Some Results on Elegant Graphs

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Abstract

In 1981, Chang, Hsu and Rogers [1] defined an elegant labeling \( f \) of a graph \( G \) with \( q \) edges as an injective function from the vertices of \( G \) to the set \{0, 1, 2, \ldots, q\} such that when each edge \( xy \) is assigned the label \((f(x) + f(y)) \pmod{(q+1)}\), the resulting edge labels are distinct and non-zero. In this paper, certain families of graphs are shown to be elegant.

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1. INTRODUCTION

In this paper, by a graph we mean an undirected graph without loops or multiple edges. For notations and terminology, we follow Bondy and Murthy [2].

Throughout this paper, we denote the cycle on \( n \) vertices by \( C_n \) and the path on \( n \) vertices by \( P_n \). Also, \( f \) stands for a 1 – 1 function from \( V(G) \) to a subset of the set of non-negative integers and for any edge \( e = xy \in E(G) \), \( f^*(xy) = f(x) + f(y) \). We call \( f^* \) the induced edge labeling of \( G \) (induced by \( f \)).

Chang, Hsu and Rogers [1] defined an elegant labeling \( f \) of a graph \( G \) with \( q \) edges as an injective function from the vertices of \( G \) to the set \{0, 1, 2, \ldots, q\} such that when each edge \( xy \) is assigned the label \((f(x) + f(y)) \pmod{(q+1)}\), the resulting edge labels are distinct and non-zero. In this paper, certain families of graphs are shown to be elegant.

Balakrishnan, Selvam and Yegnanarayanan [3] have shown that the bistar \( B_{n,n} \) is elegant if and only if \( n \) is even. For example, an elegant labeling of \( B_{2,2} \) is shown in Figure 1.1.

Fig. 1.1

Theorem 1.1: The total possibilities of the edge labeling in an elegant graph is \( \frac{q^2}{2} \) when \( q \) is even and \( \frac{q^2 + 1}{2} \) when \( q \) is odd.

Proof: case (i) when \( q \) is even.

Sub case (i): Let the edge label be \( k \).

The possible edge labels are \{(i, q-i+k+1) : k + 1 \leq i \leq \frac{q}{2} + \left\lfloor \frac{k}{2} \right\rfloor \} \cup \{(i, k-i) : 0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor - 1\} \) and its total is \( \frac{q}{2} \).

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Hence, the total possible edge labels is \( q \left( \frac{q}{2} \right) = \frac{q^2}{2} \).

Case (ii) when \( q \) is odd.

Sub case (i): Let the edge label be \( k \) and let \( k \) be odd with \( 1 \leq k \leq q \).

The possible edge labels are \( \{(i, q-i+k+1) : k+1 \leq i \leq \frac{q+1}{2} + \frac{k-1}{2}\} \cup \{(i, k-i) : 0 \leq i \leq \frac{k-1}{2}\} \) and its total is \( \frac{q+1}{2} \).

Hence, the total possible odd edge label is \( \left( \frac{q+1}{2} \right) \left( \frac{q+1}{2} \right) \) \( \text{ (1) } \)

Sub case (ii): Let the edge label be \( k \) and let \( k \) be even with \( 2 \leq k \leq q-1 \).

The possible edge labels are \( \{(i, q-i+k+1) : k+1 \leq i \leq \frac{q+1}{2} + \frac{k-2}{2}\} \cup \{(i, k-i) : 0 \leq i \leq \frac{k}{2}-1\} \) and its total is \( \frac{q-1}{2} \).

Hence, the total possible even edge labels is \( \left( \frac{q-1}{2} \right) \left( \frac{q-1}{2} \right) \) \( \text{ (2) } \)

Therefore, the total possible edge label is,

\[ \left( \frac{q+1}{2} \right) \left( \frac{q+1}{2} \right) + \left( \frac{q-1}{2} \right) \left( \frac{q-1}{2} \right) = \frac{q^2 + 1}{2} , \text{ when } q \text{ is odd} \]

Hence, the total possibilities of the edge labeling in an elegant graph is \( \frac{q^2 + 1}{2} \) when \( q \) is even and \( \frac{q^2 + 1}{2} \) when \( q \) is odd.

