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# ON h – CLOSED SETS IN TOPOLOGICAL SPACES

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***ABSTRACT***

***I****n this paper, we introduce a new class of sets namely, h – closed sets and their properties, Applying these sets, we introduce new spaces namely* ${}_{\alpha}T_h$ , ${}_{c}T_h$ *and* $sT_h$ *spaces.*

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## 1. INTRODUCTION

N. Levine [6] introduced the class of g – closed sets in 1970. M. K. R. S. Veera Kumar introduced several generalised closed sets namely, $g^*$ - closed sets , ${}^*g$ - closed sets , $\alpha^*g$ - closed sets , ${}^*gs$ - closed sets and ω – closed sets. In this paper, we introduce h – closed sets and applying these sets , we introduce new spaces namely ${}_{\alpha}T_h$ , ${}_{c}T_h$ and $sT_h$ spaces.

## 2. PRELIMINARIES

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation axioms unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and $A^c$ denote the closure of A, the interior of A and the complement of A in X respectively.

**2.1 Definition:** *A subset A of a space ( X ,τ ) is called*

1. a semi-open set [7] if A ⊂ cl(int(A)) and a semi closed set if int(cl(A)) ⊂ A
2. a preopen set [8] if A ⊂ int(cl(A)) and a preclosed set if cl(int(A)) ⊂ A
3. a α-open set [9] if A ⊂ $int$(cl(int(A)))and a α- closed set if cl(int(cl(A))) ⊂ A

The intersection of all semi-closed (resp., preclosed, α-closed)-sets containing A of X is called the semi closure (resp., preclosure , α-closure) of A and is denoted by scl(A) (resp, pcl(A) , αcl(A)) . The union of all semi - open sets contained in A is called semi interior of A and it is denoted by sint (A).

**2.2. Definition:**

A subset A of a space ( X ,τ ) is called a

**1.** a generalized closed (briefly g-closed) set [6] if cl(A)⊂ $U$ whenever A⊂ Uand U is open in (X, τ ) ; the compliment of a g- closed set is called a g-open set.
**2.** a generalized semi-closed (briefly gs-closed) set [3] if scl(A) ⊂ U whenever A⊂ U and U is open in( X ,τ) .
**3.** a generalized semi pre-closed (briefly gsp-closed) set [ 5 ] if spcl(A) ⊂ U whenever A⊂ Uand U is open in( X ,τ) .
**4.** a semi- generalized closed (briefly sg-closed) set [4 ] if scl(A) ⊂ U whenever A⊂ Uand U is semi-open in( X ,τ); the compliment of a sg- closed set is called a sg-open set.
**5.** a ψ - closed set [ 10 ] if scl(A) ⊂ U whenever A⊂ U and U is sg -open set in (X, τ);
**6.** a $g^*$-closed set [12 ] if cl(A) ⊂ U whenever A⊂ U and U is g -open set in (X, τ).

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**7.** a ω - closed set [11 ] if cl(A) ⊂ U whenever A⊂ U and U is semi -open set in (X, τ); the compliment of a ω - closed set is called a ω -open set.

**8.** a $^*$g -closed set [12] if cl(A) ⊂ U whenever A⊂ U and U is ω -open set in (X, τ).

**9.** a *g –semi closed ($^*$g s –closed) set [12] if scl(A) ⊂ U whenever A⊂ U and U is g -open setIn (X, τ).

**10.** a α $^*$g -closed set [12] if αcl(A) ⊂ U whenever A⊂ U and U is ω -open set in (X,τ).

**2.3 Definition:**
1. a $T_{1/2}$ space [4 ] if every g- closed set is closed.
2 a $T^*_{1/2}$ space [13] if every $g^*$- closed set is closed.
3. a$^*$ $T_{1/2}$ space [13 ] if every g-closed set is $g^*$- closed.

## 3. BASIC PROPERTIES OF h – CLOSED SETS

**3.1 Definition:** A subset A of (X, τ) is called a h- closed set if scl (A) $\subset U$ whenever A $\subset U$ and U is ω - open in X. The class of all h- closed subsets of (X, τ) is denoted by hC (X, τ).

**3.2 Proposition:** Every closed set (resp. α – closed set, semi – closed set) is h-closed. But the converses are not true as can be seen from the following examples.

**Proof:** Follows from the definition.

**3.3 Example:** Let X={a, b, c} and τ=$\{\varphi, X, \{a\}\}$. hC (X ,τ ) = { φ, X, {b),{c},{a, b}{b, c}{c, a}}. The set {a, b} is h-closed but not closed (resp. not α – closed set, not semi – closed)

**3.4 Proposition:** Let X be a topological space and A$\subset X$ . Then the following are true.
1. If A is g$^*$ - closed, then A is h- closed set.
2. If A is * g - closed, then A is h- closed set.
3. If A is α $^*$ g - closed, then A is h- closed set.
4. If A is * g s - closed, then A is h- closed set.
5. If A is ψ - closed, then A is h- closed set.

**Proof:** (1) Since every ω – open set is g – open and scl (A) ⊂ cl (A) the proof follows.

(2), (3), and (4): Since scl (A) ⊂ αcl (A) ⊂ cl (A), the proof is clear.

(5): Since every ω – open set is sg – open, the proof is clear.

**3.5 Remarks:** The converse of Proposition 3.4 is not true. For,

**3.6 Example** Let X= {a, b, c, d} and τ= $\{\varphi, X, \{a\}\}$. The set {a} is h closed but it is not a g$^*$- closed set.

**3.7 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}\ \{b\}\ \{a, b\}\}$. The set {a, b} is h - closed but it is not $^*$ g - closed also it is not α$^*$ g – closed.

**3.8 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}\ \}$. The set { a, b} is a h - closed but it is not $^*$ g s- closed set. Also it is not ψ - closed.

**3.9 Proposition:** Let X be a topological space and A $\subset X$ . Then the following are true.
1. If A is h - closed, then A is gs - closed set.
2. If A is h - closed, then A is gsp - closed set.

**Proof:** (1) Since every open set is ω – open, the proof is clear.
(2): Since every ω – open set is open and spcl (A) ⊂ scl (A) every h - closed set is gs – closed.

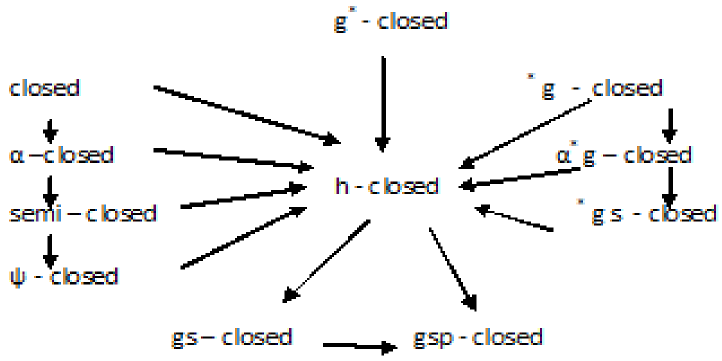
**3.10 Remarks:** The converse of Proposition 3.4 is not true. For,

**3.11 Example:** Let X= {a, b, c} and τ= $\{\varphi, X, \{a\}, \{b, c\}\}$.The set {a, b} is gs- closed but not h-closed.

**3.12 Example**: Let X= {a, b, c} and τ= $\{\varphi, X, \{a, b\}\}$. the set {a} is gsp- closed but not h-closed.

The following diagram shows the relationships considerd in this paper. (A $\longrightarrow$ B reprents A implies B but B need not imply A.)




**3.13 DEFINITION:** Let (X, τ) be a topological space and A ⊂ X. The set of all limit points of A is said to be the Semi – derived set of A and is denoted by $D_s$ (A).

**3.14 Theorem:** Let A and B be two h – closed sets such that D (A) ⊂ $D_s$ (A) and D (B) ⊂ $D_s$ (B).Then A ∪ B is h – closed.

**Proof:** Let U be a ω – open set such that A ∪ B ⊂ U . A ⊂ U and U being ω – open scl (A ) ⊂ U .B ⊂ U and U being ω – open scl (B ) ⊂ U. Therefore scl (A) ∪ scl (B) ⊂ U. Given D (A) ⊂ $D_s$ (A) but always we have $D_s$ (A) ⊂ D(A) .Therefore D (A)= $D_s$ (A). Hence cl(A) = scl(A) and cl(B) = scl(B). scl( AUB) ⊂ $U$ .Therefore AUB is h – closed.

