COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS
IN $\mathcal{M}$ - FUZZY METRIC SPACE

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(Received on: 16-03-12; Accepted on: 31-03-12)

ABSTRACT

In this paper we prove a common fixed point theorem for four mappings in $\mathcal{M}$ – fuzzy metric space using the notion of semi compatibility. Also, we prove a common fixed point theorem for four weakly compatible mappings in $\mathcal{M}$ – fuzzy metric space.

Mathematics Subject Classification: 47H10, 54H25.

Keywords: Complete $\mathcal{M}$ – Fuzzy metric space, Semi compatible mappings, weakly compatible mappings, Common fixed point.

INTRODUCTION AND PRELIMINARIES

Zadeh [16] introduced the concept of fuzzy sets in 1965. George and Veeramani [2] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7] and defined the Hausdorff topology of fuzzy metric spaces. Many authors [4, 8] have proved fixed point theorems in fuzzy metric space. Recently Sedghi and Shobe [13] introduced $D^*$ - metric space as a probable modification of the definition of $D$ - metric introduced by Dhage [1], and prove some basic properties in $D^*$ - metric spaces. Using $D^*$ - metric concepts, Sedghi and Shobe define $\mathcal{M}$ – fuzzy metric space and proved a common fixed point theorem in it. Jong Seo Park [5] introduced the concept of semi compatible and weak compatible in $\mathcal{M}$ – fuzzy metric space and prove some fixed point theorems satisfying some conditions in $\mathcal{M}$ – fuzzy metric space. In this paper we prove a common fixed point theorem for four mappings in $\mathcal{M}$ – fuzzy metric space using the notion of semi compatibility. Also, we prove a common fixed point theorem for four weakly compatible mappings in $\mathcal{M}$ – fuzzy metric space.

Definition: 1.1 Let $X$ be a nonempty set. A generalized metric (or $D^*$ - metric) on $X$ is a function: $D^* : X^3 \rightarrow [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$

(i) $D^*(x, y, z) \geq 0$,
(ii) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
(iii) $D^*(x, y, z) = D^*(p(x, y, z))$ (symmetry) where $p$ is a permutation function,
(iv) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z)$. The pair $(X, D^*)$, is called a generalized metric (or $D^*$ - metric) space.

Example: 1.2 Examples of $D^*$ - metric are
(a) $D^*(x, y, z) = \max \{d(x, y), d(y, z), d(z, x) \}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.
Here, $d$ is the ordinary metric on $X$.

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International Journal of Mathematical Archive- 3 (3), Mar. – 2012
Definition: 1.3 A fuzzy set \( M \) in an arbitrary set \( X \) is a function with domain \( X \) and values in \([0, 1]\).

Definition: 1.4 A binary operation \( * \): \([0, 1] \times [0, 1] \rightarrow [0, 1]\) is a continuous \( t \)-norm if it satisfies the following conditions

(i) \( * \) is associative and commutative,
(ii) \( * \) is continuous,
(iii) \( a * 1 = a \) for all \( a \in [0, 1] \),
(iv) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0, 1] \).

Examples for continuous \( t \)-norm are \( a * b = ab \) and \( a * b = \min \{a, b \} \).

Definition: 1.5 A 3-tuple \((X, M, *)\) is called \( M \)-fuzzy metric space if \( X \) is an arbitrary non-empty set, \( * \) is a continuous \( t \)-norm, and \( M \) is a fuzzy set on \( X^2 \times (0, \infty) \), satisfying the following conditions for each \( x, y, z, a \in X \) and \( t > 0 \)

\[
\begin{align*}
(FM-1) & \quad M(x, y, z, t) > 0 \\
(FM-2) & \quad M(x, y, z, t) = 1 \iff x = y = z \\
(FM-3) & \quad M(x, y, z, t) = M(p \{x, y, z\}, t), \text{ where } p \text{ is a permutation function} \\
(FM-4) & \quad M(x, y, a, t) * M(a, z, s) \leq M(x, y, z, t+s) \\
(FM-5) & \quad M(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1] \text{ is continuous} \\
(FM-6) & \quad \lim_{n \to \infty} M(x, y, z, t) = 1.
\end{align*}
\]

