, since $x \neq y$ and $X$ is $\alpha g$-US, $<x_n>$ cannot $\alpha g$-converge to $y$, i.e, there exists a $\alpha g$-open set $V$ containing $y$ but not $x$. Similarly, for the sequence $<y_n>$ where $y_n = y$ for all $n$, and proceeding as above we get a $\alpha g$-open set $U$ containing $x$ but not $y$. Thus, the space $X$ is $\alpha g$-$T_1$.

Theorem 7.2: Every $\alpha g$-$T_2$ space is $\alpha g$-US.

Proof: Let $X$ be $\alpha g$-$T_2$ space and $<x_n>$ be a sequence in $X$. If possible suppose that $<x_n>$ $\alpha g$-converge to two distinct points $x$ and $y$. That is, $<x_n>$ is eventually in every $\alpha g$-open set containing $x$ and also in every $\alpha g$-open set containing $y$.

This is contradiction since $X$ is $\alpha g$-$T_2$ space. Hence the space $X$ is $\alpha g$-US.

Definition 7.3: A set $F$ is sequentially $\alpha g$-closed if every sequence in $F$ $\alpha g$-converges to a point in $F$.

Theorem 7.3: $X$ is $\alpha g$-US iff the diagonal set is a sequentially $\alpha g$-closed subset of $X \times X$.

Proof: Let $X$ be $\alpha g$-US. Let $<x_n, y_n>$ be a sequence in $\Delta$. Then $<x_n>$ is a sequence in $X$. As $X$ is $\alpha g$-US, $<x_n> \to^g x$ for a unique $x \in X$. i.e., if $<x_n> \to^g x$ and $y$. Thus, $x = y$. Hence $\Delta$ is sequentially $\alpha g$-closed.

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Conversely, let $\Delta$ be sequentially $\alpha g$-closed and let $<x_n> \to^\alpha x$ and $y$. Hence $<x_n, x_n> \to^\alpha (x,y)$. Since $\Delta$ is sequentially $\alpha g$-closed, $(x,y) \in \Delta$ which means that $x = y$ implies space $X$ is $\alpha g$-US.

**Definition 7.4:** A subset $G$ of a space $X$ is said to be sequentially $\alpha g$-compact if every sequence in $G$ has a subsequence which $\alpha g$-converges to a point in $G$.

**Theorem 7.4:** In an $\alpha g$-US space every sequentially $\alpha g$-compact set is sequentially $\alpha g$-closed.

**Proof:** Let $X$ be an $\alpha g$-US space. Let $Y$ be a sequentially $\alpha g$-compact subset of $X$. Let $<x_n>$ be a sequence in $Y$. Suppose that $<x_n>$ $\alpha g$-converges to a point in $X-Y$. Let $<x_{n_0}>$ be subsequence of $<x_n>$ that $\alpha g$-converges to a point $y \in Y$ since $Y$ is sequentially $\alpha g$-compact. Also, let a subsequence $<x_{n_p}>$ of $<x_{n_0}>$ $\alpha g$-converge to $x \in X-Y$. Since $<x_{n_p}>$ is a sequence in the $\alpha g$-US space $X$, $x = y$. Thus, $Y$ is sequentially $\alpha g$-closed set.

Next, we give a hereditary property of $\alpha g$-US spaces.

**Theorem 7.5:** Every regular open subset of a $\alpha g$-US space is $\alpha g$-US.

**Proof:** Let $X$ be a $\alpha g$-US space and $Y \subset X$ be a regular open set. Let $<x_n>$ be a sequence in $Y$. Suppose that $<x_n>$ $\alpha g$-converges to $x$ and $y$ in $Y$. We shall prove that $<x_n>$ $\alpha g$-converges to $x$ and $y$ in $X$. Let $U$ be any $\alpha g$-open subset of $X$ containing $x$ and $V$ be any $\alpha g$-open set of $X$ containing $y$. Then, $U \cap Y$ and $V \cap Y$ are $\alpha g$-open sets in $Y$. Therefore, $<x_n>$ is eventually in $U \cap Y$ and $V \cap Y$ and so in $U$ and $V$. Since $X$ is $\alpha g$-US, this implies that $x = y$. Hence the subspace $Y$ is $\alpha g$-US.

**Theorem 7.6:** A space $X$ is $\alpha g$-T$_2$ iff it is both $\alpha g$-R$_1$ and $\alpha g$-US.

**Proof:** Let $X$ be $\alpha g$-T$_2$ space. Then $X$ is $\alpha g$-R$_1$ and $\alpha g$-US by Theorem 7.2.

Conversely, let $X$ be both $\alpha g$-R$_1$ and $\alpha g$-US space. By Theorem 7.1, $X$ is both $\alpha g$-T$_1$ and $\alpha g$-R$_1$ and, it follows that space $X$ is $\alpha g$-T$_2$.

