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SMALL PQ-PRINCIPALLY INJECTIVE MODULES

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ABSTRACT

Let M be a right $R-$module. A right $R-$module N is called small pseudo $M-$principally injective (briefly, small $PM-$principally injective) if, every $R-$monomorphism from an $M-$cyclic small submodule of M to N can be extended to an $R-$homomorphism from M to N. In this paper, we give some characterizations and properties of small pseudo quasi-principally injective modules.

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1. INTRODUCTION

Let R be a ring. A right $R-$module M is called *principally injective* (or $P-$*injective*) [6], if every $R-$homomorphism from a principal right ideal of R to M can be extended to an $R-$homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$. Following [9], a right $R-$module M is called *quasi-principally injective*, if every $R-$homomorphism from an $M-$cyclic submodule of M to M can be extended to M.

In [14], a right $R-$module M is called $PPQ-$*injective* if, every $R-$monomorphism from a principal submodule of M to M extends to an endomorphism of M. A right $R-$module N is called small *principally* $M-$*injective* (briefly, $SP-M-$*injective*) [13] if, every $R-$homomorphism from a small and principal submodule of M to N can be extended to an $R-$homomorphism from M to N. A right $R-$module M is called *small principally quasi-injective* (briefly, $SPQ-$*injective*) if it is $SP-M-$injective. In this note we introduce the definition of small $PQ-$principally injective modules and give some interesting results on these modules.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right $R-$modules. For right $R-$modules M and N, $Hom_R(M,N)$ denotes the set of all $R-$homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. A submodule X of M is said to be $M-$cyclic submodule of M if it is the image of an element of S. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notations, $N \subset^{\oplus} M$, $N \subset^{e} M$, and $N \ll M$ we mean that N is a direct summand, an essential submodule and a superfluous submodule of M, respectively. We denote the Jacobson radical of M by $J(M)$.

Following [1], a submodule K of a right $R-$module M is *superfluous* (or *small*) in M, abbreviated $K \ll M$, in case for every submodule L of M, $K + L = M$ implies $L = M$.

It is clear that $kR \ll R$ if and only if $k \in J(R)$.

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2. SMALL PQ-PRINCIPALLY INJECTIVE MODULES

Definition 2.1: Let M be a right $R-$module. A right $R-$module N is called *small pseudo* $M-$*principally injective* (briefly, *small* $PM-$*principally injective*) if, every $R-$monomorphism from an $M-$cyclic small submodule of M to N can be extended to an $R-$homomorphism from M to N. M is called *small pseudo quasi-principally injective* (briefly, *small PQ-principally injective*) if, it is small $PM-$principally injective.

Example 2.2: Let $R=\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field, $M_R = R_R$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$.
Then N is small $PM-$principally injective.

Proof: It is clear that only $X=\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is the nonzero $M-$cyclic small submodule of M.

Let $\varphi : X \to N$ be an $R-$monomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in X$, there exists $x_{11}, x_{12} \in F$ such that

$$\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

It follows that $x_{11} = 0$.

Define $\hat{\varphi} : M \to N$ by $\hat{\varphi}\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} ax_{12} & bx_{12} \\ 0 & 0 \end{pmatrix}$ for every $a,b,c \in F$.

It is clear that $\hat{\varphi}$ is an $R-$homomorphism and $\hat{\varphi}$ extends φ.

Then N is small $PM-$principally injective.

Clearly, every $X-$cyclic submodule of X is an $M-$cyclic submodule of M for every $M-$cyclic submodule X of M. Then we have the following

Lemma 2.3:
(1) N is small $PM-$principally injective if and only if N is small $PX-$principally injective for any $M-$cyclic submodule X of M.
(2) Every direct summand of small $PM-$principally injective is also small $PM-$principally injective.

Proof: (1) The sufficiency is trivial. For the necessity, let $f(X)$ be an $M-$cyclic small submodule of X and let $\alpha : f(X) \to N$ be an $R-$monomorphism. Since $f(X) \ll M$ [1, Lemma 5.18], there exists an $R-$homomorphism $\hat{\alpha} : M \to N$ such that $\alpha = \hat{\alpha}\iota_2\iota_1$ where $\iota_1 : f(X) \to X$ and $\iota_2 : X \to M$ are the inclusion maps. Then $\hat{\alpha}\iota_2$ extends α.

(2) Let N be a small $PM-$principally injective module, $X \subset^{\oplus} N$, $s \in S$ with $s(M) \ll M$ and let $\alpha : s(M) \to X$ be an $R-$monomorphism. Let $\varphi : X \to N$ be the injection map. Since $\varphi\alpha$ is monic, there exists an $R-$homomorphism $\beta : M \to N$ such that $\varphi\alpha = \beta\iota$ where $\iota : s(M) \to M$ is the inclusion map. Then $\pi\beta$ extends α where $\pi : N \to X$ is the projection map.

Theorem 2.4: Let M be a right $R-$module. If every $M-$cyclic small submodule of M is projective, then every factor module of a small $PM-$principally injective module is small $PM-$principally injective.