Example: 1.6 Let \( X \) be a nonempty set and \( D^* \) is the \( D^* \) - metric on \( X \). Denote \( a*b = ab \) for all \( a, b \in [0, 1] \). For each \( t \in (0, \infty) \), define

\[
M(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}
\]

for all \( x, y, z \in X \), then \((X, M, *)\) is a \( M \)-fuzzy metric space. We call this \( M \)-fuzzy metric induced by \( D^* \) - metric.

Thus every \( D^* \) - metric induces a \( M \)-fuzzy metric.

Lemma: 1.7 ([13]) Let \((X, M, *)\) be a \( M \)-fuzzy metric space. Then for every \( t > 0 \) and for every \( x, y \in X \), we have

\[
M(x, y, t) = M(x, y, t).
\]

Lemma: 1.8 ([13]) Let \((X, M, *)\) be a \( M \)-fuzzy metric space. Then \( M(x, y, z, t) \) is non-decreasing with respect to \( t \), for all \( x, y, z \in X \).

Definition: 1.9 Let \((X, M, *)\) be a \( M \)-fuzzy metric space and \( \{x_n\} \) be a sequence in \( X \)

(a) \( \{x_n\} \) is said to be converges to a point \( x \in X \) if \( \lim_{n \to \infty} M(x, x, x_n, t) = 1 \) for all \( t > 0 \)

(b) \( \{x_n\} \) is called Cauchy sequence if \( \lim_{n \to \infty} M(x_{n+p}, x_{n+p}, t) = 1 \) for all \( t > 0 \) and \( p > 0 \)

(c) A \( M \)-fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark: 1.10 Since \( * \) is continuous, it follows from \((FM-4)\) that the limit of the sequence is uniquely determined.

Definition: 1.11 Let \( S \) and \( T \) be two self mappings of a \( M \)-fuzzy metric space \((X, M, *)\). Then the mappings are said to be compatible if \( \lim_{n \to \infty} M(STx_n, TStx_n, TStx_n, t) = 1 \), for all \( t > 0 \), whenever \( \{x_n\} \) be a sequence in \( X \) such that \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} Sx_n = z \) for some \( z \in X \).

Definition: 1.12 Let \( S \) and \( T \) be two self mappings of a \( M \)-fuzzy metric space \((X, M, *)\). Then the mappings are called semi compatible if \( \lim_{n \to \infty} M(STx_n, Tz, Tz, t) = 1 \), \( \lim_{n \to \infty} M(TSx_n, Sx, Sz, t) = 1 \) for all \( t > 0 \), whenever \( \{x_n\} \) be a sequence in \( X \) such that \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tz_n = z \) for some \( z \in X \).

Definition: 1.13 Let \( S \) and \( T \) be two self mappings of a \( M \)-fuzzy metric space \((X, M, *)\). Then the mappings \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence points; that is, if \( Sx = Tx \) for some \( x \in X \), then \( STx = TSx \).

Lemma: 1.14 ([11]) Let \( \{x_n\} \) be a sequence in a \( M \)-fuzzy metric space \((X, M, *)\) with the condition \((FM-6)\). If there exists a number \( k \in (0, 1) \) such that

\[
M(x_n, x_{n+1}, x_{n+1}, kt) \geq M(x_{n+1}, x_{n+1}, x_{n+1}, t)
\]

for all \( t > 0 \) and \( n = 1, 2, 3 \ldots \), then \( \{x_n\} \) is a Cauchy sequence.

Lemma 1.15 ([11]) Let \((X, M, *)\) be a \( M \)-fuzzy metric space with condition \((FM-6)\). If there exists a number \( k \in (0, 1) \) such that \( M(x, y, z, kt) \geq M(x, y, z, t) \), for all \( x, y, z \in X \) and \( t > 0 \), then \( x = y = z \).
MAIN RESULTS:

Theorem 2.1 Let $S$ and $T$ be two continuous self mappings of a complete $\mathcal{M}$–fuzzy metric space $(X, \mathcal{M}, *)$. Let $A$ and $B$ be two mappings of $X$ satisfying

1. $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$.
2. $(A, S)$ and $(B, T)$ are semi compatible.
3. there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,
\[ \mathcal{M}(Ax, By, By, kt) \geq \min \{ \mathcal{M}(By, Ty, Ty, t), \mathcal{M}(Sx, Tx, Ty, t), \mathcal{M}(Ax, By, By, t) \}. \]

Then $A, B, S$ and $T$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.

Since $A(X) \subseteq T(X)$, then there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$.

Also, since $B(X) \subseteq S(X)$, then there exists another point $x_2 \in X$ such that $Bx_1 = Sx_2$.

Then by induction, we can define a sequence $\{y_n\}$ in $X$ such that $y_{2n+1} = Ax_{2n}$ and $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$

Now using condition (3) we get
\[ \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) = \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt) \]
\[ \geq \min \{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t), \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \} \]
\[ = \min \{ \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, 1), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1) \} \]
\[ = \min \{ \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, y_{2n+2}, 1), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1) \} \]
\[ = \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, 1). \]

Therefore $\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq \mathcal{M}(y_{2n+1}, y_{2n+1}, 1)$.

Also, $\mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+1}, y_{2n+1}, kt) = \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1)$
\[ = \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, 1) = \mathcal{M}(Ax_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt) \]
\[ \geq \min \{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n+2}, Sx_{2n+2}, Sx_{2n+2}, t), \mathcal{M}(Ax_{2n+2}, Bx_{2n+1}, Bx_{2n+1}, t) \} \]
\[ = \min \{ \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+1}, 1), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1) \} \]
\[ = \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, y_{2n+2}, 1) \]
\[ = \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, y_{2n+2}, 1) \]
\[ = \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, y_{2n+2}, 1) \]
\[ = \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, y_{2n+2}, 1). \]

Therefore $\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, 1)$.

Hence $\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq \mathcal{M}(y_{2n}, y_{2n}, 1)$, for all $n$.

By lemma 1.14, $\{y_n\}$ is a Cauchy sequence in $\mathcal{M}$–fuzzy metric space $X$.

Since $X$ is $\mathcal{M}$–fuzzy complete, sequence $\{y_n\}$ converges to the point $z \in X$.

Also, since $\{Ax_{2n}\}$, $\{Bx_{2n}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ are subsequences of $\{y_n\}$, they also converge to the point $z$.

Case I: Since $S$ is continuous, we have $SAx_{2n} \rightarrow Sz$, $SSx_{2n} \rightarrow Sz$.

Also $(A, S)$ is semi compatible, we have $ASx_{2n} \rightarrow Sz$.
Let \( x = Sx_{2n}, y = x_{2n+1} \) in condition (3) we get
\[
\mathcal{M}(ASx_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt) \geq \min \left\{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(SSx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(ASx_{2n}, SSx_{2n}, SSx_{2n}, t), \mathcal{M}(ASx_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \right\}
\]
Taking limit as \( n \to \infty \) we get
\[
\mathcal{M}(Sz, z, z, kt) \geq \min \left\{ \mathcal{M}(z, z, z, t), \mathcal{M}(Sz, z, z, t), \mathcal{M}(Sz, Sz, Sz, t), \mathcal{M}(Sz, z, z, t) \right\}
\]
\[
= \mathcal{M}(Sz, z, z, t)
\]
Therefore by lemma 1.15, \( Sz = z \).

Now let \( x = z, y = x_{2n+1} \) in condition (3) we get
\[
\mathcal{M}(Az, Bx_{2n+1}, Bx_{2n+1}, kt) \geq \min \left\{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sz, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, Bx_{2n+1}, Bx_{2n+1}, t) \right\}
\]
Taking limit as \( n \to \infty \) we get
\[
\mathcal{M}(Az, z, z, t) \geq \min \left\{ \mathcal{M}(z, z, z, t), \mathcal{M}(Sz, z, z, t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, z, z, t) \right\}
\]
\[
= \min \left\{ \mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(Az, z, z, t), \mathcal{M}(Az, z, z, t) \right\}
\]
\[
= \mathcal{M}(Az, z, z, t)
\]
Therefore by lemma 1.15, \( Az = z \).

\textbf{Case II:} Since \( T \) is continuous, we have \( TTx_{2n+1} \to Tz, TTX_{2n+1} \to Tz \).

Also \( (B, T) \) is semi compatible; we have \( BTx_{2n+1} \to Tz \).

Let \( x = x_{2n}, y = Tx_{2n+1} \) in condition (3) we get
\[
\mathcal{M}(Ax_{2n}, BTx_{2n+1}, BTx_{2n+1}, kt) \geq \min \left\{ \mathcal{M}(BTx_{2n+1}, TTX_{2n+1}, TTX_{2n+1}, t), \mathcal{M}(Sx_{2n}, TTX_{2n+1}, TTX_{2n+1}, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t), \mathcal{M}(Ax_{2n}, BTx_{2n+1}, BTx_{2n+1}, t) \right\}
\]
Taking limit as \( n \to \infty \) we get
\[
\mathcal{M}(z, Tz, Tz, kt) \geq \min \left\{ \mathcal{M}(Tz, Tz, Tz, t), \mathcal{M}(z, Tz, Tz, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Tz, Tz, t) \right\}
\]
\[
= \mathcal{M}(z, Tz, Tz, t)
\]
Therefore by lemma 1.15, \( Tz = z \).

Now let \( x = x_{2n}, y = z \) in condition (3) we get
\[
\mathcal{M}(Ax_{2n}, Bz, Bz, kt) \geq \min \left\{ \mathcal{M}(Bz, Tz, Tz, t), \mathcal{M}(Sx_{2n}, Tz, Tz, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t), \mathcal{M}(Ax_{2n}, Bz, Bz, t) \right\}
\]
Taking limit as \( n \to \infty \) we get
\[
\mathcal{M}(z, Bz, Bz, kt) \geq \min \left\{ \mathcal{M}(Bz, Tz, Tz, t), \mathcal{M}(z, Tz, Tz, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Bz, Bz, t) \right\}
\]
\[
= \min \left\{ \mathcal{M}(z, Bz, Bz, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t) \right\}
\]
\[
= \mathcal{M}(z, Bz, Bz, t)
\]
Therefore by lemma 1.15, \( Bz = z \).

Therefore \( Bz = z = Tz \).

Thus we have \( Az = Sz = Bz = Tz = z \).

Hence \( z \) is a common fixed point of \( A, B, S, \) and \( T \).
Uniqueness: Suppose \( z' \) \( (\neq z) \) is another common fixed point of \( A, B, S, \) and \( T. \)

Now \( \mathcal{M}(z', z'; z', k) = \mathcal{M}(Az, Bz', Bz', kt) \)
\[\geq \min \{ \mathcal{M}(Bz', Tz', Tz', t), \mathcal{M}(Sz, Tz', Tz', t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, Bz', Bz', t) \} \]
\[= \min \{ \mathcal{M}(z', z', z', t), \mathcal{M}(z, z', z', t), \mathcal{M}(z, z, t), \mathcal{M}(z, z', z', t) \} \]
\[= \mathcal{M}(z, z', z', t) \]

Therefore by lemma 1.15, \( z = z'. \)

This completes the proof.