**Definition 7.5:** A point $y$ is a $\alpha g$-cluster point of sequence $<x_n>$ iff $<x_n>$ is frequently in every $\alpha g$-open set containing $x$. The set of all $\alpha g$-cluster points of $<x_n>$ will be denoted by $\alpha g$-cl($x_n$).

**Definition 7.6:** A point $y$ is $\alpha g$-side point of a sequence $<x_n>$ if $y$ is a $\alpha g$-cluster point of $<x_n>$ but no subsequence of $<x_n>$ $\alpha g$-converges to $y$.

Now, we define the following.

**Definition 7.7:** A space $X$ is said to be

(i) $\alpha g$-S$_1$ if it is $\alpha g$-US and every sequence $<x_n>$ $\alpha g$-converges with subsequence of $<x_n>$ $\alpha g$-side points.

(ii) $\alpha g$-S$_2$ if it is $\alpha g$-US and every sequence $<x_n>$ in $X$ $\alpha g$-converges which has no $\alpha g$-side point.

**Lemma 7.1:** Every $\alpha g$-S$_2$ space is $\alpha g$-S$_1$ and Every $\alpha g$-S$_1$ space is $\alpha g$-US.

Now using the notion of sequentially continuous functions, we define the notion of sequentially $\alpha g$-continuous functions.

**Definition 7.8:** A function $f$ is said to be sequentially $\alpha g$-continuous at $x \in X$ if $f(<x_n>) \to^\alpha f(x)$ whenever $<x_n> \to^\alpha x$. If $f$ is sequentially $\alpha g$-continuous at all $x \in X$, then $f$ is said to be sequentially $\alpha g$-continuous.

**Theorem 7.7:** Let $f$ and $g$ be two sequentially $\alpha g$-continuous functions. If $Y$ is $\alpha g$-US, then the set $A = \{x \mid f(x) = g(x)\}$ is sequentially $\alpha g$-closed.

**Proof:** Let $Y$ be $\alpha g$-US and suppose that there is a sequence $<x_n>$ in $A$ $\alpha g$-converging to $x \in X$. Since $f$ and $g$ are sequentially $\alpha g$-continuous functions, $f(<x_n>) \to^\alpha f(x)$ and $g(<x_n>) \to^\alpha g(x)$. Hence $f(x) = g(x)$ and $x \in A$. Therefore, $A$ is sequentially $\alpha g$-closed.

Next, we prove the product theorem for $\alpha g$-US spaces.

**Theorem 7.8:** Product of arbitrary family of $\alpha g$-US spaces is $\alpha g$-US.
8. Sequentially sub-$\alpha$-continuity:

Definition 8.1: A function $f$ is said to be
(i) sequentially nearly $\alpha$-continuous if for each point $x \in X$ and each sequence $<x_{n}> \rightarrow^{\alpha} x$ in $X$, there exists a subsequence $<x_{n_{k}}>$ of $<x_{n}>$ such that $f(x_{n_{k}}) \rightarrow^{\alpha} f(x)$.
(ii) sequentially sub-$\alpha$-continuous if for each point $x \in X$ and each sequence $<x_{n}> \rightarrow^{\alpha} x$ in $X$, there exists a subsequence $<x_{n_{k}}>$ of $<x_{n}>$ and a point $y \in Y$ such that $f(x_{n_{k}}) \rightarrow^{\alpha} y$.
(iii) sequentially $\alpha$-compact preserving if $f(K)$ is sequentially $\alpha$-compact in $Y$ for every sequentially $\alpha$-compact set $K$ of $X$.

Lemma 8.1: Every function $f$ is sequentially sub-$\alpha$-continuous if $Y$ is a sequentially $\alpha$-compact.

Proof: Let $<x_{n}> \rightarrow^{\alpha} x$ in $X$. Since $Y$ is sequentially $\alpha$-compact, there exists a subsequence $<f(x_{n})>$ of $<f(x_{n})>$ $\alpha$-converging to a point $y \in Y$. Hence $f$ is sequentially sub-$\alpha$-continuous.

Theorem 8.1: Every sequentially nearly $\alpha$-continuous function is sequentially $\alpha$-compact preserving.