2. DEFINITIONS:

Definition 2.1: Consider the graph \( C_n \times P_m \). Let \( C_n^i, 1 \leq i \leq m \) denote the \( m \) cycles in the graph \( C_n \times P_m \), corresponding to each vertex of \( v_i \) of \( P_m \). Add a new vertex \( v \) and join it to all the vertices of \( C_n^1, C_n^2, C_n^3, \ldots, C_n^m \). The resulting graph be called as \( C_{n,m} \).

Definition 2.2: \([4]\) Let \( C_{n,m}^\dagger \) denotes the graph obtained from \( C_n \times P_m \) by taking two new distinct vertices, say \( u \) and \( v \) and joining \( u \) to all the vertices of \( C_n^1 \) and \( v \) to all the vertices of \( C_n^m \).

Definition 2.3: The total graph \( T(G) \) of \( G \) has the vertex set \( V(G) \cup E(G) \) in which two vertices are adjacent whenever they are either adjacent or incident in \( G \). The vertex set of \( T(P_n) \) is \( \{u_i, v_j : 1 \leq i \leq n, \ 1 \leq j \leq n - 1\} \) and the edge set of \( T(P_n) \) is \( \{u_i u_{i+1}, v_{j-1}, u_i v_i, u_{i+1} v_{j+1} : 1 \leq i \leq n - 1, \ 2 \leq j \leq n - 2\} \).

For example, \( T(P_3) \) is shown in Figure 2.1.

![Fig. 2.1](image)

Definition 2.4: The graph \( P_n^2 \) is a graph with vertex set \( V(P_n^2) = \{u_i : 1 \leq i \leq n\} \) and \( E(P_n^2) = \{u_i u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_{i+2} : 1 \leq i \leq n - 2\} \).

Definition 2.5: The graph \( K_2 + mK_1 \) is the join of the graph \( K_2 \) and \( m \) disjoint copies of \( K_1 \). Some authors call this graph a Book with triangular pages.
For example, \( K_2 + 4K_1 \) is shown in Figure 2.2.

![Fig. 2.2](image)

**Definition 2.6:** The graph \((P_2 \cup mK_1) + N_2\) is a graph with vertex set \(\{z_1, z_2, x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2\}\) and the edge set \(\{z_1z_2, y_1z_1, y_1z_2, y_2z_1, y_2z_2\} \cup \{y_1x_i, y_2x_i / 1 \leq i \leq m\}\).

For example, \((P_2 \cup 2K_1) + N_2\) is shown in Figure 2.3.

![Fig. 2.3](image)

**Definition 2.7:** For integers \(m, n \geq 0\), we consider the graph **Jelly Fish** \(J(m, n)\) with vertex set \(V(J(m, n)) = \{u, v, x, y\} \cup \{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}\) and the edge set \(E(J(m, n)) = \{(u, x), (u, y), (v, x), (v, y)\} \cup \{(x_i, x) / 1 \leq i \leq m\} \cup \{(y_j, y) / 1 \leq j \leq n\}\).

For example, \(J(3, 4)\) is shown in Figure 2.4.

![Fig. 2.4](image)

**Definition 2.8:** \(\langle C_3, K_{1,m} \rangle (m \geq 1)\) be the graph obtained by attaching \(K_{1,m}\) to one vertex of the cycle \(C_3\).

For example, \(\langle C_3, K_{1,4} \rangle\) is shown in Figure 2.5.

![Fig. 2.5](image)
Definition 2.9: \((K_4 - \{e\})_t\) is the one edge union of \(K_4 - \{e\}\).

