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ON SOME RELIABILITY CHARACTERIZATIONS OF MEMBER OF THE PEARSON AND ORD FAMILIES OF DISTRIBUTIONS

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ABSTRACT

In this paper we characterize the Pearson family of distributions by finding a relationship between the reversed hazard rate (RHR) and the mean inactivity time(MIT). we present characterization of discrete distribution using Ord -Carver system (discrete Pearson family).

Keywords: *Reversed Hazard Rate, Mean inactivity time, Characterizations, Exponential Distribution. Induced Distributions.*

In Reliability analysis the hazard rate, also known as failure rate, of a life distribution plays an important role for stochastic modeling and classification. Being a ratio of probability density function and the corresponding survival function, it uniquely determines the underlying distributions and exhibits different monotonic behaviors.

let X be a random variable usually representing the life length for a certain unit (where this unit can have multiple interpretations) or component in a system with reliability (survival) function and probability density function as follows

$$F(t) = P(X \le t), f(t) = P(X = t).$$

In life testing situations, the additional lifetime given that the component has survived up to time t is called the residual life function (RLF) of the component. More specifically if X is the life of component, then the random variable

$X_t = (X - t \mid X \ge t)$ is called the residual life random variable. In the insurance business, this random variable represents the amount of claim if the deductible for a particular policy is t and X is the random variable representing the loss. The quantity $\mu(t) = E(X_t) = E(X - t \mid X \ge t)$ is called the mean residual life function (MRLF) or the life expectancy at age t and has been employed in life length studies by various authors e.g. Hollander and Proschan [10], Hall and Wellner [9] have characterized the class of mean residual life functions. Limiting properties (behavior).of the MRLF have been studied by Bradley and Gupta [6]. Several functions are defined related to the residual life and it plays a crucial role in reliability and survival analysis. The failure (hazard) rate function, defined by

$\lambda(t) = \dfrac{f(t)}{\overline{F}(t)} = -\dfrac{d}{dt}\ln \overline{F}(t)$ for all t such that F(t) > 0 , and it can be easily verified that $\lambda(t) = \dfrac{1 + \frac{d}{dt}\mu(t)}{\mu(t)}$

However, it is reasonable to presume that in many realistic situations the random variable is not necessarily related to the future but can also refer to the past. For instant, consider a system whose state is observed only at certain reassigned inspection times. If at time t the system is inspected for the first time and it is found to be "down" , then the failure relies on the past, i.e. on which instant in (0,t) it has failed. It thus seems natural to study a notion that is dual to the residual life, in the sense that it refers to past time and not to future (see Di Crescenzo and Longobardi [7]).

The concept of reversed hazard rate (RHR) was initially introduced as the hazard rate in the negative direction and received a cold reception in the literature at the early stage. This was because RHR , being the ratio of probability density function and the corresponding distribution function, was conceived as a dual measure of hazard rate. Keilson and Sumita [14] were among the first to define RHR and called it the “Duality" is well explained by Block at al [5].The reversed hazard rate τ(t) is defined by the following equation

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$$\tau(t) = \frac{f(t)}{F(t)}.$$

Reversed hazard rate function is useful among other ways in the estimation of the survival function for left censored lifetimes. Making simple transformations we arrive at an important relation between τ(t) and λ(t) given by the following

$$\tau(t) = \frac{\lambda(t)\overline{F}(t)}{1-\overline{F}(t)}$$

Applications of hazard functions are quite well known in the statistical literature. Recently the reversed hazard functions also become quite popular among the statisticians, see for example Gupta and Han [11]. Anderson et al. [3] show that the reversed hazard function plays the same role in the analysis of left-censored data as the hazard function plays in the analysis of right-censored data. Interestingly, it is observed that there exists a relation between the proportional reversed hazard class of distributions and the exponentiated class of distributions. However, it is reasonable to presume that in many realistic situations, the random variables are not necessarily related only to the future, but they can also refer to the past. In fact, in many reliability problems, it is of interest to consider variables of the kind $X_{(t)} = (X - t \mid X \le t)$ for fixed t >0, and known in literature as the inactivity time (IT) Also the mean inactivity time (MIT) is given by

$$M_X(t) = E(X - t \mid \mathrm{X} \le \mathrm{t}),\ \mathrm{t} > 0.$$

The properties of the mean inactivity time have been considered by many authors , see, eg., Kayid and Ahmad [13] and Ahmad , Kayid and Pellery [2], Di Crescenzo and Longobardi [7]. Several characterizations of probability models have been obtained in the last 30 years based on the univariate failure rate or mean residual life functions. The problems of characterization of distributions are today a substantial part of probability theory and Mathematical Statistics.

The mean inactivity time and mean residual life are applicable in both biostatistics and many other actuarial science, engineering, economics, biometry and applied probability areas. They also are useful in survival analysis studies when we take are faced with left or right censored data.

Several authors have considered different techniques to characterize probability distributions of interest, see, e.g., Arnold [4], EL-Arishi [8] presented the conditional variance characterization of some discrete probability distributions. Ahmed, A.N. [1] presented the characterization of beta, binomial and Poisson distributions. Unnikrishnan and Sudheesh [19].studied the characterization of continuous distributions by properties of conditional variance. Nair and Sankaran [16] introduce a characterization of the Pearson family of distributions. Gupta and Bradley [12] introduced representing the mean residual life in terms of the failure rate. and the references cited therein.

To our knowledge, non has characterized distributions in terms of their MIT. Our aim in this paper we characterize the Pearson family of distributions by finding a relationship between the reversed failure rate and mean inactivity time for some continuous probability distributions. Finally, we present an analogue of the Pearson system of distributions for discrete case which sometimes known as the Ord- Carver system of distributions (O.s.d).

2: A GENERAL FAMILY OF DISTRIBUTIONS, INCLUDING THE PEARSON FAMILY.

Consider the family of distributions whose probability density function f is differentiable. Let

$$\frac{d}{dx}\log f(x) = \frac{\mu - x}{g(x)} - \frac{d}{dx}\log g(x) \tag{2.1}$$

Where $\mu = E(X)$ is a constant and g(x) satisfies the first order linear differential equation, and

$$xf(x) = \mu f(x) - \frac{d}{dx}\left[f(x)g(x)\right]. \tag{2.2}$$

The integrating of the above equation yields

$$\int_0^t xf(x)dx = \int_0^t \mu\, f(x)dx - \int_0^t \frac{d}{dx}\left[f(x)\ g(x)\right]$$

and $\frac{1}{F(t)}\int_0^t xf(x)dx = \mu - \frac{1}{F(t)}\left[f(t)g(t)\right]$. In other words,

$$E(X \mid X \le t) = \mu - g(t)\,\tau(t) \tag{2.3}$$

Or equivalently, The mean inactivity time as follows

$$\begin{aligned} M_X(t) &= t - E(X \mid X \le t) \\ &= t - \mu(t) + g(t)\tau(t) \end{aligned} \tag{2.4}$$

Thus the mean inactivity time has been expressed in terms of the reversed failure rate , the given function g and the constant μ. By appropriately specializing g in (2.4) , on can obtain many of the important cases that have appeared in the literature.

3: THE MAIN RESULT

In this section we shall characterize some well known probability distributions. To this let the quadratic function $g(x) = a_0 + a_1 x + a_2 x^2 \;\; with \;\; a_2 \ne -\frac{1}{2}$. This yields the Pearson family, with these notation. We establish the characterization of Beta, the Gamma, the Power , the Exponential , the Normal and Maxwell distributions. From equation (2.1) , we get

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{\mu - x}{a_0 + a_1 x + a_2 x^2} - \frac{a_1 + 2a_2 x}{a_0 + a_1 x + a_2 x^2} \\ &= - \frac{x + d}{A_0 + A_1 x + A_2 x^2} \end{aligned} \tag{3.1}$$

taking $A_i = \frac{a_i}{1 + 2a_2},\ i = 0,1,2 \;\; and \;\; d = \frac{a_1 - \mu}{1 + 2a_2}$. For the Pearson family, the equation (2.3) becomes

$$E(X \mid X \le t) = \frac{A_1 - d}{1 - 2A_2} - \frac{A_0 + A_1 t + A_2 t^2}{1 - 2A_2} \tag{3.2}$$

3.1: The Beta distribution

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Let X has beta distribution of the first kind with probability density function

$$f(x,m,n) = \frac{1}{\beta(m,n)}\, x^{m-1}\,(1-x)^{n-1} \;\; , 0 < x < 1 , m,n > 0 .$$

$With \quad g(x) = \frac{x(1-x)}{m+n}$, the beta distribution satisfies the following

$\frac{f'(x)}{f(x)} = - \frac{\left[x + (1-m)\right]l(m+n-2)}{\left[x(1-x)\right]l(m+n-2)}$, $m + n \ne 2$, and hence , belongs to the Pearson family given by (3.1) with $d = \frac{1-m}{(m+n-2)}$, $A_0 = 0$, $A_1 = \frac{1}{(m+n-2)}$ and $A_2 = \frac{-1}{(m+n-2)}$. Thus

$E(X \mid X \le t) = \mu - \frac{t(1-t)}{m+n}\tau(t)$, $0 < t < 1.$ Also

$M_X(t) = t - \mu + \frac{t(1-t)}{m+n}\tau(t)$. Where $\mu = \frac{m}{m+n}$. If $m + n = 2$, we have the following

$f(x) = \frac{1}{\Gamma(m)\Gamma(2-m)}\left(\frac{x}{1-x}\right)^{m-1}$, $0 < m < 2$, consequently

$E(X \mid X \le t) = \frac{m}{2} - \frac{t(1-t)}{2}\tau(t)$, and $M_X(t) = t - \frac{m}{2} + \frac{t(1-t)}{2}\tau(t)$, $0<t<1$.

3.2: The Gamma distribution

Let g(x) = αx, α > 0, yields the gamma distribution. In this case (2.1) gives $\frac{f'(x)}{f(x)} = -\frac{x+\alpha-\mu}{\alpha x}$, so that f belongs to the Pearson family (3.1) with $d = \alpha - \mu$, $A_0 = A_1 = 0$, *and* $A_2 = \alpha$
In fact

$f(x) = \frac{\alpha^{-\frac{\mu}{\alpha}}}{\Gamma(\frac{\mu}{\alpha})} x^{\frac{\mu}{\alpha}-1} e^{\frac{\mu}{\alpha}}$ is a gamma distribution with mean μ. Hence equation (2.3) takes the form

$$E(X \mid X \le t) = \mu - \alpha t\, \tau(t), \quad and \quad M_X(t) = t - \mu + \alpha t \tau(t)$$

3.3: The Power distribution

With $g(x) = \frac{x(\alpha - x)}{\theta + 1}$ we have the power distribution $f(x) = \left(\frac{x}{\alpha}\right)^{\theta}$, $0 < x < \alpha$, $\theta > 0$. It satisfies $\frac{f'(x)}{f(x)} = \frac{\theta - 1}{x}$ and hence belongs to the Pearson family given by 3.1 with d = $A_0 = A_1 = 0$ and $A_2 = (1/(1-\theta))$. Thus

$$E(X \mid X \le t) = \mu - g(t)\ \tau(t)$$
$$= \mu - \left[\frac{t(\alpha - t)}{\theta + 1}\right]\tau(t)$$

Where $\mu = \frac{\alpha\theta}{1+\theta}$ and $M_X(t) = t - \mu + \left[\frac{t(\alpha - t)}{\theta + 1}\right]\tau(t)$

3.4: The Exponential distribution

The function $g(x) = \frac{x}{\lambda}$, $\lambda > 0$ and $x > 0$ yields the Exponential distribution. In this case the equation 2.1 gives $\frac{f'(x)}{f(x)} = -\lambda$

so that f belongs to the Pearson family 3.1 with $d = A_0 = A_2 = 0$, *and* $A_1 = \frac{1}{\lambda}$. It follows that

$f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$ *and* $x > 0$. Hence 2.3 can be written as

$$E(X \mid X \le t) = \mu - g(t)\ \tau(t)$$
$$= \mu - \left(\frac{t}{\lambda}\right)\tau(t)$$

and mean inactivity time is given by the following $M_X(t) = t - \mu + \left(\frac{t}{\lambda}\right)\tau(t)$.

3.5: The Normal distribution

Let the constant function g(x) = σ^2, $\sigma > 0$, for $-\infty < x < \infty$ yields the normal distribution. In this case, (2.1) gives

$\dfrac{f'(x)}{f(x)} = \dfrac{\mu - x}{\sigma^2}$, so that f belongs to the Pearson family (3.1) with

$$d = -\mu \ , \ A_0 = \sigma^2 \quad and \ A_1 = A_2 = 0.$$

It follows that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \ \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

a normal distribution with mean μ and variance σ^2. Hence (2.3) and (2.4) can be written as

$$\begin{aligned} E(X \mid X \leq t) &= \mu - g(t)\ \tau(t) \\ &= \mu - \sigma^2 \ \tau(t) \end{aligned}$$

and mean inactivity time is given by the following

$$M_X(t) = t - \mu + \sigma^2 \ \tau(t).$$

3.6: The Maxwell distribution

The Maxwell distribution has the form

$$f(x) = \frac{4}{\theta^3\sqrt{\pi}} \ \exp\left(\frac{-x^2}{\theta^2}\right), \ x > 0 \ , \ \theta > 0.$$

Then we have

$$\frac{f'(x)}{f(x)} = \frac{2}{x} - \frac{2\ x}{\theta^2} \ ,$$

which can be written in the form (2.1) with μ=0, and $g(x) = \dfrac{\theta^2(x^2 + \theta^2)}{2x^2}$. Hence (2.3) and (2.4) can be written as

$$\begin{aligned} E(X \mid X \leq t) &= \mu - g(t)\ \tau(t) \\ &= -\frac{\theta^2\left(t^2 + \theta^2\right)}{2\ t^2}\tau(t) \end{aligned}$$

and mean inactivity time is given by $M_X(t) = \ t + \dfrac{\theta^2\left(t^2 + \theta^2\right)}{2\ t^2}\tau(t).$

4: DISCRETE CASE

Modeling lifetimes through relations between conditional expectations and failure rates in the discrete time domain have been initiated by Osaki and Li [18] when they proved such a result for the negative binomial distribution. This was followed by a similar result by Ahmed [1] concerning the binomial and Poisson distributions. In a more general framework Nair and Sankaran [16] introduced a characterization of the pearson family of distributions.

In this section , we consider the analogues of the results in the Section 3 in the discrete case. First we define an analogue of the Pearson system of distributions for discrete case which is known as the Ord- Carver system of distributions (O.s.d), see Kotz et al.[15]. We establish the characterization of some discrete distribution such as binomial, Poisson and Negative binomial distributions.

Definition 4.1: Let X be a nonnegative integer valued, non-degenerate random variable with probability mass function f . The distribution corresponding to f is said to be a member of O.s.d if f satisfies the equation

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{x+d}{b_0+b_1x+b_2x^2}\ ,\ x=0,1,2,....$$

where b_0, b_1, b_2 *and* d are real valued.

Lemma 4.2: If

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{x+d}{b_0+b_1x+b_2x^2}$$
$$= \frac{\mu-x}{g(x)} - \frac{g(x+1)-g(x)}{g(x)}$$

then

$$E\ (X \mid X \geq x) = \mu + (a_0+a_1x+a_2x^2)\lambda(x)$$

where

$$a_i = \frac{b_i}{1-2b_2}\ ,\ i=0,1,2. \qquad and \qquad \mu = \frac{b_1-d}{1-2b_2}.$$

4.1: The Binomial distribution

In this subsection we characterize the binomial distribution through Ord- Carver system.

Let X be an integer valued random variable with probability density function

$f(x) = \binom{n}{x} p^x\ q^{n-x}\ ,\ x = 0,1,2,...,n\quad ,\ 0<p<1$ take $g(x) = q(1+x)$ we have

$$\frac{f(x+1)-f(x)}{f(x)} = \frac{p}{q}\left(\frac{n-x}{x+1}\right) - 1 = -\frac{x+q-np}{q(1+x)}$$
$$= \frac{\mu-x}{g(x)} - \frac{g(x+1)-g(x)}{g(x)}$$

this gives that $b_0 = b_1 = q\ ,\ b_2 = 0\ ,\ d = q-np\ ,\ a_0 = a_1 = q\quad and\quad \mu = np$

Thus $E(X \mid X \geq x) = \mu + q(1+x)\lambda(x)$ and the mean residual life gives by

$$\mu(x) = E(X \mid X \geq x) = \mu + q(1+x)\lambda(x) - x$$

4.2: The Poisson distribution

Let X be an integer valued random variable with probability density function

$f(x) = \frac{\lambda^x\ e^{-\lambda}}{x!}\ ,\ x = 0,1,2,....,\ \lambda > 0.$ take $g(x) = x+1$ then

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{x+(1-\lambda)}{x+1}$$

this gives that $b_0 = b_1 = 1\ ,\ b_2 = 0\ ,\ d = 1-\lambda\ ,\ a_0 = a_1 = 1\quad and\quad \mu = \lambda$. Thus

$E(X \mid X \geq x) = \mu + (1+x)\lambda(x)$ and $\mu(x) = E(X \mid X \geq x) = \mu + (1+x)\lambda(x) - x.$

4.3: The Negative Binomial distribution

Let X be an integer valued random variable with probability density function

$$f(x)=\binom{x-1}{r-1}\ p^x\ q^{x-r}, x=r, r+1, \ldots .$$

Take $g(x)=\frac{1+x}{p}$ thus

$$\frac{f(x+1)-f(x)}{f(x)}=-\frac{\left(x+\frac{1-qr}{1-q}\right)}{\frac{x+1}{1-q}}$$

this gives that $b_0=b_1=\frac{1}{1-q}, b_2=0 \quad and \quad d=\frac{1-rq}{1-q}$

also $a_0=a_1=\frac{1}{1-q}\ , a_2=0\ \ and\ \ \mu=b_1-d=\frac{rq}{p}$

Thus $E(X \mid X \geq x)=\mu+\left(\frac{1+x}{p}\right)\lambda(x)$ and the mean residual life is given by

$$\mu(x)=\mu+\left(\frac{1+x}{p}\right)\lambda(x)-x.$$

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