Proof: Assume $f$ is sequentially nearly $\alpha$-continuous and $K$ any sequentially $\alpha$-compact subset of $X$. Let $<y_{n}>$ be any sequence in $f(K)$. Then for each positive integer $n$, there exists a point $x_{n} \in K$ such that $f(x_{n}) = y_{n}$. Since $<x_{n}>$ is a sequence in the sequentially $\alpha$-compact set $K$, there exists a subsequence $<x_{n_{k}}>$ of $<x_{n}>$ $\alpha$-converging to a point $x \in K$. By hypothesis, $f$ is sequentially nearly $\alpha$-continuous and hence there exists a subsequence $<x_{n_{k}}>$ of $<x_{n}>$ such that $f(x_{n_{k}}) \rightarrow^{\alpha} f(x)$. Thus, there exists a subsequence $<y_{n_{k}}>$ of $<y_{n}>$ $\alpha$-converging to $f(x) \in f(K)$. This shows that $f(K)$ is sequentially $\alpha$-compact set in $Y$.

Theorem 8.2: Every sequentially $\alpha$-continuous function is sequentially $\alpha$-continuous.

Proof: Let $f$ be a sequentially $\alpha$-continuous and $<x_{n}> \rightarrow^{\alpha} x \in X$. Then $<x_{n}> \rightarrow^{\alpha} x$. Since $f$ is sequentially $\alpha$-continuous, $f(x_{n}) \rightarrow^{\alpha} f(x)$. But we know that $<x_{n}> \rightarrow^{\alpha} x$ implies $<x_{n}> \rightarrow^{\alpha} x$ and hence $f(x_{n}) \rightarrow^{\alpha} f(x)$ implies $f$ is sequentially $\alpha$-continuous.

Theorem 8.3: Every sequentially $\alpha$-compact preserving function is sequentially sub-$\alpha$-continuous.

Proof: Suppose $f$ is a sequentially $\alpha$-compact preserving function. Let $x$ be any point of $X$ and $<x_{n}>$ any sequence in $X$ $\alpha$-converging to $x$. We shall denote the set $\{x_{n} | n = 1, 2, 3 \ldots \}$ by $A$ and $K = A \cup \{x\}$. Then $K$ is sequentially $\alpha$-compact since $<x_{n}> \rightarrow^{\alpha} x$. By hypothesis, $f$ is sequentially $\alpha$-compact preserving and hence $f(K)$ is a sequentially $\alpha$-compact set of $Y$. Since $<f(x_{n})>$ is a sequence in $f(K)$, there exists a subsequence $<f(x_{n_{k}})> \alpha$-converging to a point $y \in f(K)$. This implies that $f$ is sequentially sub-$\alpha$-continuous.

Theorem 8.4: A function $f: X \rightarrow Y$ is sequentially $\alpha$-compact preserving iff $f_{K}: K \rightarrow f(K)$ is sequentially sub-$\alpha$-continuous for each sequentially $\alpha$-compact subset $K$ of $X$.

Proof: Suppose $f$ is a sequentially $\alpha$-compact preserving function. Then $f(K)$ is sequentially $\alpha$-compact set in $Y$ for each sequentially $\alpha$-compact set $K$ of $X$. Therefore, by Lemma 8.1 above, $f_{K}: K \rightarrow f(K)$ is sequentially $\alpha$-continuous function.

Conversely, let $K$ be any sequentially $\alpha$-compact set of $X$. Let $<y_{n}>$ be any sequence in $f(K)$. Then for each positive integer $n$, there exists a point $x_{n} \in K$ such that $f(x_{n}) = y_{n}$. Since $<x_{n}>$ is a sequence in the sequentially $\alpha$-compact set $K$, there exists a subsequence $<x_{n_{k}}>$ of $<x_{n}>$ $\alpha$-converging to a point $x \in K$. By hypothesis, $f_{K}: K \rightarrow f(K)$ is sequentially sub-$\alpha$-continuous and hence there exists a subsequence $<y_{n_{k}}>$ of $<y_{n}>$ $\alpha$-converging to a point $y \in f(K)$. This implies that $f(K)$ is sequentially $\alpha$-compact set in $Y$. Thus, $f$ is sequentially $\alpha$-compact preserving function.
The following corollary gives a sufficient condition for a sequentially sub-$\alpha g$-continuous function to be sequentially $\alpha g$-compact preserving.

**Corollary 8.1:** If $f$ is sequentially sub-$\alpha g$-continuous and $f(K)$ is sequentially $\alpha g$-closed set in $Y$ for each sequentially $\alpha g$-compact set $K$ of $X$, then $f$ is sequentially $\alpha g$-compact preserving function.

**Proof:** Omitted.

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