**Remark: 2.2** Putting \( B = A \) in theorem 2.1, we get the following result.

**Corollary: 2.3** Let \( S \) and \( T \) be two continuous self mappings of a complete \( \mathcal{M} - \) fuzzy metric space \((X, \mathcal{M}, *)\). Let \( A \) be a self mapping of \( X \) satisfying

1. \( A(X) \subset T(X), A(X) \subset S(X) \).
2. \((A, S)\) and \((A, T)\) are semi compatible.
3. there exists \( k \in (0, 1) \) such that for all \( x, y \in X \) and \( t > 0 \),
\[\mathcal{M}(Ax, Ay, Ay, kt) \geq \min \{ \mathcal{M}(Ay, Ty, Ty, t), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, Ay, Ay, t) \}.\]

Then \( A, S \) and \( T \) have a unique common fixed point.

**Remark: 2.4** Putting \( B = A, T = S \) in theorem 2.1, we get the following result.

**Corollary: 2.5** Let \( S \) be continuous self mapping of a complete \( \mathcal{M} - \) fuzzy metric space \((X, \mathcal{M}, *)\). Let \( A \) be a self mapping of \( X \) satisfying

1. \( A(X) \subset S(X) \)
2. \((A, S)\) semi compatible pair of mappings
3. there exists \( k \in (0, 1) \) such that for all \( x, y \in X \) and \( t > 0 \),
\[\mathcal{M}(Ax, Ay, Ay, kt) \geq \min \{ \mathcal{M}(Ay, Sy, Sy, t), \mathcal{M}(Sx, Sy, Sy, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, Ay, Ay, t) \}.\]

Then \( A \) and \( S \) have a unique common fixed point.

**Remark: 2.6** Putting \( B = A, T = S = I \) in theorem 2.1, we get the following result.

**Corollary: 2.7** Let \( A \) be a self mapping of a complete \( \mathcal{M} - \) fuzzy metric space \((X, \mathcal{M}, *)\) satisfying
\[\mathcal{M}(Ax, Ay, Ay, kt) \geq \min \{ \mathcal{M}(Ay, y, y, t), \mathcal{M}(x, y, y, t), \mathcal{M}(Ax, x, x, t), \mathcal{M}(Ax, Ay, Ay, t) \} \]
for all \( x, y \in X, t > 0 \) and \( 0 < k < 1 \). Then \( A \) has a unique fixed point.

**Theorem: 2.8** Let \( A, B, S \) and \( T \) be self mappings of a complete \( \mathcal{M} - \) fuzzy metric space \((X, \mathcal{M}, *)\) satisfying the following conditions

1. \( A(X) \subset T(X), B(X) \subset S(X). \)
2. \((A, S)\) and \((B, T)\) are weakly compatible.
3. there exists \( k \in (0, 1) \) such that for all \( x, y \in X \) and \( t > 0 \),
\[\mathcal{M}(Ax, By, By, kt) \geq \min \{ \mathcal{M}(By, Ty, Ty, t), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, By, By, t) \}.\]

Then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof:** Let \( x_0 \in X \) be any arbitrary element.

Since \( A(X) \subset T(X), \) then there exists a point \( x_1 \in X \) such that \( Ax_0 = Tx_1. \)

Also, since \( B(X) \subset S(X) \), then there exists another point \( x_2 \in X \) such that \( Bx_1 = Sx_2. \)
Therefore by lemma l.15, Similarly, since 
\[ y_{2n+1} = Ax_{2n} = Tx_{2n+1} \] and 
\[ y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \] for \( n = 0, 1, 2, \ldots \)

Now using condition (3) we get
\[
\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+3}, k) = \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt) \\
\geq \min \{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Tx_{2n+1}, t), \\
\mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \}
\]
\[
= \min \{ \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t) \}
\]
\[
= \min \{ \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t) \}
\]
\[
= \min \{ \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t) \}
\]
\[
= \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t)
\]

Therefore \( \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+3}, k) \geq \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+1}, t) \).

Also, \( \mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+4}, k) = \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+3}, k) \\
= \mathcal{M}(Bx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt) \\
\geq \min \{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Bx_{2n+1}, Sx_{2n+2}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n+2}, Ax_{2n+2}, Bx_{2n+1}, t) \}
\]
\[
= \min \{ \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t) \}
\]
\[
= \min \{ \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t) \}
\]
\[
= \min \{ \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t) \}
\]
\[
= \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+2}, t)
\]

Therefore \( \mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+4}, k) \geq \mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+2}, t) \).

Hence \( \mathcal{M}(y_{n+1}, y_{n}, y_{n+1}, k) \geq \mathcal{M}(y_{n}, y_{n+1}, y_{n+1}, t) \), for all \( n \).

By lemma 1.14, \( \{y_n\} \) is a Cauchy sequence in \( \mathcal{M} \)-fuzzy metric space \( X \).

Since \( X \) is \( \mathcal{M} \)-fuzzy complete, sequence \( \{y_n\} \) converges to the point \( z \in X \).

Also, since \( \{Ax_n\} \), \( \{Bx_{2n+1}\} \), \( \{Sx_{2n}\} \) and \( \{Tx_{2n+1}\} \) are subsequences of \( \{y_n\} \), they also converge to the point \( z \).

Since \( B(X) \subset S(X) \), there exists a point \( u \in X \) such that \( z = Su \).

Then by condition (3) we have
\[
\mathcal{M}(Au, Bx_{2n+1}, Bx_{2n+1}, kt) \geq \min \{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Su, Tx_{2n+1}, Tx_{2n+1}, t), \\
\mathcal{M}(Au, Su, t), \mathcal{M}(Au, Bx_{2n+1}, Bx_{2n+1}, t) \}
\]

Taking limit as \( n \to \infty \) we get
\[
\mathcal{M}(Au, z, z, kt) \geq \min \{ \mathcal{M}(z, z, t), \mathcal{M}(Su, z, t), \mathcal{M}(Au, Su, t), \mathcal{M}(Au, z, z, t), \mathcal{M}(Au, z, z, t) \}
\]
\[
= \mathcal{M}(Au, z, z, t)
\]

Therefore by lemma 1.15, \( Au = z \).

Therefore \( Au = z = Su \).

Similarly, since \( A(X) \subset T(X) \), there exists a point \( v \in X \) such that \( z = Tv \).

Then by condition (3) we have
\[
\mathcal{M}(z, Bv, Bv, kt) = \mathcal{M}(Au, Bv, Bv, kt) \\
\geq \min \{ \mathcal{M}(Bv, Tv, Bv, t), \mathcal{M}(Su, Tv, Bv, t), \mathcal{M}(Au, Su, Su, t), \mathcal{M}(Au, Bv, Bv, t) \}
\]
\[
= \min \{ \mathcal{M}(Bv, z, z, t), \mathcal{M}(z, z, t), \mathcal{M}(z, z, t) \}
\]
\[
= \mathcal{M}(z, Bv, Bv, t).
\]
Therefore by lemma 1.15, $Bv = z$.

Therefore $Bv = z = T v$.

Hence $Au = z = Su = Bv = T v$.

Since the pair of mappings $(A, S)$ is weakly compatible, so $ASu = SAu$ gives $Az = Sz$.

Now we prove $z$ is a fixed point of $A$.

$$M(Az, z, z, kt) = M(Az, Bv, Bv, kt) \geq \min \{ M(Bv, Tv, Tv, t), M(Sz, Tv, Tv, t), M(Az, Sz, Sz, t), M(Az, Bv, Bv, t) \}$$

$$= \min \{ M(z, z, z, t), M(Az, Az, z, t), M(Az, Az, Az, t), M(Az, z, z, t) \}$$

$$= M(Az, z, z, t).$$

Therefore by lemma 1.15, $Az = z$.

Hence $Az = z = Sz$.

Since the pair of mappings $(B, T)$ is weakly compatible, so $BTv = T Bv$ gives $Bz = T z$.

Now we prove $z$ is a fixed point of $B$.

$$M(z, Bz, Bz, kt) = M(Az, Bz, Bz, kt) \geq \min \{ M(Bz, Tz, Tz, t), M(Sz, Tz, Tz, t), M(Az, Sz, Sz, t), M(Az, Bz, Bz, t) \}$$

$$= \min \{ M(z, Bz, Bz, t), M(z, Bz, Bz, t), M(z, z, z, t), M(z, Bz, Bz, t) \}$$

$$= M(z, Bz, Bz, t).$$

Therefore by lemma 1.15, $Bz = z$.

Hence $Bz = z = T z$.

Thus we have $Az = Bz = Sz = T z = z$.

Hence $z$ is a common fixed point of $A$, $B$, $S$ and $T$.

**Uniqueness:** Suppose $z^\prime$ $(\neq z)$ is another common fixed point of $A$, $B$, $S$, and $T$.

Now $M(z, z^\prime, z^\prime, kt) = M(Az, Bz^\prime, Bz^\prime, kt)$

$$\geq \min \{ M(Bz^\prime, Tz^\prime, Tz^\prime, t), M(Sz, Tz^\prime, Tz^\prime, t), M(Az, Sz, Sz, t), M(Az, Bz^\prime, Bz^\prime, t) \}$$

$$= \min \{ M(z, z^\prime, z^\prime, t), M(z, z^\prime, z^\prime, t), M(z, z, z, t), M(z, z^\prime, z^\prime, t) \}$$

$$= M(z, z^\prime, z^\prime, t).$$

Therefore by lemma 1.15, $z = z^\prime$.

This completes the proof.

**Remark:** Putting $B = A$ in theorem 2.8, we get the following result.

**Corollary:** Let $A$, $S$ and $T$ be self mappings of a complete $M$ – fuzzy metric space $(X, \mathcal{M}, *)$ satisfying the following conditions

1. $A(X) \subset T(X)$, $A(X) \subset S(X)$.
2. $(A, S)$ and $(A, T)$ are weakly compatible.
3. there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$M(Ax, Ay, Ay, kt) \geq \min \{ M(Ay, Ty, Ty, t), M(Sx, Ty, Ty, t), M(Ax, Sx, Sx, t), M(Ax, Ay, Ay, t) \}.$$  

Then $A$, $S$ and $T$ have a unique common fixed point.

**Remark:** Putting $B = A$, $T = S$ in theorem 2.8, we get the following result.
Corollary: 2.12 Let $A$ and $S$ be self mappings of a complete $\mathcal{M}$– fuzzy metric space $(X, \mathcal{M}, *)$ satisfying the following conditions

1. $A(X) \subseteq S(X)$.
2. $(A, S)$ weakly compatible pair of mappings.
3. There exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min \{ \mathcal{M}(Ay, Sy, Sy, t), \mathcal{M}(Sx, Sy, Sy, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, Ay, Ay, t) \}.$$ 

Then $A$ and $S$ have a unique common fixed point.

Remark: 2.13 Putting $B = A$, $T = S = I$ in theorem 2.8, we get the following result.

Corollary: 2.14 Let $A$ be a self mapping of a complete $\mathcal{M}$– fuzzy metric space $(X, \mathcal{M}, *)$ satisfying

$$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min \{ \mathcal{M}(Ay, y, y, t), \mathcal{M}(x, y, y, t), \mathcal{M}(Ax, x, x, t), \mathcal{M}(Ax, Ay, Ay, t) \}$$

for all $x, y \in X$, $t > 0$ and $0 < k < 1$. Then $A$ has a unique fixed point.

REFERENCES: