DOMINATION NUMBERS OF GRID GRAPHS $P_{15} \times P_n$

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**ABSTRACT**

In this paper we deal with the domination numbers of the complete grid graphs $P_k \times P_n$ for $k = 15$ and $n \geq 1$. Establishing this numbers for various class of graphs is the content of many papers in literature. Some of the papers which deal with the dominating number of grid graphs are \[3\], \[4\], \[5\]. There is no general formula for the domination number of a graph. In this paper, we use the concept of transforming the domination from a vertex in a dominating set $D$ of a graph $G = (V, E)$ to a vertex in $V - D$, where $G$ is a simple connected graph. We give an algorithm using this transformation to obtain a domination set of a graph $G$.

**Key words:** Dominating set, Domination number, Transformation of a dominating set.

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1. INTRODUCTION

A graph $G = (V, E)$ is a mathematical structure consisting of two sets, $V$ (of vertices), and $E$ (of edges) where $V$ is a finite and nonempty and elements of $E$ are unordered pairs $\{u, v\}$ of distinct element of $V$. We simply write $uv$ instead of $\{u, v\}$. The order of the graph $G$ is the cardinality of its vertex set $V$, the size of the graph $G$ is the cardinality of its edge set $E$ (see \[1\]). Two vertices $u$ and $v$ of a graph $G$ are said to be adjacent if $uv \in E$. For a vertex $v$ of $G$, the neighborhood of $v$, $N(v)$, is defined as the set of all vertices of $G$ which are adjacent to $v$. The closed neighborhood of $v$ is $\overline{N}(v) = N(v) \cup \{v\}$. If $S$ is a set of vertices of $G$, then the neighborhood of $S$ is given by $N(S) = \bigcup_{v \in S} N(v)$ and $\overline{N}(S) = \bigcup_{v \in S} \overline{N}(v)$. The degree of vertex $v$, is given by $d(v) = |N(v)|$, the number of vertices which are adjacent to $v$.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is called a dominating set of $G$ if every vertex in $V-D$ is adjacent to at least one vertex of $D$. A dominating set $D$ of $G$ is said to be a minimal dominating set of $G$ if $|D| \leq |D'|$ for any dominating set $D'$ of $G$. In otherwords, dominating set is minimal if it contains no other dominating set as a proper subset. The cardinality of a minimal dominating set of $G$ is known as the domination number of $G$, and is denoted by $\gamma(G)$.

Next we give some more definitions.

Let $D$ be a dominating set of a graph $G = (V, E)$.

1. We define the weight function $F_D$ on $V$ as follows: $F_D : V \rightarrow \mathbb{N}$ given by $F_D(v)$ = the cardinality of the closed neighborhood of $v$, ie., $F_D(v) = |\overline{N}(v)|$.
   (Here $\mathbb{N}$ denotes the set of natural numbers).

2. We say that $v \in D$ has a moving domination if there exits a vertex $w \in N(v) - D$ such that $wu \in E$ for every vertex $u \in \{y \in N(v) : F_D(y)\}$. Let $D_1$ be another dominating set of $G$.

3. We say that a vertex $v \in D$ is redundant of $D$ if $F_D(w) \geq 2$ for every vertex $w \in \overline{N}(v)$.

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We say that $v$ has moving domination, we say that $v$ is inefficient if transforming the domination from $v$ to any vertex in $N(v)$ would not produce any redundant vertex.

For two vertices $v_0$ and $v_n$ of a graph $G$, a $v_0 - v_n$ walk is an alternating sequence of $n + 1$ vertices and $n$ edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that consecutive vertices and edges are incident.

A path is defined to be a walk in which no vertices are repeated. A path with $n$ vertices is denoted by $P_n$. It has $n - 1$ edges and the length of $P_n$ is $n - 1$. The grid graph $P_k \times P_n$ is defined by $V(P_k \times P_n) = \{v_{ij} : 1 \leq i \leq k, 1 \leq j \leq n\}$ and two vertices $v_{ij}$ and $v_{rs}$ are adjacent either $i = t, j = s \pm 1$ or $i = t \pm 1, j = s$.

