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SOME FIXED POINT THEOREMS FOR EXPANSION MAPPINGS TAKING SELF MAPPINGS

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ABSTRACT

In the present paper we shall establish some fixed point theorems for expansion mappings in complete metric spaces and complete 2-metric spaces taking self mappings. Our results are generalization of some well known results.

Keywords: *Fixed Point, Complete Metric spaces. Expansion mappings.*

2. INTRODUCTION & PRELIMINARY:

This paper is divided into two parts

Section I: Some fixed point theorems for expansion mappings in complete metric spaces

Section II: Some fixed point theorems for expansion mappings in complete 2- Metric spaces Before starting main result we write some definitions.

Definition 2.1: (Metric Space) A metric space is an ordered pair (X, d) where X is a set and d a function on X ×X with the properties of a metric, namely:

1. d (x, y) ≥ 0. (*non-negative*) ,
2. *d*(*x*, *y*) = *d*(*y*, *x*) (*symmetry*),
2. d (x, y) = 0 if &only if x = y. (identity of indiscernible)
3. The triangle inequality holds:

d (x, z) ≤ d(x, y)+d(y, z), for all x, y, z in X &$x < y < z$.

Example 2.1: Let $E_n (or\ R^n) = \{x = (x_1, x_2, x_3 ..., x_n), x_i \in R, R \text{ the set of real numbers}\}$ and let d be defined as follows:

If $y = (y_1, y_2, y_3 ..., y_n)$ then $d(x, y) = \left(\sum_1^n |x_i - y_i|^p \right)^{1/p} = d_p(x, y)$ where p is a fixed number in [0, ∞).

The fact that d is metric follows from the well-known Minkowski inequality. Also another metric on S considered above can be defined as follows

$$d(x, y) = \sup_i \{|x_i - y_i|\} = d_\infty(x, y)$$

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Example 2.2: Let s be the set of all sequences of real numbers $x = (x_i)_1^{\infty}$ such that for some fixed

$p \in [0, \infty)$, $\sum_1^{\infty} |x_i|^p \prec \infty$ In this case if y=y_i is another point in S, we define

$d(x, y) = \left(\sum |x_i - y_i|^p \right)^{1/p} = d_p(x, y)$, and from Minkowski inequality it follows that this is a metric on S.

Example 2.3: For *x*, *y* ∈R, define *d*(*x*, *y*) = |*x*-*y*|. Then (R *d*) is a metric space. In general, for $x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n) \in$ R^n, define

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

Then (R^n, *d*) is a metric space. As this *d* is usually used, we called it the usual metric.

Definition 2.2: (convergent sequence in Metric space)

A sequence in metric space (X, d) is convergent to x∈ X, if $\lim_{n\to\infty} d(x_n, x) \to 0$

Definition 2.3: (Cauchy sequence in Metric space) Let M= (X, d) be a metric space, let {x_n} be a sequence in M, then {x_n}is a Cauchy sequence if and only if

$$\forall\, \epsilon \in R: \epsilon>0: \exists N: \forall\, m, n \in N: m, n \geq N: d(x_n, x_m) < \epsilon$$

Definition 2.4: (Complete Metric space)

A metric space (X, d) is complete if every Cauchy sequence is convergent.

Definition (2.5): A 2- metric space is a space X in which for each triple of points x, y, z, there exists a real function d (x, y, z,) such that [M_1] to each pair of distinct points x, y, z, d (x, y, z) ≠ 0
[M_2] d (x, y, z) = 0 when at lest two of x, y, z are equal
[M_3] d (x, y, z) = d (y, z, x) = d (x, z, y)
[M_4] d (x, y, z) ≤ d (x, y, v) + d (x, v, z) + d (v, y, z) for all x, y, z, v in X.

Definition (2.6): A sequence {x_n} in a 2-metic space (X, d) is said to be convergent at x if

$\lim_{n\to\infty}$ d (x_n, x, z) = 0 for all z in X.

Definition (2.7): A sequence {x_n} in a 2-metric space, (x, d) is said to be Cauchy sequence if limit

$\lim_{m,n\to\infty}$ d (x_n, x, z) = 0 for all z in X.

Definition (2.7): A 2-metic space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Basic Theorems

In 1975, Fisher [4], proved the following results:

Theorem (A): Let T be a self mapping of a metric spaces X such that,

$$d(Tx, Ty) \geq \frac{1}{2}[d(x, Tx) + d(y, Ty)] \forall\, x, y \in X, Then\, T\, is\, an\, identity\, mappings.$$

Theorem (B): Let X be a compact metric space and T: X → X satisfies (4.A a)
And $x \neq y\, and\;\; x, y\, \varepsilon\, X. Then\, T^r\;\; has\, a\;\; fixed\;\; po\text{int}\;\; for\; some\; positive\, \text{int}\, eger\; r$, and T is invertible.

In 1984 the first known result for expansion mapping was proved by Wang, Li, Gao and Iseki [13].

Theorem (C): "Let T be a self map of complete metric space X into itself and if there is a constant $\alpha > 1$ such that,
$d(Tx, Ty) \geq \alpha\, d(x, y)$ For all $x, y \in X$.

Then T has a unique fixed point in X.

Theorem (D): If there exist non negative real numbers α, β, γ with $\alpha+\beta+\gamma > 1$ and $\alpha < 1$ such that
$d(Tx, Ty) \geq \alpha\, d(x, Tx) + \beta\, d(y, Ty) + \gamma\, d(x, y)$, For each x, y in X with $x \neq y$ and T is onto then T has a fixed point in X.

Theorem (E): If there exist non negative real numbers $\alpha > 1$ such that
$$d(Tx,Ty) \geq \alpha \min\{d(x,Tx), d(y,Ty), d(x,y)\} \forall\, x, y \in X,$$

T is continuous on X onto itself, then T has a fixed point.

Theorem (F): If there exist non negative real numbers $\alpha > 1$ such that $d(Tx, T^2x) \geq \alpha\, d(x, Tx) \forall\, x \in X$, T is onto and continuous then T has a fixed point

In 1988, Park and Rhoades [8] shows that the above theorems are all consequence of a theorem of park [7]. In 1991, Rhoades [10] generalized the result of Iseki and others for pair of mappings:

Theorem (G): If there exist non negative real numbers $\alpha > 1$ and T, S be surjective self –map on a complete metric space (X, d) such that;

$d(Tx, Sy) \geq \alpha\, d(x, y) \forall\, x \in X$, Then T and S have a unique common fixed point.

In 1989 Taniguchi [12] extended some results of Iseki .Later, the results of expansion mappings were extended to 2-metric spaces, introduced by Sharma, Sharma and Iseki [11] for contractive mappings. Many other mathematicians worked on this way.

Rhoades [10] summarized contractive mapping of different types and discussed on their fixed-point theorems. He considered many types of mappings and analyzed the relationship amongst them, where d (Tx, Ty) is governed by,
d (x, y), d (x, Tx), d (y, Ty), d (x, Ty), d (y, Tx) ,d (x, y), d (x, Tx), d (y, Ty), d (x, Ty), d (y, Tx)

Many other mathematicians like Wang, Gao, Isekey [13], Popa [9], Jain and Jain [5], Jain and Yadav [6] worked on expansion mappings. Recently, Agrawal and Chouhan [1, 2], Bhardwaj, Rajput and Yadava [1] worked for common fixed point for expansion mapping.

Our object in this paper is, to obtain some result on fixed-point theorems of expansion type's maps on complete metric space, which are motivated by Rhoades [1], Wang, Gao, Iskey [2]. Also we are finding some results in 2- metric spaces for expansion mappings.

Now in section I, we will find some fixed point theorems for expansion mappings in complete metric spaces.

3 MAIN RESULTS:

Theorem 3.1: Let X denotes the complete metric space with metric d and f is a mapping of X into itself. If there exist non negative real's, $\alpha, \beta, \gamma, \eta, \delta > 1$ with $\alpha + 2\beta + \delta > 1$ such that

$$\begin{aligned} d(fx, fy) \geq\ & \alpha \frac{d(x, fx).d(y, fy).d(x, fy) + d(x,y).d(y, fx).d(x, fx)}{d(y, fx).d(y, fy)} \\ & + \beta[d(x, fx) + d(y, fy)] \\ & + \gamma[d(x, fy) + d(y, fx)] \\ & + \delta[d(x, y)] \end{aligned}$$

For each x, y in X with $x \neq y$ and fis onto then f has a fixed point.

Proof: Let $x_0 \in X$. since f is onto, there is an element x_1 satisfying $x_1 = f^{-1}(x_0)$. Similarly we can write

$$x_n = f^{-1}(x_{n-1}), \ (n = 1, 2, 3, \ldots,)$$

From the hypothesis

$$d(x_{n-1}, x_n) = d(f\,x_n, f\,x_{n+1})$$
$$\geq \alpha.\frac{d(x_n, f\,x_n)d(x_{n+1}, f\,x_{n+1})d(x_n, f\,x_{n+1}) + d(x_n, x_{n+1})d(x_n, f\,x_n)d(x_n, f\,x_n)}{d(x_{n+1}, f\,x_n)d(x_{n+1}, f\,x_{n+1})}$$
$$+\beta\left[d(x_n, f\,x_n) + d(x_{n+1}, f\,x_{n+1})\right]$$
$$+\gamma\left[d(x_n, f\,x_{n+1}) + d(x_{n+1}, f\,x_n)\right]$$
$$+\delta\left[d(x_n, x_{n+1})\right]$$

$$\geq \alpha.\frac{d(x_n, x_{n-1})d(x_{n+1}, x_n)d(x_n, x_n) + d(x_n, x_{n+1})d(x_{n+1}, x_{n-1})d(x_n, x_{n-1})}{d(x_{n+1}, x_{n-1})d(x_{n+1}, x_n)}$$
$$+\beta\left[d(x_n, x_{n-1}) + d(x_{n+1}, x_n)\right] + \gamma\left[d(x_n, x_n) + d(x_{n+1}, x_{n-1})\right]$$
$$+\delta\left[d(x_n, x_{n+1})\right]$$

$$\geq \alpha.\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n-1})d(x_n, x_{n-1})}{d(x_{n+1}, x_{n-1})d(x_{n+1}, x_n)}$$
$$+\beta\left[d(x_n, x_{n-1}) + d(x_{n+1}, x_n)\right]$$
$$+\gamma\left[d(x_{n+1}, x_{n-1})\right]$$
$$+\delta\left[d(x_n, x_{n+1})\right]$$

$$\geq \alpha.d(x_n, x_{n-1}) + \beta\left[d(x_n, x_{n-1}) + d(x_{n+1}, x_n)\right]$$
$$+\gamma\left[d(x_{n+1}, x_n) - d(x_n, x_{n-1})\right]$$
$$+\delta\left[d(x_n, x_{n+1})\right]$$

$$\Rightarrow (1-\alpha-\beta+\gamma)d(x_n, x_{n-1}) \geq (\beta+\gamma+\delta)d(x_{n+1}, x_n)$$

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{(1-\alpha-\beta+\gamma)}{\beta+\gamma+\delta}.d(x_n, x_{n-1})$$

Therefore $\{X_n\}$ converges to x in X. Let y є f^1 (x), for infinitely many n , $x_n \neq x$ for such n,

$$d(x_n, x) = d(f\,x_{n+1}, fy)$$
$$\geq \alpha.\frac{d(x_{n+1}, f\,x_{n+1})d(y, fy)d(x_{n+1}, fy) + d(x_n, y)d(y, f\,x_{n+1})d(x_{n+1}, f\,x_{n+1})}{d(y, f\,x_{n+1})d(y, fy)}$$
$$+\beta\left[d(x_{n+1}, f\,x_{n+1}) + d(y, fy)\right]$$
$$+\gamma\left[d(x_{n+1}, fy) + d(y, f\,x_{n+1})\right]$$
$$+\delta.d(x_{n+1}, y)$$

$$\geq \alpha.\frac{d(x_{n+1}, x_n)d(y, x)d(x_{n+1}, x) + d(x_n, y)d(y, x_n)d(x_{n+1}, x_n)}{d(y, x_n)d(y, x)}$$
$$+\beta\left[d(x_{n+1}, x_n) + d(y, x)\right]$$
$$+\gamma\left[d(x_{n+1}, x) + d(y, x_n)\right]$$
$$+\delta.d(x_{n+1}, y)$$

Since $d(x_n, x) \to \infty$ as $n \to \infty$, we have

$$d(x_{n+1}, x_n) = d(x_{n+1}, x) = 0$$

Therefore $d(x, y) = 0 \Rightarrow x = y$

i.e. y = f(x) = x

This completes the proof of the theorem 3.1.

Theorem 3.2: Let X denotes the complete metric space with metric d and f is a mapping of X into itself.

If there exist non negative real's, α, β, γ, η, δ > 1 with $\alpha + \beta + \gamma + 2\eta - \delta > 1$ such that

$$\begin{aligned} d(fx, fy) \geq\ & \alpha \frac{d(x, fx).d(y, fy).d(x, fy) + d(x, y).d(y, fx).d(x, fx)}{d(y, fx).d(y, fy)} \\ & + \beta \frac{d(x, fy).d(x, y) + d(y, fx).d(x, fx)}{d(y, fx)} \\ & + \gamma \left[\frac{d(x, fx).d(y, fy)}{d(x, y)} \right] \\ & + \eta . \frac{d(x, y).d(fx, fy) + d(x, fx).d(y, fy)}{d(y, fy)} \\ & + \delta . d(y, fy \end{aligned}$$

For each x, y in X with x ≠ y, & $d(y, fx).d(y, fy) \neq 0$ and f is onto then f has a fixed point.

Proof: Let $x_0 \in X$. since f is onto, there is an element x_1 satisfying $x_1 = f^{-1}(x_0)$. Similarly we can write

$$x_n = f^{-1}(x_{n-1}), \ (n = 1, 2, 3, \ldots,)$$

From the hypothesis

$$\begin{aligned} d(x_{n-1}, x_n) &= d(f x_n, f x_{n+1}) \\ & \geq \alpha . \frac{d(x_n, f x_n) d(x_{n+1}, f x_{n+1}) d(x_n, f x_{n+1}) + d(x_n, x_{n+1}) d(x_{n+1}, f x_n) d(x_n, f x_n)}{d(x_{n+1}, f x_n) d(x_{n+1}, f x_{n+1})} \\ & + \beta . \frac{d(x_n, f x_{n+1}) d(x_n, x_{n+1}) + d(x_{n+1}, f x_n) d(x_n, f x_n)}{d(x_{n+1}, f x_n)} \\ & + \gamma \left[\frac{d(x_n, f x_n) d(x_{n+1}, f x_{n+1})}{d(x_n, x_{n+1})} \right] \\ & + \eta \frac{d(x_n, x_{n+1}) d(fx_n, f x_{n+1}) + d(x_n, f x_n) d(x_{n+1}, f x_{n+1})}{d(x_{n+1}, f x_{n+1})} \\ & + \delta . d(x_{n+1}, f x_{n+1}) \end{aligned}$$

$$\geq \alpha.\frac{d(x_n,x_{n-1})d(x_{n+1},x_n)d(x_n,x_n)+d(x_n,x_{n+1})d(x_{n+1},x_{n-1})d(x_n,x_{n-1})}{d(x_{n+1},x_{n-1})d(x_{n+1},x_n)}$$
$$+\beta.\frac{d(x_n,x_n)d(x_n,x_{n+1})+d(x_{n+1},x_{n-1})d(x_n,x_{n-1})}{d(x_{n+1},x_{n-1})}$$
$$+\gamma\left[\frac{d(x_n,x_{n-1})d(x_{n+1},x_n)}{d(x_n,x_{n+1})}\right]$$
$$+\eta\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)+d(x_n,x_{n-1})d(x_{n+1},x_n)}{d(x_n,x_{n+1})}$$
$$+\delta.d(x_{n+1},x_n)$$

$$\geq \alpha.\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n-1})d(x_n,x_{n-1})}{d(x_{n+1},x_{n-1})d(x_{n+1},x_n)}$$
$$+\beta.\frac{d(x_{n+1},x_{n-1})d(x_n,x_{n-1})}{d(x_{n+1},x_{n-1})}$$
$$+\gamma[d(x_n,x_{n-1})]$$
$$+2\eta.d(x_{n-1},x_n)+\delta.d(x_{n+1},x_n)$$

$$\geq \alpha.d(x_n,x_{n-1})+\beta.d(x_n,x_{n-1})$$
$$+\gamma[d(x_n,x_{n-1})]+2\eta.d(x_{n-1},x_n)$$
$$+\delta.d(x_{n+1},x_n)$$

$$\geq (\alpha.+\beta+\gamma+2\eta).d(x_n,x_{n-1})+\delta.d(x_{n+1},x_n)(1-\alpha-\beta-\gamma-2\eta).d(x_n,x_{n-1})\geq \delta.d(x_{n+1},x_n)$$

$$\Rightarrow \qquad d(x_{n+1},x_n)\leq \frac{1-\alpha-\beta-\gamma-2\eta}{\delta}.d(x_n,x_{n-1})$$

Since $\frac{1-\alpha-\beta-\gamma-2\eta}{\delta}<1$

Therefore $\{X_n\}$ converges to x in X. Let y є f^1 (x) , for infinitely many n , $x_n \neq x$ for such n,

$$d(x_n,x)=d(f\,x_{n+1},fy)$$
$$\geq \alpha.\frac{d(x_{n+1},f\,x_{n+1})d(y,fy)d(x_{n+1},fy)+d(x_{n+1},y)d(y,f\,x_{n+1})d(x_{n+1},f\,x_{n+1})}{d(y,f\,x_{n+1})d(y,fy)}$$

$$+\beta\left[\frac{d(x_{n+1},fy)+d(x_{n+1},y)+d(y,f\,x_{n+1})d(x_{n+1},f\,x_{n+1})}{d(y,f\,x_{n+1})}\right]$$
$$+\gamma\left[\frac{d(x_{n+1},f\,x_{n+1})d(y,fy)}{d(x_{n+1},y)}\right]$$
$$+\eta\frac{d(x_{n+1},y)d(x_{n+1},fy)+d(x_{n+1},f\,x_{n+1})d(y,fy)}{d(y,fy)}$$
$$+\delta.d(y,fy)$$

$$\geq \alpha.\frac{d(x_{n+1},x_n)d(y,x)d(x_{n+1},x)+d(x_{n+1},y)d(y,x_n)d(x_{n+1},x_n)}{d(y,x_n)d(y,x)}$$
$$+\beta\left[\frac{d(x_{n+1},x)+d(x_{n+1},y)+d(y,x_n)d(x_{n+1},x_n)}{d(y,x_n)}\right]$$
$$+\gamma\left[\frac{d(x_{n+1},x_n)d(y,x)}{d(x_{n+1},y)}\right]$$
$$+\eta\frac{d(x_{n+1},y)d(x_{n+1},x)+d(x_{n+1},x_n)d(y,x)}{d(y,x)}$$
$$+\delta.d(y,x)$$

Since $d(x_n,x)\to 0,$ as $n\to\infty$, we have

$$d(x_{n+1},x)=d(x_{n+1},x_n)=0$$

Therefore $d(x,y)=0\Rightarrow x=y$

i.e. y = f(x)= x

This completes the proof of the theorem 3.2

Now in section II we will find some fixed point theorems in 2-metrc spaces for expansion mappings.

Section II: Some fixed point theorems in 2-Metric spaces for expansion mappings

Theorem 3.3: Let X denotes the complete 2- metric space with metric d and f is a mapping of X into itself. If there exist non negative real's, α, β, γ, η, δ, a (real) > 1 with $\alpha+2\beta+\delta>1$ such that

$$d(fx,fy,a)\geq\alpha\frac{d(x,fx,a).d(y,fy,a).d(x,fy,a)+d(x,y,a).d(y,fx,a).d(x,fx,a)}{d(y,fx,a).d(y,fy,a)}$$
$$+\beta\left[d(x,fx,a)+d(y,fy,a)\right]$$
$$+\gamma\left[d(x,fy,a)+d(y,fx,a)\right]$$
$$+\delta\left[d(x,y,a)\right]$$

For each x, y in X with x ≠ y and f is onto then f has a fixed point.

Proof: Let x_0 ϵ X. since f is onto, there is an element x_1 satisfying $x_1= f^{-1}(x_0)$. Similarly we can write

$$x_n = f^{-1}(x_{n-1}),\ (n = 1, 2, 3, . ..,)$$

From the hypothesis

$$d(x_{n-1},x_{n,a})=d(fx_n,fx_{n+1},a)$$
$$\geq\alpha.\frac{d(x_n,fx_n,a)d(x_{n+1},fx_{n+1,a})d(x_n,fx_{n+1},a)+d(x_n,x_{n+1,a})d(x_n,fx_n,a)d(x_n,fx_n,a)}{d(x_{n+1},fx_n,a)d(x_{n+1},fx_{n+1,a})}$$
$$+\beta\left[d(x_n,fx_{n,a})+d(x_{n+1},fx_{n+1},a)\right]$$
$$+\gamma\left[d(x_n,fx_{n+1},a)+d(x_{n+1},fx_n,a)\right]$$
$$+\delta\left[d(x_n,x_{n+1,a})\right]$$

$$\geq \alpha.\frac{d(x_n,x_{n-1},a)d(x_{n+1},x_{n},a)d(x_n,x_n,a)+d(x_n,x_{n+1},a)d(x_{n+1},x_{n-1},a)d(x_n,x_{n-1},a)}{d(x_{n+1},x_{n-1},a)d(x_{n+1},x_n,a)}$$
$$+\beta\left[d(x_n,x_{n-1},a)+d(x_{n+1},x_n,a)\right]+\gamma\left[d(x_n,x_{n},a)+d(x_{n+1},x_{n-1},a)\right]$$
$$+\delta\left[d(x_n,x_{n+1},a)\right]$$

$$\geq \alpha.\frac{d(x_n,x_{n+1},a)d(x_{n+1},x_{n-1},a)d(x_n,x_{n-1},a)}{d(x_{n+1},x_{n-1},a)d(x_{n+1},x_n,a)}$$
$$+\beta\left[d(x_n,x_{n-1},a)+d(x_{n+1},x_n,a)\right]$$
$$+\gamma\left[d(x_{n+1},x_{n-1},a)\right]$$
$$+\delta\left[d(x_n,x_{n+1},a)\right]$$

$$\geq \alpha.d(x_n,x_{n-1},a)+\beta\left[d(x_n,x_{n-1},a)+d(x_{n+1},x_{n},a)\right]$$
$$+\gamma\left[d(x_{n+1},x_{n},a)-d(x_n,x_{n-1},a)\right]$$
$$+\delta\left[d(x_n,x_{n+1},a)\right]$$

$$\Rightarrow (1-\alpha-\beta+\gamma)d(x_n,x_{n-1},a)\geq(\beta+\gamma+\delta)d(x_{n+1},x_n,a)$$

$$\Rightarrow d(x_{n+1},x_n,a)\leq\frac{(1-\alpha-\beta+\gamma)}{\beta+\gamma+\delta}.d(x_n,x_{n-1},a)$$

Therefore $\{X_n\}$ converges to x in X. Let y ϵ f^1 (x), for infinitely many n , $x_n \neq x$ for such n,

$$d(x_n,x,a)=d(f x_{n+1},fy,a)$$

$$\geq \alpha.\frac{d(x_{n+1},f x_{n+1},a)d(y,fy,a)d(x_{n+1},fy,a)+d(x_n,y,a)d(y,f x_{n+1},a)d(x_{n+1},f x_{n+1},a)}{d(y,f x_{n+1},a)d(y,fy,a)}$$
$$+\beta\left[d(x_{n+1},f x_{n+1},a)+d(y,fy,a)\right]+\gamma\left[d(x_{n+1},fy,a)+d(y,f x_{n+1},a)\right]$$
$$+\delta.d(x_{n+1},y,a)$$

$$\geq \alpha.\frac{d(x_{n+1},x_{n},a)d(y,x,a)d(x_{n+1},x,a)+d(x_n,y,a)d(y,x_n,a)d(x_{n+1},x_n,a)}{d(y,x_{n},a)d(y,x,a)}$$
$$+\beta\left[d(x_{n+1},x_n,a)+d(y,x,a)\right]$$
$$+\gamma\left[d(x_{n+1},x,a)+d(y,x_{n},a)\right]$$
$$+\delta.d(x_{n+1},y,a)$$

Since $d(x_n,x,a)\to\infty$ as $n\to\infty$, we have

$$d(x_{n+1},x_n,a)=d(x_{n+1},x,a)=0$$

Therefore $d(x,y,a)=0\Rightarrow x=y$

i.e. y = f(x) = x

This completes the proof of the theorem 3.3.

Theorem 3.4: Let X denotes the complete 2-metric space with metric d and f is a mapping of X into itself.

If there exist non negative real's, α, β, γ, η, δ,a(real) > 1 with $\alpha+\beta+\gamma+2\eta-\delta>1$ such that

$$d(fx, fy,a) \geq \alpha \frac{d(x, fx,a).d(y, fy,a).d(x, fy,a)+d(x,y,a).d(y, fx,a).d(x, fx,a)}{d(y, fx,a).d(y, fy,a)}$$
$$+\beta \frac{d(x, fy,a).d(x,y,a)+d(y, fx,a).d(x, fx,a)}{d(y, fx,a)}+\gamma\left[\frac{d(x, fx,a).d(y, fy,a)}{d(x,y,a)}\right]$$
$$+\eta.\frac{d(x,y,a).d(fx, fy,a)+d(x, fx,a).d(y, fy,a)}{d(y, fy,a)}$$
$$+\delta.d(y, fy,a)$$

For each x, y in X with x ≠ y, & $d(y, fx,a).d(y, fy,a) \neq 0$ and f is onto then f has a fixed point.

Proof: Let x_0 ϵ X. since f is onto, there is an element x_1 satisfying $x_1 = f^{-1}(x_0)$. Similarly we can write

$$x_n = f^{-1}(x_{n-1}), \ (n = 1, 2, 3, \ldots,)$$

From the hypothesis

$$d(x_{n-1}, x_n, a) = d(f x_n, f x_{n+1}, a)$$
$$\geq \alpha.\frac{d(x_n, f x_n, a)d(x_{n+1}, f x_{n+1}, a)d(x_n, f x_{n+1}, a)+d(x_n, x_{n+1}, a)d(x_{n+1}, f x_n, a)d(x_n, f x_{n,}a)}{d(x_{n+1}, f x_n, a)d(x_{n+1}, f x_{n+1}, a)}$$
$$+\beta.\frac{d(x_n, f x_{n+1}, a)d(x_n, x_{n+1,}a)+d(x_{n+1}, f x_n, a)d(x_n, f x_n, a)}{d(x_{n+1}, f x_n, a)}$$
$$+\gamma\left[\frac{d(x_n, f x_n, a)d(x_{n+1}, f x_{n+1}, a)}{d(x_n, x_{n+1,}a)}\right]$$
$$+\eta\frac{d(x_n, x_{n+1}, a)d(fx_n, f x_{n+1,}a)+d(x_n, f x_n, a)d(x_{n+1}, f x_{n+1,}a)}{d(x_{n+1}, f x_{n+1,}a)}$$
$$+\delta.d(x_{n+1}, f x_{n+1}, a)$$

$$\geq \alpha.\frac{d(x_n, x_{n-1}, a)d(x_{n+1}, x_n, a)d(x_n, x_n, a)+d(x_n, x_{n+1,}a)d(x_{n+1}, x_{n-1}, a)d(x_n, x_{n-1}, a)}{d(x_{n+1}, x_{n-1}, a)d(x_{n+1}, x_n, a)}$$
$$+\beta.\frac{d(x_n, x_{n,}a)d(x_n, x_{n+1,}a)+d(x_{n+1}, x_{n-1}, a)d(x_n, x_{n-1,}a)}{d(x_{n+1}, x_{n-1}, a)}$$
$$+\gamma\left[\frac{d(x_n, x_{n-1,}a)d(x_{n+1}, x_n, a)}{d(x_n, x_{n+1}, a)}\right]$$
$$+\eta\frac{d(x_n, x_{n+1}, a)d(x_{n-1}, x_n, a)+d(x_n, x_{n-1,}a)d(x_{n+1}, x_n, a)}{d(x_n, x_{n+1}, a)}$$
$$+\delta.d(x_{n+1}, x_n, a)$$

$$\geq \alpha.\frac{d(x_n, x_{n+1}, a)d(x_{n+1}, x_{n-1}, a)d(x_n, x_{n-1}, a)}{d(x_{n+1}, x_{n-1}, a)d(x_{n+1}, x_n, a)}$$
$$+\beta.\frac{d(x_{n+1}, x_{n-1}, a)d(x_n, x_{n-1}, a)}{d(x_{n+1}, x_{n-1}, a)}$$
$$+\gamma\left[d(x_n, x_{n-1}, a)\right] + 2\eta.d(x_{n-1}, x_n, a) + \delta.d(x_{n+1}, x_n, a)$$

$$\geq \alpha.d(x_n, x_{n-1}, a) + \beta.d(x_n, x_{n-1}, a)$$
$$+\gamma\left[d(x_n, x_{n-1}, a)\right] + 2\eta.d(x_{n-1}, x_n, a)$$
$$+\delta.d(x_{n+1}, x_n, a)$$

$$\geq (\alpha. + \beta + \gamma + 2\eta).d(x_n, x_{n-1}, a) + \delta.d(x_{n+1}, x_n, a)(1 - \alpha - \beta - \gamma - 2\eta).d(x_n, x_{n-1}, a) \geq \delta.d(x_{n+1}, x_n, a)$$

$$\Rightarrow d(x_{n+1}, x_n, a) \leq \frac{1 - \alpha - \beta - \gamma - 2\eta}{\delta}.d(x_n, x_{n-1}, a)$$

Since $\frac{1 - \alpha - \beta - \gamma - 2\eta}{\delta} < 1$

Therefore $\{X_n\}$ converges to x in X. Let y є f^1 (x), for infinitely many n , $x_n \neq x$ for such n,

$$d(x_n, x, a) = d(f x_{n+1}, fy, a)$$
$$\geq \alpha.\frac{d(x_{n+1}, f x_{n+1}, a)d(y, fy, a)d(x_{n+1}, fy, a) + d(x_{n+1}, y, a)d(y, f x_{n+1}, a)d(x_{n+1}, f x_{n+1}, a)}{d(y, f x_{n+1}, a)d(y, fy, a)}$$

$$+\beta\left[\frac{d(x_{n+1}, fy, a) + d(x_{n+1}, y, a) + d(y, f x_{n+1}, a)d(x_{n+1}, f x_{n+1}, a)}{d(y, f x_{n+1}, a)}\right]$$
$$+\gamma\left[\frac{d(x_{n+1}, f x_{n+1}, a)d(y, fy, a)}{d(x_{n+1}, y, a)}\right]$$
$$+\eta\frac{d(x_{n+1}, y, a)d(x_{n+1}, fy, a) + d(x_{n+1}, f x_{n+1}, a)d(y, fy, a)}{d(y, fy, a)}$$
$$+\delta.d(y, fy, a)$$

$$\geq \alpha.\frac{d(x_{n+1}, x_n, a)d(y, x, a)d(x_{n+1}, x, a) + d(x_{n+1}, y, a)d(y, x_n, a)d(x_{n+1}, x_n, a)}{d(y, x_n, a)d(y, x, a)}$$
$$+\beta\left[\frac{d(x_{n+1}, x, a) + d(x_{n+1}, y, a) + d(y, x_n, a)d(x_{n+1}, x_n, a)}{d(y, x_n, a)}\right]$$
$$+\gamma\left[\frac{d(x_{n+1}, x_n, a)d(y, x, a)}{d(x_{n+1}, y, a)}\right]$$
$$+\eta\frac{d(x_{n+1}, y, a)d(x_{n+1}, x, a) + d(x_{n+1}, x_n, a)d(y, x, a)}{d(y, x, a)}$$
$$+\delta.d(y, x, a)$$

Since $d(x_n, x, a) \to 0$, as $n \to \infty$, we have

$$d(x_{n+1}, x, a) = d(x_{n+1}, x_{n,a}) = 0$$

Therefore $d(x, y, a) = 0 \Rightarrow x = y$

i.e. y = f(x)= x

This completes the proof of the theorem 3.4

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