New classes of sets called $\beta^*$-closed sets are introduced and studied some of their properties.

Keywords: $^*$g-closed, $^*$g-open, $\tilde{g}$-open, $p$-closed, $^*$-$\eta^*$-closed, $\beta^*$-closed

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1. INTRODUCTION

The study of generalized closed sets in topological space was initiated by Levin [4]. In 1986 Andrijevic [2] defined semi pre open sets and it is also known under the name $\beta$-closed sets. In 1996, Julian Dontchev [3] introduced the notion of generalized semi-pre closed (briefly gsp-closed sets) via the concept of semi pre open sets. Generalised closed sets namely g-closed sets, gs closed sets, r-g closed sets, s-g closed sets, $\alpha$-closed sets, $\alpha$-g closed sets were introduced and studied by various authors. The class of gsp- closed sets contains properly the classes of all the above mentioned generalised closed sets except r-g closed sets. The class of $\omega$-closed set was introduced by M. Shiek John[11] in 2002. In this paper we introduce a new classes of sets called $\beta^*$-Closed sets. This class lies between the class of open and semi pre closed sets and the class of $^*$-$\eta^*$-closed sets [9].

2. PRELIMINARIES

Throught this paper $(X, \tau)$, $(Y, \sigma)$ and $(Z, \eta)$ will always denote topological spaces, on which no separation axioms are assumed unless otherwise mentioned. When $A$ is a subset of $(X, \tau)$, $\text{Cl}(A)$, $\text{Int}(A)$ and $D[A]$ denote the closure, the interior and the derived set of $A$, respectively.

We recall some known definitions needed.

**Definitions 2.1:** Let $(X, \tau)$ be topological space. A subset $A$ of $X$ is said to be
1. Preopen [7] if $A \subseteq \text{Int}(\text{cl}(A))$ and preclosed if $\text{cl}(\text{Int}(A)) \subseteq A$.
2. Semi open[6] if $A \subseteq \text{Cl}(\text{Int}(A))$ and semi closed if $\text{Int}(\text{Cl}(A)) \subseteq A$.
3. Semi pre open[1] if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$ and semi pre closed if $\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq A$.

**Definition 2.2:** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be
1. generalised closed (briefly g-closed ) [5] if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
2. generalized pre closed (briefly gp-closed) [8] if $\text{Pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
3. generalized semi pre closed (briefly gsp closed ) [3] if $\text{Spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
4. $\omega$-closed if $[11]$ if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi open in $X$.
5. $^*$g-closed if $[12]$ if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega$-open in $X$.
6. $^*$gs-closed [13] if $\text{Sc}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^*$g-open in $X$.
7. $\tilde{g}$-closed [4] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tilde{g}$-open.
8. $p$-closed [10] if $\text{Pcl}(A) \subseteq \text{Int}(U)$ whenever $A \subseteq U$ and $U$ is $p$-open in $X$.
9. $^*$-$\eta^*$ -closed [9] if $\text{Spcl}(A) \subseteq \text{Int}(\text{Cl}(U))$ whenever $A \subseteq U$ and $U$ is $\omega$-open in $X$. 

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The compliments of above mentioned sets are called their respective closed sets.

**Basic Properties Of β*Closed Sets**

We introduce the following **Definition**

**Definition 3.1**: A subset A of a space \((X, \tau)\) is said to be \(\beta^*\)-closed in \((X, \tau)\) if \(\text{spcl}(A) \subseteq \text{Int}(U)\) whenever \(A \subseteq U\) and \(U\) is \(\omega\)-open in \((X, \tau)\).

**Theorem 3.2**: Every open and semi preclosed subset of \((X, \tau)\) is \(\beta^*\)-closed but not conversely.

**Proof**: Let \(A\) be an open and semi preclosed subset of \((X, \tau)\)

Let \(A \subseteq U\) and \(U\) be \(\omega\)-open in \(X\)

Then \(\text{spcl}(A) = A = \text{Int}(A) \subseteq \text{Int}(U)\)

Hence \(A\) is \(\beta^*\)-closed

The converse of the above **Theorem** need not be true as seen from the following example.