For example, \((K_4 - \{e\})_3\) is shown in Figure 2.6.

![Fig. 2.6](image)

Definition 2.10: Let \(T\) be any tree. Denote the tree obtained from \(T\) by considering 2 copies of \(T\) by adding an edge between them by \(T(2)\) and in general, the graph obtained from \(T_{n-1}\) and \(T\) by adding an edge between them is denoted by \(T(n)\). Note that \(T(1)\) is nothing but \(T\).

For example, \(T\) and \(T(2)\) are shown in Figure 2.7.

![Fig. 2.7](image)

Definition 2.11: Let \(G\) be a graph with a fixed vertex \(v_0\) and let \(v_1, v_2, \ldots, v_m\) be the vertices in \(m\) copies of \(G\) respectively corresponding to the vertex \(v_0\). The graph \([P_m, G]\) is a graph obtained from \(m\) copies of \(G\) by joining \(v_i\) and \(v_{i+1}\) by an edge for each \(i, 1 \leq i \leq m-1\).

For example, \([P_2, C_3]\) is shown in Figure 2.8.

![Fig. 2.8](image)

3. MAIN RESULTS:

Theorem 3.1: \([P_{2m-1}, C_3]\) is an elegant graph for \(m \geq 1\).

Proof: Let \(u_1^j, u_2^j, u_3^j\) be the vertices of \(j\)th copy of \(C_3\).

Define a function \(f: V \rightarrow \{0, 1, 2, \ldots, q = 8m - 5\}\) as follows:

\[
f(u_1^j) = 4(j - 1), \quad 1 \leq j \leq 2m - 1
\]

\[
f(u_2^j) = 4j - 2, \quad 1 \leq j \leq 2m - 1
\]

\[
f(u_3^j) = 4j - 1, \quad 1 \leq j \leq 2m - 1
\]
The induced edge labels are given as,

\[
\begin{align*}
    f (u^j_1 u^j_2) &= \begin{cases} 
        8j - 6, & 1 \leq j \leq m \\
        8(j - m) - 2, & m + 1 \leq j \leq 2m - 1
    \end{cases} \\
    f (u^j_2 u^j_3) &= \begin{cases} 
        8j - 3, & 1 \leq j \leq m - 1 \\
        8(j - m) + 1, & m \leq j \leq 2m - 1
    \end{cases} \\
    f (u^j_1 u^j_3) &= \begin{cases} 
        8j - 5, & 1 \leq j \leq m \\
        8(j - m) - 1, & m + 1 \leq j \leq 2m - 1
    \end{cases} \\
    f (u^j_1 u^{j+1}_1) &= \begin{cases} 
        8j, & 1 \leq j \leq m - 1 \\
        8(j - m) + 4, & m + 1 \leq j \leq 2m - 1
    \end{cases}
\end{align*}
\]

Hence, \([P_{2m} , C_3] \) is an elegant graph for \( m \geq 1 \).

For example, an elegant labeling of \([P_3 , C_3] \) is shown in Figure 3.1.

![Fig. 3.1](image)

**Theorem 3.2:** Comb \( P_n \Theta K_1 \) is an elegant graph.

**Proof:** Let \( u_1, u_2, \ldots, u_n \) be the vertices of the path \( P_n \) and \( v_1, v_2, \ldots, v_n \) be the corresponding pendant vertices.

Define an one to one function \( f : V \to \{0, 1, 2, \ldots, q = 2n - 1\} \) as follows:

\[
\begin{align*}
    f (u_i) &= 2i - 1, \quad 1 \leq i \leq n \\
    f (v_i) &= 2(i - 1), \quad 1 \leq i \leq n
\end{align*}
\]

The induced edge labels are given as,

\[
\begin{align*}
    f (u_i u_{i+1}) &= \begin{cases} 
        4i, & 1 \leq i \leq \frac{n - 1}{2} \\
        4i - 2n, & \frac{n - 1}{2} + 1 \leq i \leq n - 1
    \end{cases} \\
    f (u_i v_i) &= \begin{cases} 
        4i - 3, & 1 \leq i \leq \frac{n + 1}{2} \\
        4i - 2n - 3, & \frac{n + 1}{2} + 1 \leq i \leq n
    \end{cases}
\end{align*}
\]

It is easy to check that \( f(E) = \{1, 2, 3, \ldots, q\} \). Hence, comb \( P_n \Theta K_1 \) is an elegant graph.