Proof: Let N be a small $PM-$principally injective module, X a submodule of N, $s(M) \ll M$ and let $\varphi : s(M) \to N/X$ be an $R-$monomorphism. Then by assumption, there exists an $R-$homomorphism $\hat{\varphi} : s(M) \to N$ such that $\varphi = \eta\hat{\varphi}$ where $\eta : N \to N/X$ is the natural $R-$epimorphism. If $x \in Ker(\hat{\varphi})$, then $\varphi(x) = \eta\hat{\varphi}(x) = X$ so $x = 0$ which shows that $\hat{\varphi}$ is monic. Since N is small $PM-$principally injective, there exists an $R-$homomorphism $\beta : M \to N$ which is an extension of $\hat{\varphi}$ to M. Then $\eta\beta$ is an extension of φ to M.

Let M be a right $R-$module with $S = End_R(M)$. Following [8], write

$$W(S) = \{s \in S : \ Ker(s) \subset^e M\}.$$

It is known that $W(S)$ is an ideal of S. A right $R-$module M is called a *principal self-generator* if every element $m \in M$ has the form $m = \gamma(m_1)$ for some $\gamma : M \to mR$.

Lemma 2.5: Let M be a small $PQ-$principally injective module. If $Ker(s) = Ker(t)$, where $s, t \in S$ with $s(M) \ll M$, then $St \subset Ss$.

Proof: Let $Ker(s) = Ker(t)$, where $s, t \in S$ with $s(M) \ll M$. Define $\varphi : s(M) \to M$ by $\varphi(s(m)) = t(m)$ for every $m \in M$. It is obvious that φ is an $R-$monomorphism.

Since M is small $PQ-$principally injective, let $\hat{\varphi} \in S$ be an extension of φ.

Then
$t = \varphi s = \hat{\varphi} s \in Ss$ so $St \subset Ss$.

Proposition 2.6: Let M be a principal module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is small $PQ-$principally injective, then $J(S) \subset W(S)$.

Proof: Let $s \in J(S)$. If $Ker(s) \not\subset^e M$, then $Ker(s) \cap K = 0$ for some nonzero submodule K of M. Since $Soc(M_R) \subset^e M$, $Soc(M_R) \cap K \neq 0$. Then there exists a simple submodule kR of M such that $kR \subset Soc(M_R) \cap K$ [1, Corollary 9.10]. As M is a principal self-generator and kR is simple, $kR = t(M)$ for some $t \in S$. It follows that $Ker(st) = Ker(t)$. Since M is a principal module, $J(M) \ll M$ [11, 21.6] and we have $J(S)M \subset J(M)$, it follows that $st(M)$ is a small submodule of M. Since M is small $PQ-$principally injective, $St \subset Sst$ by Lemma 2.5. Write $t = gst$ where $g \in S$. It follows that $(1-gs)t = 0$ so $t = (1-gs)^{-1}0 = 0$, a contradiction.

Proposition 2.7: Let M be a principal nonsingular module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is small $PQ-$principally injective, then $J(S) = 0$.

Proof: Since $J(S) \subset W(S)$ by Proposition 2.6, we show that $W(S) = 0$.

Let $s \in W(S)$ and let $m \in M$. Define $\varphi : R \to M$ by $\varphi(r) = mr$ for every $r \in R$.

It is clear that φ is an $R-$homomorphism. Thus

$$\begin{aligned} r_R(s(m)) &= \{r \in R : \ s(mr) = 0\} \\ &= \{r \in R : \ mr \in Ker(s)\} \end{aligned}$$

$$= \{ r \in R : \varphi(r) \in Ker(s) \}$$
$$= \varphi^{-1}(Ker(s)).$$

It follows that $\varphi^{-1}(Ker(s)) \subset^e R$ [3, Lemma 5.8(a)] so $r_R(s(m)) \subset^e R$. Thus $s(m) \in Z(M_R) = 0$ because M is nonsingular. As this is true for all $m \in M$,

we have $s = 0$. Hence $W(S) = 0$ as required.

Proposition 2.8: Let M be a small $PQ-$principally injective module and $s \in S$.
(1) If $s(M)$ is a simple and small right $R-$module, then Ss is a simple left $S-$module.
(2) If $Ss_1 \oplus ... \oplus Ss_n$ is direct, $s_i \in S$ with $s_i(M) \ll M$, $(1 \le i \le n)$, then any $R-$monomorphism $\alpha : s_1(M) + ... + s_n(M) \to M$ has an extension in S.

Proof: (1) If A is a nonzero submodule of Ss and $0 \ne \alpha s \in A$, then $S\alpha s \subset A$. Note that $\alpha s(M)$ is a nonzero homomorphic image of the simple module $s(M)$, then $\alpha s(M)$ is simple.

It is clear that $\alpha s(M) \ll M$. Define $\varphi : \alpha s(M) \to M$ by $\varphi(\alpha s(m)) = s(m)$ for every $m \in M$. Since $Ker(\alpha) \cap s(M) = 0$, φ is well-defined. It is clear that φ is an $R-$homomorphism. Since $\alpha s(M)$ is simple and φ is nonzero, $Ker(\varphi) = 0$.

Then there exists an $R-$homomorphism $\hat{\varphi} \in S$ is an extension of φ. Hence $s = \hat{\varphi}\alpha s \in S\alpha s$. It follows that $Ss = S\alpha s$ so $A = Ss$.

(2) Since $\alpha|_{s_i(M)}$ is monic, for each i, there exists an $R-$homomorphism $\varphi_i : M \to M$ such that $\varphi_i s_i(m) = \alpha s_i(m)$ for all $m \in M$.

Since $(\sum_{i=1}^n s_i)(M) \ll M$, $(\sum_{i=1}^n s_i)(M) \subset \sum_{i=1}^n s_i(M)$ and $\alpha|_{(\sum_{i=1}^n s_i)(M)}$ is monic, α can be extended to $\varphi : M \to M$ such that, for any $m \in M$,

$$\varphi(\sum_{i=1}^n s_i)(m) = \alpha(\sum_{i=1}^n s_i)(m).$$

It follows that $\sum_{i=1}^n \varphi s_i = \sum_{i=1}^n \varphi_i s_i$. Since $Ss_1 \oplus ... \oplus Ss_n$ is direct, $\varphi s_i = \varphi_i s_i$ for all $1 \le i \le n$. Therefore φ is an extension of α.

Theorem 2.9: Let M be a small $PQ-$principally injective module, $s, t \in S$ with $s(M) \ll M$.
(1) If $s(M)$ embeds in $t(M)$, then Ss is an image of St.
(2) If $s(M) \simeq t(M)$, then $Ss \simeq St$.

Proof: (1) Let $f : s(M) \to t(M)$ be an $R-$monomorphism. Since M is small $PQ-$principally injective, there exists $\hat{f} \in S$ such that $\hat{f}$ extends f.

Let $\sigma : St \to Ss$ defined by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. Since $\hat{f}s(M) \subset t(M)$, σ is well-defined. It is clear that σ is an $S-$homomorphism. Note that $fs(M) = \hat{f}s(M) \ll M$. Since f is monic, $Ker(s) = Ker(fs)$ and hence by Lemma 2.5, $Ss \subset Sfs$. Then $s \in Sfs \subset \sigma(St)$.

(2) Let $f: s(M) \to t(M)$ be an $R-$isomorphism. Since M is small $PQ-$principally injective, f can be extended to $\hat{f}: M \to M$. Define $\sigma: St \to Ss$ by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. It is clear that σ is an $S-$epimorphism. If $ut \in Ker(\sigma)$, then $0 = \sigma(ut) = u\hat{f}s = ufs$. Since $Im(fs) = Im(t)$, $ut = 0$. This shows that σ is monic.

Proposition 2.10: Let M be a principal, small $PQ-$principally injective module which is a principal self-generator. Then $Soc(M_R) \subset r_M(J(S))$.

Proof: Let mR be a simple submodule of M. Suppose $\alpha(m) \neq 0$ for some $\alpha \in J(S)$. As M is a principal self-generator, $mR = \sum_{s \in I} s(M)$ for some $I \subset S$.

Since mR is simple, $mR = s(M)$ for some $0 \neq s \in I$. Then $\alpha s \neq 0$ and $Ker(\alpha s) = Ker(s)$. Since M is small $PQ-$principally injective and $\alpha s(M)$ is a small submodule of M, $Ss \subset S\alpha s$ by Lemma 2.5. Write $s = \beta\alpha s$ where $\beta \in S$. Then $(1-\beta\alpha)s = 0$ so $s = (1-\beta\alpha)^{-1}0 = 0$, a contradiction.

Following [5], a ring R is called *semiregular* if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. Equivalently, R is semiregular if and only if for each element $a \in R$, there exists $e^2 = e \in Ra$ such that $a(1-e) \in J(R)$.

Proposition 2.11: Let M be a principal, small $PQ-$principally injective module.

(1) If S is local, then $J(S) = \{s \in S: \ Ker(s) \neq 0\}$.

(2) If S is semiregular, then for every $s \in S \setminus J(S)$, there exists a nonzero idempotent $\alpha \in Ss$ such that $Ker(s) \subset Ker(\alpha)$ and $Ker(s(1-\alpha)) \neq 0$.

Proof: (1) Since S is local, $Ss \neq S$ for any $s \in J(S)$. If $s \in J(S)$ and $Ker(s) = 0$, then by Lemma 2.5, $S \subset Ss$ because $s(M) \ll M$. It follows that $S = Ss$, which is a contradiction. This shows that $J(S) \subset \{s \in S: \ Ker(s) \neq 0\}$. The other inclusion is clear.

(2) Let $s \in S \setminus J(S)$. Then there exists $\alpha^2 = \alpha \in Ss$ such that $s(1-\alpha) \in J(S)$. Then $\alpha \neq 0$ and $Ker(s) \subset Ker(\alpha)$. If $Ker(s(1-\alpha)) = 0$, then $S \subset Ss(1-\alpha)$ by Lemma 2.5. It follows that $gs(1-\alpha) = 1_M$ for some $g \in S$. It follows that $\alpha = 0$, a contradiction.

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