**3.15 Result:** Intersection of two h – closed sets need not be a h – closed set.

**3.16 Example:** X = {a, b, c} τ= $\{\varphi, X, \{a\}\}$ . {a, b} and {c, a} are h – closed sets but {a, b} ∩ {c, a} = {a} is not a h – closed set.

**3 .17 Proposition:** Let A and B be any two subsets of a space X.
(1). If A is h – closed then scl (A) – A contains no non empty ω - closed set.
(2). If A is h – closed and A ⊆ B ⊆ scl (A) then B is h – closed.

**Proof:** (1) Let F be a ω - closed subset of scl (A) – A. Then F ⊂ scl (A) that is $(\text{scl }(A))^c$ ⊂ $F^c$ and $F^c$ ⊃ A. Since F is ω – closed $F^c$ is ω – open and A ⊂ $F^c$. Hence (scl (A)) ⊂ $F^c$. Therefore F ⊂ scl (A)⊂ X – F = $F^c$. This is possible only if F = φ.

(2): Let U be a ω – open set such that B ⊆ U. Then A ⊆ U given A is h – closed. Therefore scl (A) ⊆ U. Now scl (B) ⊆ scl(scl(A) ) = scl (A) ⊆ U. Hence B is h – closed.

**3.18 DEFINITION:** A subset A of a topological space (X, τ) is said to be h – open if $A^c$ is h – closed. The class of all h – open subsets of X is denoted by hO (τ).

**3.19 Theorem:** A subset A of a topological space (X, τ) is h – open if and only if F ⊆ sint (A) whenever A ⊃ F and F is ω – closed in X.

**Proof**: Suppose that is A h – open in X and A ⊃ F, where F is ω – closed in X. $A^c \subseteq F^c$ and $F^c$ is ω – open in X . Hence $scl(A^c) \subseteq F^c$ implies $(\text{sint }(A))^c \subseteq F^c$.Thus we have sint (A) ⊃ F that is F ⊆ sint (A).

**Converse**

Suppose $A^c \subseteq$ U and U is ω – open. A ⊃ $U^c$ and $U^c$ is ω – closed. Hence by the given hypothesis sint (A) ⊃ $U^c$. Therefore (sint (A) )$^c$ ⊆ U .Hence $scl(A^c) \subseteq$ U .

**3.20 Theorem:** In a topological space (X, τ) for each x ∈ X either {x} is ω – closed or h – open in X.

**Proof:** Suppose {x} is not ω – closed in X then $\{x\}^c$ is ω – closed and the only ω – open set containing $\{x\}^c$ is the space X itself. Therefore scl ($\{x\}^c$) ⊆ X and so $\{x\}^c$is h – closed– open.

**4. APPLICATION OF h – CLOSED SETS**

**4.1 Definition:** A space (X, τ) is called $_\alpha T_h$ space if every h- closed set is a α closed set.

**4.2 Example**: Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{a, b\}, \{a, c\}\}$. hC (X, τ) = $\{\varphi, X, \{b\}, \{c\}, \{b, c\}\}$ = αC (X, τ).Thus (X, τ) is a $_\alpha T_h$ space. The space in the following example is not a $_\alpha T_h$ space.

4.3 **Example**: Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$. hC (X, τ) = $\{\varphi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{c, a\}\}$. αC (X, τ) = $\{\varphi, X, \{c\}, \{b, c\}, \{a, c\}\}$.

Thus (X, τ) is not a $_\alpha T_h$ space.

**4.4 Theorem:** If (X, τ) is a $_\alpha T_h$ space, then every singleton of X is either ω – closed or α – open.

**Proof**: Let x ∈ X and suppose that {x} is not ω – closed. Then X − {x} is not ω – open. This implies that X is the only ω - open set containing X − {x}.So X − {x} is a h – closed set of (X, τ). Since (X, τ) is an $_\alpha T_h$ space, X − {x} is α – closed or equivalently {x} is α – open.

The converse of the above theorem is not true as it can be seen by the following example.

**4.5 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$. {a} and {b} are α – open sets of (X ,τ) and {c} is a ω – closed set of (X, τ). But the space (X, τ) is not an $_\alpha T_h$ space .

**4.6 Definition:** A space (X, τ) is called $_c T_h$ space if every h- closed set is a closed set.

**4.7 Theorem:** If (X, τ) is a $_c T_h$ space, then every singleton of X is either ω – closed or open.

**Proof:** Proof is similar to theorem 4.4

The converses of the above theorem is not true as it can be seen by the following examples.

**4.8 Example**: Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$ {a} and {b} are open sets of (X, τ) and {c} is a ω – closed set of (X, τ). But the space (X, τ) is not a $_c T_h$ space.

**4.9 Theorem:** Every $_c T_h$ space is an $_\alpha T_h$ space .

**Proof:** The Proof follows since every closed set is α – closed.

**4.10 Theorem:** $_c T_h$ is independent of $T_{1/2}$ ness and $^*T_{1/2}$ ness. It is shown by the following examples.

**4.11 Example**: Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b, c\}\}$.

Then (X, τ) is a $_c T_h$ space but not a $T_{1/2}$ space and a $^*T_{1/2}$ space.

**4.12 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$.
Then (X, τ) is a $T_{1/2}$ space and a $^*T_{1/2}$ space but not a $_c T_h$ space.

**4.13 Theorem:** Every ${}_cT_h$ space is a $T^*_{1/2}$ space.

**Proof:** Let (X, τ) be a ${}_cT_h$ space. Let A be a $g^*$ - closed set. Every $g^*$ - closed set is h – closed.

Hence A is closed. Therefore (X, τ) is a $T^*_{1/2}$ space.

The converse of the above theorem is not true as it can be seen by the following example.

**4.14 Example**: Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$. (X, τ ) is a $T^*_{1/2}$ space but not a ${}_cT_h$ space.

**4.15 Definition:** A space (X, τ) is called a $T_b$ space if every gs - closed set is a closed set.

**4.16 Theorem:** Every $T_b$ space is a $sT_h$ space.

**Proof:** Since every h – closed set is gs – closed set, the proof is clear.

The converse of the above theorem is not true as it can be seen by the following examples.

**4.17 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b, c\}\}$. hC (X, τ) = $\{\varphi, X, \{a\}, \{b, c\} =$ C (X, τ). gsC (X, τ) =$\{\varphi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$. The set {b} is gs – closed but not closed.

**4.18 Definition**: A space (X, τ) is called $sT_h$ space if every h- closed set is a semi closed set.

**4.19 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is a $sT_h$ space.

**4.20 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}\}$. (X, τ) is not a $sT_h$ space.

**4.21 Theorem:** If (X, τ) is a $sT_h$ space then every singleton of X is either ω – closed or semi - open.

**Proof:** Proof is similar to theorem 4.4.

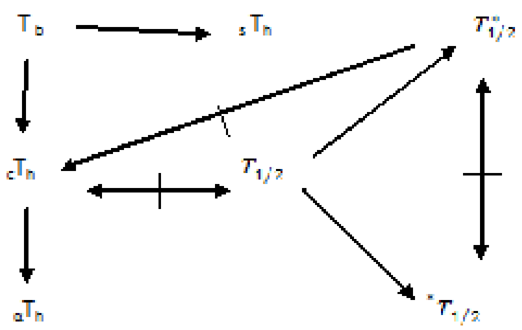
**4.22 Theorem**: Every $T_b$ *space is a* ${}_sT_h$ space.

**Proof**: Since every h – closed set is gs – closed set, the proof is clear.

The converse of the above theorem is not true as it can be seen by the following example.

**4.23 Example:** Let X = {a, b, c} and τ = $\{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$. hC (X, τ) = $\{\varphi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{c, a\}\}$= sC (X, τ) = gsC (X, τ). The set {a} is gs – closed but not closed.

**4.27 Remark:** The following diagram shows some relationships considerd in this paper. (A $\nleftrightarrow$ B reprents A implies B but B need not imply A. A $\longrightarrow$ B means A and B are independent).



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