For example, an elegant labeling of \( P_5 \Theta K_1 \) is shown in Figure 3.2.

![Fig. 3.2](image)
Theorem 3.3: The graph $K_2 + mK_1$ is an elegant graph for all $m$.

Proof: Let $u, v$ be the vertices of $K_2$ and $u_1, u_2, \ldots, u_m$ be the remaining vertices of the graph $K_2 + mK_1$ with edges $(u_i, v_i), 1 \leq i \leq m$.

Define an one to one function $f: V \rightarrow \{0, 1, 2, 3, \ldots, q = 2m + 1\}$ by

$$f(u) = 0, \quad f(v) = 2m + 1, \quad f(u_i) = 2i, 1 \leq i \leq m.$$

The induced edge labels are given as,

$$f(u_iu_j) = 2i, \quad 1 \leq i \leq m$$
$$f(uvu_i) = 2i - 1, \quad 1 \leq i \leq m$$

Hence, the graph $K_2 + mK_1$ is an elegant graph for all $m$.

For example, an elegant labeling of $K_2 + 4K_1$ is shown in Figure 3.3.

![Fig. 3.3](image-url)

Lemma 3.4: $C_3 \times P_n$ is an elegant graph.

Proof: Let $V(C_3 \times P_n) = \{u_{ij} / 1 \leq i \leq 3 \text{ and } 1 \leq j \leq n\}$ and $E(C_3 \times P_n) = \{(u_{ij}, u_{i+1,j}), \{(u_{ij}, u_{i+1,j}), \{(u_{ij}, u_{i,j}), 1 \leq j \leq n) \cup \{(u_{ij}, u_{i,j+1}) : 1 \leq j \leq n - 1}\}$.

Define an one to one function $f: V \rightarrow \{0, 1, 2, \ldots, q = 6n - 3\}$ as follows:

$$f(u_{ij}) = i - 1, \quad 1 \leq i \leq 3 \text{ for } j = 1$$
$$f(u_{i1}) = 4, \quad f(u_{i2}) = 5, \quad f(u_{i3}) = 6 \text{ and}$$

Let $a = i + j$ where the summation is taken modulo 3 with residues 1,2,3.

$$f(u_{aj}) = f(u_{a+1,j-1}) + i, \quad 1 \leq i \leq 3 \text{ for } 3 \leq j \leq n$$

Clearly, the edge labels $1, 2, 3, \ldots, q = 6n - 3$. 
For example, an elegant labeling of $C_3 \times P_4$ is shown in Figure 3.4.

**Theorem 3.5**: $C_{3,n}$ is an elegant graph for any $n$.

**Proof**: $C_3 \times P_n$ is an elegant graph by lemma 3.4. Let $V(C_{3,n}) = \{v, u_{ij} : 1 \leq i \leq 3, 1 \leq j \leq n\}$ and $E(C_{3,n}) = E(C_3 \times P_n) \cup \{v u_{ij} : 1 \leq i \leq 3, 1 \leq j \leq n\}$

Define $f(u_{ij})$ as in lemma 3.4 and $f(v) = 6n - 2$

The edge labels of $u_{ij} v$ is $6n - 2 + f(u_{ij}), 1 \leq i \leq 3$ and $1 \leq j \leq n$.

Clearly, the edge labels of $C_3 \times P_n$ are distinct and non-zero.

For example, an elegant labeling of $C_{3,4}$ is shown in Figure 3.5.

**Theorem 3.6**: $C_{3,n}^\dagger$ is an elegant graph for any $n$.

**Proof**: $C_3 \times P_n$ is an elegant graph by lemma 3.4. Let $V(C_{3,n}^\dagger) = V(C_3 \times P_n) \cup \{u, v\}$ and $E(C_{3,n}^\dagger) = E(C_3 \times P_n) \cup \{(u u_{i1}), (v u_{in}) : 1 \leq i \leq 3, 1 \leq j \leq n\}$

Define $f(u_{ij})$ as in lemma 3.4 and $f(u) = 6n - 2, f(v) = 3n + 4$

The labels of the edges $u_{11} u, u_{12} u, u_{13} u, u_{n1} v, u_{n2} v, u_{n3} v$ as $6n - 2, 6n - 1, \ldots 6n + 3$.

Hence, the edge labels of $C_{3,n}^\dagger$ distinct and non-zero.
For example, an elegant labeling of $C_{3,4}$ is shown in Figure 3.6.

\[ \begin{align*}
\text{Fig. 3.6} \\
\end{align*} \]

**Theorem 3.7:** The total graph $T(P_n)$ is an elegant for any positive integer $n$.

**Proof:** Let $P_n = u_1, u_2, \ldots, u_n$ and let $V(T(P_n)) = V(P_n) \cup \{v_i : 1 \leq i \leq n-1\}$ and $E(T(P_n)) = E(P_n) \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. The total number of edges is $3n - 4$.

Define an one to one function $f : V \rightarrow \{0, 1, 2, 3, \ldots q = 4n - 5\}$ by

\[
\begin{align*}
f(u_i) &= \begin{cases} 
  i, & 1 \leq i \leq 2 \\
  2i - 3, & 3 \leq i \leq n 
\end{cases} \\
f(v_j) &= \begin{cases} 
  0, & j = 1 \\
  2j, & 2 \leq j \leq n - 1 
\end{cases}
\end{align*}
\]

The labels of the edges are given as:

\[
\begin{align*}
f(u_i u_{i+1}) &= \begin{cases} 
  2i + 1, & 1 \leq i \leq 2 \\
  4(i - 1), & 3 \leq i \leq n - 1 
\end{cases} \\
f(v_j v_{j+1}) &= \begin{cases} 
  4, & i = 1 \\
  4i + 2, & 2 \leq i \leq n - 2 
\end{cases} \\
f(u_i v_j) &= \begin{cases} 
  5i - 4, & 1 \leq i \leq 2 \\
  4i - 3, & 3 \leq i \leq n - 1 
\end{cases} \\
f(u_{i-1} v_i) &= \begin{cases} 
  5i - 3, & 1 \leq i \leq 2 \\
  4i - 1, & 3 \leq i \leq n - 1 
\end{cases}
\end{align*}
\]

Hence, the total graph $T(P_n)$ is an elegant for any positive integer $n$.

For example, an elegant labeling of $T(P_5)$ is shown in Figure 3.7.

\[ \begin{align*}
\text{Fig. 3.7} \\
\end{align*} \]
Theorem 3.8: The graph $P_n^2$ is an elegant graph.

Proof: Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$.

Define an one to one function $f: V \to \{0, 1, 2, 3, \ldots, q\}$ by
\[ f(u_i) = i - 1, \quad 1 \leq i \leq n \]
The labels of the edges are given as:
\[ f(u_i u_{i+1}) = 2i - 1, \quad 1 \leq i \leq n - 1 \]
\[ f(u_i u_{i+2}) = 2i, \quad 1 \leq i \leq n - 2 \]
Hence, the graph $P_n^2$ is an almost elegant graph.

For example, the elegant labeling of $P_5^2$ is given in the Figure 3.8.

![Fig. 3.8](image)

Theorem 3.9: $(P_2 \cup mK_1) + N_2$ is an elegant graph.

Proof: Let $z_1$ and $z_2$ and $y_1$ and $y_2$ and $x_j, 1 \leq j \leq m$ be the vertices of $(P_2 \cup mK_1) + N_2$.

Define an one to one function $f: V \to \{0, 1, 2, 3, \ldots, q = 2m + 5\}$ by
\[ f(z_i) = 3(i-1), \quad 1 \leq i \leq 2 \]
\[ f(y_i) = i, \quad 1 \leq i \leq 2 \]
\[ f(x_j) = 2j + 3, \quad 1 \leq j \leq m \]
The labels of the edges are given as:
\[ f(y_1 z_1) = 1, \]
\[ f(z_1 y_2) = 2, \]
\[ f(z_1 z_2) = 3, \]
\[ f(y_1 z_2) = 4, \]
\[ f(y_2 z_2) = 5, \]
\[ f(y_1 x_j) = 2j + 4, \quad 1 \leq j \leq m \]
\[ f(y_2 x_j) = 2j + 5, \quad 1 \leq j \leq m \]
Hence, $(P_2 \cup mK_1) + N_2$ is an elegant graph.
For example, an elegant labeling of \((P_2 \cup 2K_1) + N_2\) is shown in Figure 3.9.

**Theorem 3.10:** Jelly fish \(J(m, n)\) is an elegant graph for any positive integers \(m, n\).

**Proof:** Let \(u, v, x, y, x_i, 1 \leq i \leq m\) and \(y_j, 1 \leq j \leq n\) be the vertices of Jelly fish. Let \(V(J(m, n)) = \{u, v, x, y\} \cup \{x_i : 1 \leq i \leq m\} \cup \{y_j : 1 \leq j \leq n\}\) and \(E(J(m, n)) = \{(u, x), (u, y), (u, v), (v, x), (v, y)\} \cup \{(x_i, x) : 1 \leq i \leq m\} \cup \{(y_j, y) : 1 \leq j \leq n\}\).

Define an one to one function \(f : V \rightarrow \{0, 1, 2, 3, \ldots, q = m + n + 5\}\) by

- \(f(u) = 0\),
- \(f(v) = 3\),
- \(f(x) = 1\),
- \(f(y) = 2\),
- \(f(y_j) = 3 + j, \ 1 \leq j \leq n\),
- \(f(x_i) = n + 4 + i, \ 1 \leq i \leq m\)

The labels of the edges are given as follows:

- \(f(ux) = 1\),
- \(f(uy) = 2\),
- \(f(uv) = 3\),
- \(f(xv) = 4\),
- \(f(yy_j) = 5 + j, \ 1 \leq j \leq n\)
- \(f(xx_i) = n + 5 + i, \ 1 \leq i \leq m\)
- \(f(yy_j) = 5 + j, \ 1 \leq j \leq n\)

Clearly, the edge values are distinct and non-zero. Hence, Jelly fish \(J(m, n)\) is an elegant graph for any positive integers \(m, n\).

For example, an elegant labeling of \(J(3, 4)\) is shown in Figure 3.10.
Proposition 3.11: $C_3 \circ K_{1,m}$ ($m \geq 1$) is an elegant graph.

Proof: Let $V(C_3 \circ K_{1,m})=\{u_1, u_2, u_3, v_1, v_2, v_3, \ldots, v_m\}$ and $E(C_3 \circ K_{1,m}) = \{(u_1 u_2), (u_2 u_3), (u_3 u_1)\} \cup \{u_2 v_i : 1 \leq i \leq m\}$.

Let $u_2$ be the common vertex (centre vertex) of $K_{1,m}$.

Define an one to one function $f: V \rightarrow \{0, 1, 2, 3, \ldots, q = m + 3\}$ by

\[
\begin{align*}
    f(u_i) &= i - 1, & & 1 \leq i \leq 3 \\
    f(v_j) &= 2 + j, & & 1 \leq j \leq m
\end{align*}
\]

The labels of the edges are given as:

\[
\begin{align*}
    f(u_1u_2) &= 1, & f(u_2u_3) &= 3, & f(u_3u_1) &= 2, \\
    f(u_2v_i) &= 3 + j, & 1 \leq j \leq m
\end{align*}
\]

Clearly, the edge labels are distinct and non-zero. Hence, $C_3 \circ K_{1,m}$ ($m \geq 1$) is an elegant graph.

For example, an elegant labeling of $C_3 \circ K_{1,6}$ is shown in Figure 3.11.

\[\text{Fig. 3.11}\]

Theorem 3.12: $(K_4 - \{e\})_t$ is an elegant graph for $t \geq 1$.

Proof: Let $V((K_4 - \{e\})_t) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E((K_4 - \{e\})_t) = \{(u_i u_{i+1}), (v_i v_{i+1}), (u_i v_{i+1}) : 1 \leq i \leq n - 1\} \cup \{u_i v_i : 1 \leq i \leq n\}$.

Define an one to one function $f: V \rightarrow \{0, 1, 2, 3, \ldots, q\}$ by

\[
\begin{align*}
    f(u_i) &= 2i - 2, & & 1 \leq i \leq n \\
    f(v_i) &= 2i - 1, & & 1 \leq i \leq n
\end{align*}
\]

The labels of the edges are given as:

\[
\begin{align*}
    f(u_i u_{i+1}) &= 4i - 2, & & 1 \leq i \leq n - 1, \\
    f(v_i v_{i+1}) &= 4i, & & 1 \leq i \leq n - 1, \\
    f(u_i v_i) &= 4i - 3, & & 1 \leq i \leq n, \\
    f(u_i v_{i+1}) &= 4i - 1, & & 1 \leq i \leq n - 1.
\end{align*}
\]

Clearly, the edge labels are distinct and non-zero. Hence, $(K_4 - \{e\})_t$ is a near felicitous graph for $t \geq 1$.

For example, an elegant labeling of $(K_4 - \{e\})_4$ is shown in Figure 3.12.

\[\text{Fig. 3.12}\]
Theorem 3.13: Let \( n \)-armed crown \( C_3 \otimes K_m \), \( m \geq 1 \) is an elegant graph.

Proof: Let \( V(C_3 \otimes K_m) = \{u_1, u_2, u_3, : 1 \leq j \leq m\} \) and \( E(C_3 \otimes K_m) = \{(u_1 u_2), (u_2 u_3), (u_3 u_1)\} \cup \{(u_1 v_j), (u_2 v_j), (u_3 v_j) : 1 \leq j \leq m\} \).

Define an one to one function \( f: V \rightarrow \{0, 1, 2, \ldots, q = 3m + 3\} \) by

\[
f(u_i) = i - 1, \quad 1 \leq i \leq 3
\]

For \( 1 \leq i \leq 2 \),

\[
f(u_{i+2j}) = i + 2j, \quad 1 \leq j \leq m
\]

For \( i = 3 \),

\[
f(u_{i+2j}) = f(u_1j) + j, \quad 1 \leq j \leq m
\]

The label of the edges are given as :

\[
f(u_3 v_j) = 2j + 1, \quad 1 \leq j \leq m
\]

\[
f(u_1 v_j) = 2(j + 1), \quad 1 \leq j \leq m
\]

\[
f(u_2 v_j) = 2m + 4 + (j - 1), \quad 1 \leq j \leq m
\]

Clearly, the edge labels are distinct and non-zero. Hence, \( n \)-armed crown \( C_3 \otimes K_m \), \( m \geq 1 \) is an elegant graph.

For example, an elegant labeling of \( C_3 \otimes K_3 \) is shown in Figure 3.13.

Fig. 3.13

REFERENCES:


