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GENERALIZED FOURIER-FINITE MELLIN TRANSFORM

V. D. Sharma[1*] & A. N. Rangari[2]

[1]*HOD, Mathematics Department, Arts, Commerce and Science College, Amravati, 444606 (M.S.), India*
E-mail: vdsharma@hotmail.co.in
[2]*Mathematics Department Adarsh college, Dhamangaon Rly., 444709 (M.S.), India*

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ABSTRACT

The Fourier-Mellin transform is a useful mathematical tool for image recognition because its resulting spectrum is invariant in rotation, translation and scale. This paper discusses an extension of Fourier finite Mellin transform in the distributional Generalized sense. Using Gelfand-Shilov technique the testing function space $FM_{f,b,c,\propto}$ is defined. Generalized Fourier- finite Mellin transform is a Frechet space is proved and some topological proportions are obtained.

Keywords: *Fourier-Mellin transforms, Generalized function, Fourier-finite Mellin transform, signal processing, watermarking.*

1. INTRODUCTION

Integral transforms provide a powerful technique for solving initial and boundary value problems arising in applied mathematics, mathematical physics and engineering. There are many integral transforms including Fourier, Laplace, Mellin, Hankel, Hilbert, Stieltjes etc. Still new integral transforms are being introduced in mathematical literature [1].

One of great importance in many applications is the Fourier transform, where the kernel takes the form of a complex exponential function. The Fourier transform are widely used for solving differential and integral equations. In physics and engineering it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices and mechanical system [2].

Time and frequency represent two fundamental physical variables of signal analysis and processing. The Fourier transform (FT), which provides a mapping between the time domain and frequently domain representation of a signal, has been used extensively in signal processing applications [3].

It is itself translation invariant and its conversion to log polar co-ordinates converts the scale and rotation differences to vertical and horizontal offsets that can be measured. A second FFT called the Mellin transform (MT) gives a transform-space image that is invariant to translation, rotation and scale.

The application of the Fourier-Mellin transform has been studies in digital signal and image processing, pattern recognition, speech processing, and ship target recognition by sonar system and radar signal analysis [4]. Also Fourier-Mellin transform is used in detecting watermarks in images regardless of scale or rotation [5].

In the present paper, Fourier-finite Mellin transform is extended in the distributional generalized sense. In section 2 we extended Fourier-finite Mellin transform to S-type space using Gelfand-Shilov technique. Section 3 defines Distributional generalized Fourier-finite Mellin transform. $FM_{f,b,c,\propto}$ is a Frachet space is proved in section 4. Topological property is proved in section 5. Section 6 concludes the conclusion.

2. FOURIER FINITE MELLIN TRANSFORM ($FM_F T$)

The Fourier finite Mellin transform is defined as

$$FM_f\{f(t,x)\} = F(s,p) = \int_0^\infty \int_0^a f(t,x)\, K(t,x) dt\, dx,$$

where $K(t,x) = e^{-ist}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)$

***Corresponding author: V. D. Sharma[1*], *E-mail: vdsharma@hotmail.co.in**

2.1. Testing function space: $FM_{f,b,c,\propto}$

The space $FM_{f,b,c,\propto}, \propto \geq 0$ consists of all infinitely differentiable functions $\emptyset(x,t)$, where $0 < t < \infty, 0 < x < a$, satisfying the inequality

$$\gamma_{l,q}\, \emptyset(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \emptyset(t,x) \right| < \infty, \quad \text{for each} \quad l,q = 0,1,2,\ldots\ldots\ldots$$

where, $\lambda_{b,c}(x) = \begin{cases} x^{+b}, & 0<x<1 \\ x^{+c}, & 1<x<a \end{cases}$

2.2. Lemma:

The function $e^{-ist}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)$ is a member of $FM_{f,b,c,\propto}$ if $-b < Rep \leq b$, for any real number *c* and *s>0*.

Proof: Let $\emptyset(t,x) = e^{-ist}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)$

Consider

$$\gamma_{l,q}\, \emptyset(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \emptyset(t,x) \right|$$

$$= \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, e^{-ist}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right) \right|$$

$$= \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k\, \lambda_{b,c}(x)\, x^{q+1}\, (-is)^l\, e^{-ist}\, [P(-p-q)x^{-p-q-1}a^{2p} - P(p-q)\, x^{p-q-1}] \right|,$$

where *P* is a polynomial in *p* and *q*.

$$= \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k\, (-is)^l\, e^{-ist}\, [P(-p-q)x^{b-p} - P(p-q)\, x^{b+p}] \right| < \infty,$$

If as $x \to 0$,

$b - p > 0$ and $b + p > 0$

i.e. $b > Rep$ $\qquad p > -b$

i.e. $Rep < b$ $\qquad -b < Rep$

i.e. if $-b < Rep \leq b$ and for any real number *c* and also $s > 0$.

Thus $\emptyset(t,x) \in FM_{f,b,c,\propto}$ if $-b < Rep \leq b$, for any real number *c* and $s > 0$.

3. Distributional Generalized Fourier finite Mellin transform

For $f(t,x) \in FM^*_{f,b,c,\propto}$, where $FM^*_{f,b,c,\propto}$ is the dual space of $FM_{f,b,c,\propto}$ and $-b < Rep \leq b$, real number $c, s > 0$, the distributional Fourier finite Mellin transform is defined as

$$FM_t\{f(t,x)\} = F(s,p) = \left\langle f(t,x), e^{-ist}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\right\rangle, \tag{3.1}$$

where for each final x $(0 < x < a, 0 < t < \infty)$, the right hand side of (3.1) has a sense as an application of

$f \in FM^*_{f,b,c,\propto}$ to $e^{-ist}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right) \in FM_{f,b,c,\propto}$

4. Theorem: $FM_{f,b,c,\propto}$; $T_{b,c,\alpha}$ is a Frechet space

Proof: As the family $D_{b,c,\alpha}$ of semi-nor $\{\gamma_{b,c,k,q,l}\}_{k,q,l=0}^{\infty}$, generating $T_{b,c,\alpha}$ is countable, it suffices to prove the completeness of the space $FM_{f,b,c,\propto}$; $T_{b,c,\alpha}$.

Let us consider a Cauchy sequence $\{\varphi_n\}$ in $FM_{f,b,c,\propto}$. Hence for a given $\in > 0$, there exist an $N = N_{k,q,l}$ such that for $m, n \geq 0$.

$$\gamma_{b,c,k,q,l}(\varphi_m - \varphi_n) = \sup_{\substack{0<x<a \\ 0<t<\infty}} \left| t^k \lambda_{b,c}(x)\, x^{q+1} D_t^l D_x^q (\varphi_m - \varphi_n) \right| < \varepsilon \tag{4.1}$$

In particular for $k = q = l = 0$, for $m, n \geq n$

$$\sup_{\substack{0<x<a \\ 0<t<\infty}} \left| \lambda_{b,c}(x)\, x \,\{\varphi_m(t,x) - \varphi_n(t,x)\} \right| < \in \tag{4.2}$$

consequently, for fixed (t,x) over $0 < x < a, 0 < t < \infty$, $\{\varphi_m(t,x)\}$ is a numerical Cauchy sequence. Let $\varphi(t,x)$ be the point wise limit of $\{\varphi_m(t,x)\}$.

Using (4.2), we can easily deduce that $\{\varphi_m\}$ converges to φ uniformly on$0 < x < a, 0 < t < \infty$. Thus φ is continuous.

Moreover, repeated use of (4.1) for different value of k, q, l yields that φ is smooth i.e. $\varphi \in E$, where E is the class of infinitely differentiable function defined over $0 < x < a, 0 < t < \infty$. Further from (4.1) we get,

$$\gamma_{b,c,k,q,l}(\varphi_m) \leq \gamma_{b,c,k,q,l}(\varphi_N) + \varepsilon \quad \forall \quad m \geq N$$

$$\leq C_{l,q} A^k k^{k\alpha} + \varepsilon$$

Taking $m \to \infty$ and ε is arbitrary we get,

$$\gamma_{b,c,k,q,l}(\varphi) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k \lambda_{b,c}(x)\, x^{q+1} D_t^l D_x^q \varphi(t,x) \right|$$

$$\leq C_{l,q} A^k k^{k\alpha}$$

Hence $\varphi \in FM_{f,b,c,\propto}$ and it is the $T_{b,c,\alpha}$ limit of φ_m by (4.1) again. This proves the completeness of $FM_{f,b,c,\propto}$.

Therefore $FM_{f,b,c,\propto}$; $T_{b,c,\alpha}$ is a Frechet space.

5. Theorem: The space D(I) is a subspace of $\mathbf{FM_{f,b,c,\propto}}$

Proof: For $\varphi(t,x) \in D(I)$, set

$L = \sup_{\substack{0<x<a \\ 0<t<\infty}} \{t : (t,x) \in supp\, \varphi\}$ and $C_{lq} = \sup_{I} \left| t^k \lambda_{b,c}(x)\, x^{q+1} D_t^l D_x^q \varphi(t,x) \right|$

$$\begin{aligned} \gamma_{b,c,k,q,l}\, \varphi(t,x) &= \sup_{I} \left| t^k \lambda_{b,c}(x)\, x^{q+1} D_t^l D_x^q \varphi(t,x) \right| \\ &\leq C_{lq} L^k \\ &= C_{lq} \left(\frac{L}{Ak^\alpha}\right)^k A^k k^{k\alpha} \end{aligned} \tag{5.1}$$

since $\left(\frac{L}{Ak^\alpha}\right)^k \leq 1$ iff $k \geq \left(\frac{L}{A}\right)^{\frac{1}{\alpha}}$,

define $k_0 = \left[\left(\frac{L}{A}\right)^{\frac{1}{\alpha}}\right] + 1$,

where $[t]$ denotes the Gaussian symbol, that is the greatest integer not exceeding t.

Therefore for $k > k_0$, we have

$$\gamma_{b,c,k,q,l}\, \varphi(t,x) \leq C_{lq} A^k k^{k\alpha} \tag{5.2}$$

If $k \leq k_0$, let us write,

$$C = max \left\{ \frac{L}{A}, \left(\frac{L}{A2^\propto}\right)^2, \dots\dots, \left(\frac{L}{Ak_0^\propto}\right)^{k_0} \right\}.$$

Then again from (5.1)

$$\gamma_{b,c,k,q,l}\, \varphi(t,x) \leq C C_{lq} A^k k^{k\alpha} \tag{5.3}$$

Hence the inequalities (5.2) and (5.3) together yield,

$$\gamma_{b,c,k,q,l}\, \varphi(t,x) \le C'_{lq} A^k k^{k\alpha} \qquad \forall\ k \ge 0$$

where $C'_{lq} = CC_{lq}$

Implying that $\varphi \in FM_{f,b,c,\propto}$.

Consequently, $D(I) \subset FM_{f,b,c,\propto}$.

6. CONCLUSION:

Fourier-finite Mellin transform is extended in the distributional generalized sense. Testing function space using Gelfand Shilov technique and distributional generalized Fourier-finite Mellin transform is defined. Fourier-finite Mellin transform is a Frechet space and its topological property is proved.

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