SOME OTHER COMMUTATIVE - TRANSITIVE FINITE RINGS

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(Received on: 10-03-12; Accepted on: 28-03-12)

ABSTRACT

The ring $R$ is said to be commutative- transitive if for each $a, b, c \in R \setminus Z(R)$, $ab = ba$ and $bc = cb$ imply $ac = ca$. In this paper, we present other examples of commutative- transitive rings.

We show that a ring $R$ is commutative- transitive iff commutative graph $\mu(R)$ is a union of complete graphs. So we show non-commutative rings of order $p^4$ are commutative- transitive.

Keywords: transitive rings, commutative- transitive, centralizer,

INTRODUCTION

In reference [1], the structure of commutative- transitive finite rings has been described, it is shown in that paper that simple and commutative- transitive finite rings are fields or $2 \times 2$ matrices rings on fields or $\frac{R}{I[R]} = F_1 \times F_2$ for two fields and . Also, the structure of irreducible commutative- transitive ring were specified.

COMMUTATIVE - TRANSITIVE FINITE RINGS

Definition 1: Ring $R$ is said to be commutative-transitive if for each $a, b, c \in R$, $ab = ba$ and $bc = cb$, imply $ac = ca$ [1].

Theorem 1: The following conditions are equivalent for ring $R$ :

A) $R$ is commutative- transitive.  
B) For each $x, y \in R \setminus Z(R)$, if $xy = yx$, then $c(x) = c(y)$. 
C) The centralizers of all non central elements of $R$ are commutative.

Definition 2: Let $R$ is a ring.Commutative graph $\mu(R)$ as set of vertices $\mu(R)$ is all elements non central of $R$ and distinct vertices $a, b$ in $\mu(R)$ adjacent iff $ab = ba$.

Result 1: The ring $R$ is commutative-transitive iff commutative graph $\mu(R)$ is union of complete graphs.

Example 1: For an arbitrary field, the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in F \right\}$ is commutative -transitive. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a non-central elements of $R$. We determine the centralizer of $A$.

If $B = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \in C(A)$, Then $AB = BA$. So $(a-c)x = (x-z)b$ we consider the following cases:

Case 1: if $a = c$, then $C(A) = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \mid x, y \in F \right\}$. 

Case 2: If $a \neq c$, then $C(A) = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \begin{bmatrix} (x-z)b(a-c)^{-1} \\ 0 \\. \end{bmatrix} \mid x, z \in F \right\}$.

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In each case $C(A)$ is commutative. So $R$ is commutative-transitive. The commutative graph of $R$ is the union of $|F|+1$ complete graphs which an of them has $|F|^2 - |F|$ vertices.

GROUP RINGS

**Definition 3:** Let $G$ is a group and $R$ is a ring. Then $RG$ is defined as $RG = \{\sum_{g \in G} r_g g | r_g \in R\}$ in which $r_g = 0$, except for some finite numbers. In $RG$, addition and multiplication are defined naturally and distributively, respectively. $RG$ is called a group ring [3].

**Example 2:** Let $Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$ is an eight-element quaternion group. We show that the group ring of $Z_2Q_8$ is commutative-transitive. For simplicity, we put $1' = -1, i' = -i, j' = -j, k' = -k$. It is shown that:

$Z(Z_2Q_8) = \{a_{11} + a_{21'} + a_{3i} + a_{4i'} + a_{5(j+j')} + a_{6(k+k')} | a_{1...a_6} \in Z_2\}.$

Regarding the symmetry of $Z_2Q_8$ elements. We must find $c(i), c(i+j)$, and $c(i+j+k)$ to prove that centralizer is commutative for each non-central element. we can get with a little calculation

$c(i) = \{a_{11} + a_{21'} + a_{3i} + a_{4i'} + a_{5(j+j')} + a_{6(k+k')} | a_{1...a_6} \in Z_2\}.$

$c(i+j) = \{a_{11} + a_{21'} + a_{3i} + a_{4j} + (a_{5-4}i') + (a_{5-3})j' + a_{6(k+k')} | a_{1...a_6} \in Z_2\}.$

and $C(i+j+k) = \{a_{11} + a_{21'} + a_{3i} + a_{4j} + a_{5-4}i' + a_{6-4}j' + a_{6-3}k + a_{6-4}k' | a_{1...a_6} \in Z_2\}.$

As all above three centralizer are 6-dimensional and each contains a 5-dimensional subspace $Z(Z_2Q_8)$, they are commutative. Consequently group ring of $Z_2Q_8$ is commutative-transitive.

**Theorem 2:** If $R$ is a non-commutative ring with identity of order $p^4$, in which $p$ is prime number, then $R$ is commutative-transitive.

**Proof:** As $|R| = p^4$, char $R$ is a power of $P$. Therefore, elements of $0, 1, ..., p-1$ are distinct. For each $a \in R$, we have $|c(a)| > p$. so for $a \notin Z(R)$, $|c(a)| = p^2$ or $p^3$. We show that $c(a)$ is commutative.

**Case 1:** If $|c(a)| = p^2$, we know each ring with identity of order $p^2$ is commutative.

**Case 2:** If $|c(a)| = p^3$, we put $S = C(a)$ and as element $a, 0, ..., p-1$ are all in the center of $S$, we get $|Z(S)| > p$ Therefore, $|Z(S)| = p^2$ or $p^3$. If $|Z(S)| = p^3$, then $Z(S) = S$. If $|Z(S)| = p^2$ for each $b \notin Z(S)$. We have $Z(S) \subseteq C(b) \subseteq S$ which is impossible, because $|S| = p^4$ and $|Z(S)| = p^3$. The proof is complete.

**Example 3:** Let the following rings have 16 elements.

$R = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b \in F_4 \}, S = \{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} | a, b, c, d \in F_2 \}.$

Therefore, based on the above theorem, they are commutative-transitive and the commutative of $R$ is the union of one $K_6$ graph and four $K_2$ graphs and the commutative graph of $S$ is the union of three $K_4$ graphs.

**Note 1:** We have a single non commutative ring of $p^3$ order with identity the following ring:

$R = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in Z_p \}$ That is also commutative-transitive [2].

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SKEW POLYNOMIAL RING

**Definition 4:** If $R$ is a ring and $\sigma : R \rightarrow R$ is an endomorphism, let $R[x; \sigma]$ denote the ring of polynomials over $R$, that is, all formal polynomials in $x$ with coefficients from $R$ with multiplication defined by $xr = \sigma(r)x$ [4].

**Theorem 3:** Let $F$ is a field and $\sigma$ is an endomorphism of $F$. Then $R = \frac{F[x; \sigma]}{<x^2>}$ is commutative-transitive.
Proof: Let $Z(R) = \text{Fix}(\sigma) = K$. Now show centralizer of each non-central elements is commutative. Let $a+bx$ is a non-central element of $R$ and $a+\mu x \in c(a+bx)$

Case 1: If $\sigma(a) \neq a$ then $C(a+bx) = \left\{ \alpha + b\frac{\sigma(\alpha)-\alpha}{\sigma(\alpha)-a} x \mid \alpha \in F \right\}$ in this case centralizer is commutative.

Case 2: If $\sigma(a) = a$ in this case $\sigma(a) = a$ and $b \neq s$ so $C(a+bx) = \{ a + \mu x \mid \mu \in F, \alpha \in k \}$.

In this case centralizer is commutative. Therefore ring $R$ is commutative-transitive. Commutative graph of $R$ is the union of $|F|$ complete graphs which an of them has $|F| - |k|$ vertices and one complete graph with $|k| |F| - |k|$ vertices. In the following, we preset a simple and short proof of a Corollary (21) of [1].

Theorem 4: Let $R$ is a local ring and $R/J(R) \cong F_{p^r}$, in which $r$ is prime. If $J(R)$ is commutative, then $R$ is commutative-transitive.

Proof: Let $a$ is a non-central element of $R$, show $C(a)$ is commutative. Since $J(C(a)) = C(a) \cap J(R)$, so $Z_p \subseteq \frac{C(a)}{C(a) \cap J(R)} \cong \frac{R}{J(R)} \cong F_{p^r}$.

Since $r$ is prime so there is not any field between $Z_p$ and $F_{p^r}$. On the other hand $\frac{C(a)}{C(a) \cap J(R)}$ is a field since $C(a)$ is a local ring with maximal ideal of $C(a) \cap J(R)$.

Therefore, $\frac{C(a)}{C(a) \cap J(R)} \cong Z_p$ or $\frac{C(a)}{C(a) \cap J(R)} \cong F_{p^r}$.

Case 1: If $\frac{C(a)}{C(a) \cap J(R)} \cong Z_p$ then $C(a) = Z_p + (C(a) \cap J(R))$ so $C(a)$ is commutative.

Case 2: If $\frac{C(a)}{C(a) \cap J(R)} \cong F_{p^r}$, in this case $a \notin Z_p + J(R)$, since if $a \in Z_p + J(R)$ then $J(R)$ is commutative therefore $J(R) \subseteq C(a)$ since $\frac{C(a)}{J(R)} \cong F_{p^r}$ and $\frac{R}{J(R)} \cong F_{p^r}$ so $C(a) \cong R$ this meaning $\alpha$ is central which is a contradiction. As $a \in Z(C(a))$ so $\frac{Z(C(a))}{Z(C(a)) \cap J[R]} \cong Z_p$.

On the other hand $Z_p \subseteq \frac{Z(C(a))}{Z(C(a)) \cap J[R]} \cong \frac{Z(C(a)) + J[R]}{J[R]} \cong \frac{R}{J[R]} \cong F_{p^r}$.

Therefore $\frac{Z(C(a)) + J[R]}{J[R]} \cong F_{p^r}$, since $\frac{C(a) + J[R]}{J[R]} \cong F_{p^r}$ so $Z(C(a)) + J(R) = C(a) + J(R)$ then $C(a) = Z(C(a)) + (C(a) \cap J(R))$.

And as $J(a)$ is commutative so $C(a)$ is commutative. Therefore $R$ is commutative-transitive.

REFERENCES


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