**Example 3.3**: Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a\}, X\}\)

Then the set \(A = \{a, c\}\) is \(\beta^*\) closed but neither open nor preclosed

**Theorem 3.4**: Every \(\beta^*\)-closed set is gsp – closed but not conversely.

**Proof**: Let \(A\) be any \(\beta^*\)-closed set in \(X\)

Let \(A \subseteq U\) and \(U\) be open in \(X\)

Since every open set is \(\omega\)-open and \(A\) is \(\beta^*\)-closed, \(\text{Spcl}(A) \subseteq \text{Int}(U) = U\)

Hence \(A\) is gsp – closed

Converse of the above **Theorem** need not be true as seen from the following example.

**Example 3.5**: Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b, c\}, X\}\)

Then the set \(A = \{a, b\}\) is gsp closed but not \(\beta^*\)-closed in \(X\)

**Theorem 3.7**: Every open and preclosed subset of \((X, \tau)\) is \(\beta^*\)-closed

**Proof**: Let \(A\) be an open and preclosed subset of \((X, \tau)\)

Let \(A \subseteq U\) and \(U\) be \(\omega\)-open in \(X\)

Then \(\text{spcl}(A) \subseteq \text{pcl}(A) = A = \text{Int}(A) \subseteq \text{Int}(U)\)

Hence \(A\) is \(\beta^*\)-closed

The converse of the above **Theorem** need not be true. It is seen from the following example.

**Example 3.8**: Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a\}, X\}\)

Then the set \(\{a, b\}\) is \(\beta^*\)-closed but neither open are preclosed.

**Remark 3.9**: \(\beta^*\)-closedness and preclosedness are independent. It is shown by the following examples.

**Example 3.10**: Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\)

Then the set \(A = \{a\}\) is \(\beta^*\)-closed but not preclosed.
Example 3.11: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}\}$

Then the set $A = \{a, b\}$ is preclosed but not $\beta^*$-closed

Remark 3.12: $\beta^*$-closedness and $\alpha$-closedness are independent. It is shown by the following examples.

Example 3.13: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$

Then the set $A = \{a\}$ is $\alpha$-closed but not $\beta^*$-closed

Example 3.14: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, X\}$

Then the set $\{a, b\}$ is $\beta^*$-closed but not $\alpha$-closed.

Remark 3.15: $\beta^*$-closed sets are independent of semi closed sets and semi preclosed sets. It is shown by the following examples.

Example 3.16: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$

Then the set $A = \{a, b\}$ is both semi preclosed and semi closed but not $\beta^*$-closed.

Example 3.17: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$

Then the set $A = \{a, b\}$ is $\beta^*$-closed but neither semi preclosed nor semi closed.

Remark 3.18: $\beta^*$-closedness and pre semi closedness are independent. It is shown by the following examples.

Example 3.19: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$

Then the set $A = \{a\}$ is pre semi closed but not $\beta^*$-closed.

Example 3.20: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, X\}$

Then the set $A = \{a, b\}$ is $\beta^*$-closed but not pre semi closed.

Remark 3.21: $\beta^*$-closedness and $g$ closedness are independent. It is shown by the following examples.

Example 3.22: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$

Then the set $A = \{a, c\}$ is $g$ closed but not $\beta^*$-closed.

Example 3.23: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$

Then the set $A = \{b\}$ is $\beta^*$-closed but not $g$ closed.

Theorem 3.24: $\beta^*$-closed set is $\eta^\ast$ closed set but not conversely.

Proof: Let $A$ be any $\beta^*$-closed set in $X$.

Let $A \subseteq U$ and $U$ be $\omega$-open in $X$.

Then $spcl(A) \subseteq Int(U) \subseteq U$

Hence $A$ is $\eta^\ast$ closed

Converse of the above Theorem need not be true. It is seen from the following example

Example 3.25: Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e\}, X\}$

Then the set $A = \{a\}$ is $\eta^\ast$ closed but not $\beta^*$-closed

Example 3.26: $\beta^*$-closedness and $\rho$ closedness are independent. It is shown by the following examples.
Example 3.27: Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e\}, X\}$

Then the set $A = \{c\}$ is $\beta^*$ closed but not $\rho$ - closed

Then the set $B = \{b, d\}$ is $\rho$ closed but not $\beta^*$ - closed.

Definition 3.28: A subset $A$ of a space $(X, \tau)$ is said to be $\beta^*$- closed in $(X, \tau)$ if $\text{spcl} \ (A) \subseteq \text{Int} \ (\text{cl}(U))$ whenever $A \subseteq U$ and $U$ is $\omega$ -open in $(X, \tau)$.

Theorem 3.29: Every $\beta^*$- closed set is $\beta s^*$- closed but not conversely.

Proof: Let $A$ be any $\beta^*$- closed set.

Let $A \subseteq U$ and $U$ be $\omega$ -open

$A$ is $\beta^*$- closed, $\text{spcl} \ (A) \subseteq \text{Int} \ (U) \subseteq \text{Int} \ (\text{cl}(U))$

Hence $A$ is $\beta s^*$- closed.

The converse of the above Theorem need not be true. It is seen from the following example.

Example 3.30: Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a, b\}, \{a, b, d\},\{a, b, c, d\},\{a, b, d, e\},X\}$

Then the set $A = \{b, d\}$ is $\beta s^*$- closed but not $\beta^*$ - closed

Remark 3.31: From the above discussion and known results we have the following implications. $A \rightarrow B$ represents $A$ implies $B$ but not conversely and $A \leftrightarrow B$ represents $A$ and $B$ are independent of each other.

Closed $\rightarrow \alpha closed \rightarrow$ semi closed $\rightarrow$ semi pre closed $\rightarrow$ pre semi closed $\rightarrow$ $\eta^*$ closed

Open and $\rightarrow$ open and $\rightarrow$ open and $\rightarrow$ $\beta^*$- closed $\rightarrow$ gsp closed

Closed $\rightarrow$ $\beta^*s$- closed $\rightarrow$ g closed $\rightarrow$ $\rho$ closed
Properties Of β*-Closed Sets

Remark 3.32: The union and intersection of two β*-closed sets need not be β*-closed. It is shown in the following examples.

Example 3.33: Let X = \{a, b, c, d\} and \(\tau = \{\emptyset, \{a\}, X\}\)

Then set A = \{a, b\} and B = \{a, c\} are β*-closed but \(A \cap B = \{a\}\) is not β*-closed

2. Let X = \{a, b, c\} and \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\)

Then set A = \{a\} and B = \{b\} are β*-closed but \(A \cup B = \{a, b\}\) is not β*-closed

Theorem 3.34: A set A is β*-closed then \(\text{spcl}(A) - A\) contains no nonempty closed set.

Proof: Suppose \(H \subseteq \text{spcl}(A) - A\) is a nonempty closed set

Then \(H \subseteq \text{spcl}(A)\) and \(A \subseteq X-H\). Since \(X-H\) is \(\omega\)-open and A is β*-closed, we have \(\text{spcl}(A) \subseteq \text{Int}(X-H) = X - \text{cl}(A)\).

Hence \(\text{cl}(H) \subseteq X - \text{spcl}(A)\), which is a contradiction.

Hence \(\text{spcl}(A) - (A)\) contains no nonempty closed set. Converse of the above

Theorem need not be true as seen from the following example.

Example 3.35: Let X = \{a, b, c\} and \(\tau = \{\emptyset, \{a\}, \{b, c\}, X\}\)

Let A = \{a, c\} Then \(\text{spcl}(A) - (A) = \{a, c\} - \{a, c\} = \emptyset\) contains no nonempty closed set but A is not β*-closed

Theorem 3.36: A set A is β*-closed then \(\text{spcl}(A) - (A)\) contains no nonempty \(\omega\)-closed sets.

Proof: Let F be a nonempty \(\omega\)-closed set of \(F \subseteq \text{spcl}(A) - (A)\). Then \(F \subseteq \text{spcl}(A)\) and \(A \subseteq X-F\). Since A is β*-closed and \(X-F\) is \(\omega\)-open. We have \(\text{spcl}(A) \subseteq \text{Int}(X-F) = X - \text{cl}(F)\).

Hence \(\text{cl}(F) \subseteq X - \text{spcl}(A)\) and so \(F \subseteq X - \text{spcl}(A)\).

Already \(F \subseteq \text{spcl}(A)\) hence we get a contradiction. Hence \(\text{spcl}(A) - (A)\) contains no nonempty \(\omega\)-closed set.

Theorem 3.37: If A is β*-closed and \(A \subseteq B \subseteq \text{spcl}(A)\) then B is β*-closed

Proof: Let U be a \(\omega\)-open set of X such that \(B \subseteq U\). Since A is open and gsp closed by the lemma 3.39; A is semi pre closed. Hence \(\text{spcl}(A) \subseteq \text{Int}(U)\).

Theorem 3.38: If a subset A of (X, \(\tau\)) is \(\omega\)-open and β*-closed then A is semipreclosed in (X, \(\tau\)).

Proof: If A is \(\omega\)-open and β*-closed, since A \(\subseteq A\), we have \(\text{spcl}(A) \subseteq \text{Int}(A) \subseteq A\) but A \(\subseteq \text{spcl}(A)\). Hence A = \(\text{spcl}(A)\). So A is semipreclosed.

Lemma 3.39: If A is open and gsp closed. Then A is semi pre closed.

Proof: Let A \(\subseteq U\) and U be open in (X, \(\tau\)), since A is open and A \(\subseteq A\), we have \(\text{spcl}(A) \subseteq A \subseteq U\), hence \(\text{spcl}(A) \subseteq U\) and so A is semi pre closed.

Theorem 3.40: A open set of (X, \(\tau\)) is gsp closed if and only if A is β*-closed.

Proof: Let A be a open set of (X, \(\tau\)) and A is gsp closed.

Let A \(\subseteq U\) and U be \(\omega\)-open in (X, \(\tau\)), since A is open and gsp closed by the lemma 3.39; A is semi pre closed. Hence \(\text{spcl}(A) = A = \text{Int}(A) \subseteq \text{Int}(U)\), therefore \(\text{spcl}(A) \subseteq \text{Int}(U)\) and so A is β*-closed. Conversely let A be a β*-closed set. Then by Theorem 3.4, A is gsp closed.

Theorem 3.41: Let A be β*-closed in (X, \(\tau\)) Then A is semi pre closed if and only if \(\text{spcl}(A) - (A)\) is \(\omega\)-closed.
Proof: Let $A$ be semi pre closed. Then $spcl(A) = (A)$ and so $spcl(A) - (A) = \emptyset$ which is $\omega$ closed. Conversely Let $spcl(A) - (A)$ is $\omega$ closed, Since $A$ is $\beta^*$- closed by Theorem 3.36 $spcl(A) - A$ contains no non empty $\omega$ – closed set.

Hence $spcl(A) - (A) = \emptyset$ which implies $spcl(A) = (A)$ and so $A$ is semi pre closed

Definition 3.42: Let $(X, \tau)$ be a topological space and $A \subseteq X$ and $x \in X$, Then $x$ is said to be a semi pre limit point of $A$ if every semi pre open set containing $x$ contains a point of $A$ different from $x$.

Definition 3.43: Let $(X,\tau)$ be a topological space and $A \subseteq X$, the set of all semi pre limit point of $A$ is said to be semi pre derived set of $A$ and is denoted by $D_{sp}[A]

Theorem 3.44: If $D[A] \subseteq D_{sp}[A]$ for each subset $A$ of a space $(X, \tau)$, Then the union of two $\beta^*$- closed set is $\beta^*$-closed.

Proof: Let $A$ and $B$ be $\beta^*$- closed subsets of $X$ and $U$ be $\omega$ – open set with $A \cup B \subseteq U$. Then $spcl(A) \subseteq Int(U)$ and $spcl(B) \subseteq Int(U)$. Since for each subset $A$ of $X$, we have $D_{sp}[A] \subseteq D[A]$, we get $cl(A) = spcl(A)$ and $cl(B) = spcl(B)$, therefore $cl(A \cup B) = cl(A) \cup cl(B) = spcl(A) \cup spcl(B) \subseteq Int(U)$, but $spcl(A \cup B) \subseteq cl(A \cup B) \subseteq Int(U)$, hence $A \cup B$ is $\beta^*$- closed

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