Thus a grid graph can be treated as the Cartesian product of two paths $P_k$ and $P_n$ of path length $k - 1$ and $n - 1$.

Observe that if $D$ is a dominating set of grid graph $P_k \times P_n$ which has no redundant vertex, then $v \in D$ has a moving domination if and only if one of the following two cases occurs:

**Case 1:** For every vertex $w \in N(v)$ we have $F_D(w) \geq 2$.

In this case the domination of $v$ can be transformed to any vertex in $N(v) - D$.

**Case 2:** There exits exactly one vertex $u \in N(v)$ such that $F_D(u) = 1$.

In this case the domination of $v$ can be transformed only to $u$.

**2. AN ALGORITHM FOR FINDING A DOMINATING SET OF $P_k \times P_n$ USING TRANSFORMATION OF DOMINATION OF VERTICES.**

In this section, we present an algorithm for finding dominating sets and consequently dominating numbers of graphs. In the next section we apply this to the grid graph $P_{15} \times P_n$.

Let $G = (V, E)$ be a graph of order greater than 1, say, $|V| = m$. Let $D = V$ be a dominating set of $G$. Then for any vertex $v \in D$ we have $F_D(v) = d(v) + 1 \geq 2$. Pick a vertex $v_1$ of $D$ and delete from $D$ all vertices $w, w \in N(v_1)$. Then for $1 \leq n \leq \frac{m}{2}$, pick a vertex $v_n \in D - \bigcup_{i=1}^{n-1} N(v_i)$ and delete from $D$ all vertices $w, w \in N(v_n) - \bigcup_{i=1}^{n-1} N(v_i)$.

Next, if $D$ contains a redundant vertex, then delete it. Repeat this process until $D$ has no redundant vertices.

Next transform domination from vertices of $D$ which have moving domination to vertices in $V - D$ to obtain redundant vertices and go to above step. If no redundant vertex can be obtained by a transformation of domination of vertices of $D$, then stop. This set is a dominating set $D_k$ satisfying that for every $v \in D$ there is a $w \in N(v)$ such that $F_D(w) = 1$.

**3. Example:** In this section we give an example to illustrate the above algorithm, in case of the grid graph $G = P_{15} \times P_{14}$.

1. We have $|V| = 210$. Let $(k, n)$ be the vertex in $k^{th}$ row and in the $n^{th}$ column of the graph $G$.

2. Let $D = V$ be a dominating set of $G$.

3. Pick a vertex $v_1 = (1, 3) \in D$ and delete from $D$ all vertices $w, w \in N(v_1)$. Then for $1 \leq n \leq \frac{210}{2}$, pick a vertex $v_n, v_n \in D - \bigcup_{i=1}^{n-1} N(v_i)$, and delete from $D$ all vertices $w, w \in N(v_n) - \bigcup_{i=1}^{n-1} N(v_i)$. We obtain a dominating set $D$ (indicated by black circles in the figures below).

4. Since for every vertex $v \in D, \exists w \in N(v)$ such that $F_D(w) = 1$, $D$ has no redundant vertices.

5. Transform the domination from the vertex $(5, 9)$ to the vertex $(6, 9)$ and delete from $D$, and the resulting redundant vertex is $(7, 9)$. Therefore, the set $D$ indicated in figure below (black circles) is a dominating set of $G = P_{15} \times P_{14}$. Note that $D$ is a minimal dominating set (see [3]).
(2). Domination numbers for $\gamma(P_{15} \times P_n)$ for $1 \leq n \leq 14$

Below we give some minimal dominating set of $P_{15} \times P_n$ up to $n = 14$ and consequently obtain the dominating numbers of these graphs

These results match with the known formula for dominating number of grid graphs $P_{15} \times P_n$,

$$\gamma(P_{15} \times P_n) = \begin{cases} \frac{44n + 27}{13} & \text{for } n \equiv 5 \pmod{26} \\ \frac{44n + 40}{13} & \text{otherwise} \end{cases}$$

See for eg [6].

REFERENCES:


