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STRUCTURE OF SEMIRINGS AND ORDERED SEMIRINGS

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ABSTRACT

In this paper, we study some properties of semirings and ordered semirings. Also PRD semirings or semirings in which (S, +) is a zeroid or C – Semiring are discussed.

Keywords: *Non-negatively ordered; Non-positively ordered; PRD; Zeroid; C – semirings.*

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1.INTRODUCTION:

A triple $(S,+,\cdot)$ is called a semiring if $(S,+)$ is a semigroup; $(S,\cdot)$ is semigroup; $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for every a, b, c *in* S. A semiring $(S,+,\cdot)$ is said to be a totally ordered semiring if the additive semigroup $(S,+)$ and multiplicative semigroup $(S,\cdot)$ are totally ordered semigroups under the same total order relation. An element x in a totally ordered semigroup $(S,\cdot)$ is non - negative (non - positive) if $x^2 \geq x(x^2 \leq x)$. A totally ordered semigroup $(S,\cdot)$ is said to be non - negatively (non - positively) ordered if every one of its elements is non-negative (non-positive). $(S,\cdot)$ is positively (negatively) ordered in strict sense if $xy \geq x \text{ and } xy \geq y$ $(xy \leq x \text{ and } xy \leq y)$ for every x and y in S. A semigroup $(S,+)$ is said to be a band if $a + a = a$ for all a in S. A semiring $(S,+,\cdot)$ is said to satisfy Integral Multiple Property (IMP) if $a^2 = na$ for all a in S where the positive integer n depends on the element a. A semiring $(S,+,\cdot)$ with additive identity zero which is multiplicative zero is said to be zero square ring if $x^2 = 0$ for all $x \in S$. In[3], the structure of semirings in which $(S,\cdot)$ contains the multiplicative identity or $(S,\cdot)$ is cancellative are studied.This motivated the author to study the structure of semirings in which $(S,+)$ is a zeroid or PRD semirings. Zeroid of a semiring $(S,+,\cdot)$ is the set of all x in S such that $x + y = y$ or $y + x = y$ for some y in S. We may also term this as the zeroid of $(S,+,\cdot)$. A semiring $(S,+,\cdot)$ is said to be a Positive Rational Domain (PRD) if and only if $(S,\cdot)$ is an abelian group.

2.PASITIVE RATIONAL DOMAIN (PRD) AND ZEROID:

Theorem 2.1: Let $(S,+,\bullet)$ be a PRD and $x \in Z$, where Z is the zeroid of $(S,+,\bullet)$. Then the multiplicative identity 1 is a zeroid.

Proof: Suppose $x \in Z$. Since Z is a zeroid of $(S,+,\cdot)$, $\exists y \in S$ such that $x + y = y$ or $y + x = y$.

Suppose $x + y = y$
$\Rightarrow x^{-1}(x + y) = x^{-1}y$
$\Rightarrow x^{-1}x + x^{-1}y = x^{-1}y$
$\Rightarrow 1 + x^{-1}y = x^{-1}y$
i.e., $1 + s = s \text{ where } s = x^{-1}y \in S$
$\Rightarrow$ 1 is the zeroed

Suppose $y + x = y$
$x^{-1}(y + x) = x^{-1}y$
$\Rightarrow x^{-1}y + x^{-1}x = x^{-1}y$
$\Rightarrow x^{-1}y + 1 = x^{-1}y$
i.e., $p + 1 = p$ where $p = x^{-1}y \in S$
$\Rightarrow$ 1 is the zeroid
$\therefore$ In both cases 1 is a zeroid.

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Theorem 2.2: Let $(S,+,\bullet)$ be a PRD semiring satisfying $ab = a + b + ab$ for all $a, b \in S$. If $(S,+)$ is right cancellative, then $(S,+)$ is commutative.

Proof: By hypothesis $ab = a + b + ab$
$ba = b + a + ba$

Since (S, •) is commutative, $ab = ba$.

Therefore, $a + b + ab = b + a + ba$
$= b + a + ab$

$\therefore a + b = b + a$, since (S, +) is right cancellative

$\therefore$ $(S,+)$ is commutative.

Theorem 2.3: If $(S,+,\bullet)$ be a PRD semiring and $(S,\bullet)$ be a rectangular band. Then S reduces to a singleton set.

Proof: Suppose $(S,\bullet)$ is a rectangular band

Let x be an arbitrary element of S

$x = x(x)x$
$x = x^3$

$i.e\ x^2 = x^4$

Also $x = x(x^2)x$

$\Rightarrow x = x^4$

Therefore $x = x^2$ i.e., $(S,\bullet)$ is a b and.

Since $(S,+,\bullet)$ is a PRD, $(S,\bullet)$ is a group and therefore the identity is the only multiplicative idempotent in S.

Hence S reduces to a singleton set.

Examples of Zeroid Semiring 2.4:

The following are the examples of semiring in which (S, +) is a zeroid. $a < b < 2a < a + b < b + a < 2b < c$
(i)

+, •	a	b	2a	a + b	b + a	2b	c
a	2a	a + b	c	c	c	c	c
b	b + a	2b	c	c	c	c	c
2a	c	c	c	c	c	c	c
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
c	c	c	c	c	c	c	c

$(S,+)$ is non-commutative. A commutative version is obtained identifying $a + b$ and $b + a$.

In example (ii) $(S,+)$ is commutative
$a < 2a < c < b < a + b < d$

+, •	a	2a	c	b	a + b	d
a	2a	c	c	a + b	d	d
2a	c	c	c	d	d	d
c	c	c	c	d	d	d

b	d	d	d	d	d	d
a + b	d	d	d	d	d	d
d	d	d	d	d	d	d

Theorem 2.5: Let $(S,+,\bullet)$ be a zero square semiring and 0 is the additive identity. If $(S,+)$ is a zeroid then $S^2 = \{0\}$.

Proof: Since $(S,+)$ is a zeroid
$x + y = y$ *or* $y + x = y$

Since S is a zero square semiring

$x^2 = 0, y^2 = 0, \forall x,y \in S$
$x + y = y$

$\Rightarrow (x + y)y = y^2$
$xy + y^2 = y^2$
$xy + 0 = 0$
$xy = 0$
$x + y = y$

$\Rightarrow y(x+y) = y^2$
$yx + y^2 = y^2$
$yx + 0 = 0$
$yx = 0$

If $y + x = y$ then $y(y+x) = y^2$

$\Rightarrow y^2 + yx = y^2$

$\Rightarrow 0 + yx = 0$

$\Rightarrow yx = 0$

Also $y + x = y$ implies $(y+x)y = y^2$

$\Rightarrow y^2 + xy = y^2$

$\Rightarrow 0 + xy = 0$

$xy = 0$

$\therefore xy = yx = 0$

Hence, $S^2 = \{0\}$

Theorem 2.6: Let $(S,+,\bullet)$ be a semiring with IMP in which $(S,+)$ is a zeroid. If $(S,+)$ is cancellative, then $(S,\bullet)$ is a band.

Proof: Let x∈S. since $(S,+)$ is a zeroid
$x + y = y$ *or* $y + x = y$ for some y in S

Suppose $x + y = y$
$x + x + y = x + y$
$2x + y = y$

Continuing like this, $nx + y = x + y$

$nx = x$ (since $(S,+)$ is right cancellative) (1)

since S satisfies IMP, $x^2 = nx$ (2)

$\therefore$ From (1)& (2), $x^2 = x$

If $y + x = y$
$y + x + x = y + x$
$y + 2x = y + x$

Continuing like this, $y + nx = y + x$

$\Rightarrow nx = x$ (since $(S, +)$ is left cancellative) (3)

S satisfies IMP, $x^2 = nx$ (4)

$\therefore$ From (3) and (4), $x^2 = x$

$\therefore (S,\bullet)$ is a band.

3.C – SEMIRING:

In this section, we characterize c-semiring A semiring is said to be Constraint over semiring (C-semiring) if

Definition: A C – semiring is a semiring in which
(i) $(S, +)$ is a commutative monoid
(ii) $(S, \cdot)$ is a commutative monoid
(iii) $a\,(b + c) = ab + ac$ *and* $(b + c)\,a = ba + ca$ for every a, b, c in S
(iv) $a.0 = 0.a = 0$
(v) $(S, +)$ is a band and 1 is the absorbing element of '$+$'

Example 3.1: Let S be a C – semiring. Define $nx + 0 = nx = 0 + nx$, $x + 1 = 1 = 1 + x$, $\forall x \in S$, then $(S, \cdot)$ is not necessarily band.

+	1	x	y	0
1	1	1	1	1
x	1	x	y	x
y	1	y	y	y
0	1	x	y	0

•	1	x	y	0
1	1	x	y	0
x	x	0	0	0
y	y	0	0	0
0	0	0	0	0

Theorem 3.2: Let $(S, +,\bullet)$ be a C – semiring and $(S, +)$ be left cancellative or right cancellative then $(S,\bullet)$ is a band.

Proof: Let $a \in S$
Consider $a + a^2 = a.1 + a^2$
$= a(1 + a)$
$= a.1$
$= a ----(I)$

Also $a + a = a ----(II)$

From (I) and (II), $a + a^2 = a + a$

$\Rightarrow a^2 = a$ (since (S,+) is left cancellative)

$\therefore (S,\bullet)$ is a band

Similarly if $(S, +)$ is right cancellative, we can prove that $(S,\cdot)$ is band.

Theorem 3.3: Let $(S, +,\bullet)$ be a C – semiring, then $a^2 = a + a^2$, $\forall a \in S$ if and only if $(S,\bullet)$ is a band.

Proof: Suppose $a^2 = a + a^2$
$= a.1 + a^2$
$= a(1 + a)$
$= a.1$ (Since 1 is an absorbing element w.r.to.'$+$')
$a^2 = a$

$\therefore$ $(S,\bullet)$ is a band

Conversely suppose $(S,\bullet)$ is a band

$i.e., a^2 = a, \forall a \in S$
$a^2 = a.1$
$= a(1 + a)$ (Since 1 is an absorbing element w.r to.'+')
$= a + a^2, \forall a \in S$
$\therefore a^2 = a + a^2$

Theorem 3.4: Let $(S,+,\bullet)$ be a C – semiring. Then (i) $(S,+)$ is regular. (ii) $(S,+)$ is Zeroid.

Proof: (i) Let $x \in S$

Then $x + 0 + x = x + x$ $(\therefore 0$ is the additive identity)
$= x$ $(\therefore (S,+)$ is a band)
$\therefore S$ is regular.

(ii) Let $x \in S$

Then $x + 1 = 1, \forall x \in S$ ($\therefore$ S is a C – Semiring)

$\therefore$ $(S,+)$ is a zeroid.

Theorem 3.5: Let $(S,+,\bullet)$ be a C – semiring. Then S contains two elements a and b such that $ab = a + b + ab$ if and only if $ab = a + b = a = b$

Proof: $ab = a + b + ab$
$= a + (1 + a)b$
$ab = a + b$ $-----$ (I)
$ab = a + b + ab$
$ab + b = a + b + ab + b$
$(a + 1)b = a + b + (a + 1)b$
$1.b = a + b + 1.b$

$\Rightarrow 1.b = a + b + b$
$b = a + b$ $----$ (II) (since $b + b = b$)
$ab = a + b + ab$

$\Rightarrow ab + a = a + b + ab + a$
$a(b + 1 = a + b + a(b + 1)$
$a.1 = a + b + a.1$
$a = a + b$ $----$ (III)

from (I), (II) and (III), $ab = a + b = a = b$

conversely, $ab = a + b$
$= a + b + b$ (since $b + b = b$)
$= a + b + ab$
$\therefore ab = a + b + ab$

Examples 3.6:

The following are the examples of ordered C – semirings.

(i)

+	1	x	0
1	1	1	1
x	1	x	x
0	1	x	0

•	1	x	0
1	1	x	0
x	x	0	0
0	0	0	0

(ii)

+	1	x	0
1	1	1	1
x	1	x	x
0	1	x	0

•	1	x	0
1	1	x	0
x	x	x	0
0	0	0	0

(i) (S, +) is p.t.o. and $o < x < 1$.

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

•	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

Theorem 3.7: Let $(S,+,\bullet)$ be a t.o. C – semiring in which $(S,+)$ is p.t.o.(n.t.o.), then for any x,y in S, $x + y = x \text{ or } y$.

Proof: Suppose $< y$, Then $x + y \leq y + y$

$x + y \leq y ----(I)$ (since $y + y = y$)

By hypothesis $(S,+)$ is p.t.o.,
i.e., $x + y \geq y ----(II)$

from (I)& (II) $x + y = y$
If $y < x$
$x + y \leq x + x$
$x + y \leq x$

Since $(S,+)$ is p.t.o.
$x + y \geq x$
$\therefore x + y = x$

Theorem 3.8: Let $(S,+,\bullet)$ be a t.o. C – semiring in which (S, •) is p.t.o. Then 1 is the minimum element and 0 is the maximum element.

Proof: $a.0 = 0.a = 0$ since 0 is the multiplicative zero
Since $(S,\bullet)$ is p.t.o. 0 is the maximum element
$a = a.1 \geq 1$
$\therefore$ 1 is the minimum element.

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