THE DETERMINATION OF THE NUMBER OF DISTINCT FUZZY SUBGROUPS OF GROUP $\mathbb{Z}_{p_1p_2\ldots p_n}$ AND THE DIHEDRAL GROUP $D_{2p_1p_2\ldots p_n}$

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ABSTRACT

In this paper, we use the natural equivalence of fuzzy subgroups studied by Iranmanesh and Naraghi [3] to determine the number of distinct fuzzy subgroups of some finite groups. We focus on the determination of the number of distinct fuzzy subgroups of group $\mathbb{Z}_{p_1p_2\ldots p_n}$ and the dihedral group $D_{2p_1p_2\ldots p_n}$ using this equivalence relation.

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1. INTRODUCTION

Zadeh introduced the notion of fuzzy sets and fuzzy set operations, in his classic paper [15] of 1965. In an analogous application with groups, Rosenfeld [13] formulated the elements of a theory of fuzzy groups. One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid evolution in the last years. Many papers have treated the particular case of finite cyclic groups. Thus, in [8] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [11, 12, 14] deal with this number for cyclic groups of order $p^nq^m$ (p, q primes). In the present paper we establish the recurrence relation verified by the number of distinct fuzzy subgroups of group $\mathbb{Z}_{p_1p_2\ldots p_n}$ and the dihedral group $D_{2p_1p_2\ldots p_n}$ such that $p_1, p_2, \ldots, p_n$ are distinct primes.

2. PRELIMINARIES

First of all, we present some basic notions and results of fuzzy sub group theory (for more details, see [4, 7, and 3]).

The dihedral group of order $2n$, for $n \geq 2$, denoted by $D_{2n}$. A fuzzy sub set of a set $X$ is a mapping $\mu : X \to [0,1]$. Fuzzy subset $\mu$ of a group $G$ is called a fuzzy subgroup of $G$ if:

$$(G_1) \mu(xy) \geq \mu(x) \land \mu(y) \text{ for all } x, y \in G;
(G_2) \mu(x^{-1}) \geq \mu(x) \text{ for all } x \in G
$$

The set of all fuzzy subgroup of a group $G$ is denoted by $F(G)$.

Definition 2.1: Let $G$ be a group and $\mu \in F(G)$. The set of $\{x \in G \mid \mu(x) > 0\}$ is called the support of $\mu$ and denoted by $\text{supp} \mu$.

Let $G$ be a group and $\mu \in F(G)$. We shall write $\text{Im} \mu$ for the image set of $\mu$ and $F(\mu)$ for the family $\{\mu_t \mid t \in \text{Im} \mu\}$.
Theorem 2.2: Let G be a fuzzy group. If $\mu$ is a fuzzy subset of G, then $\mu \in F(G)$ if and only if for all $\mu_i \in F_\mu$, $\mu_i$ is a subgroup of G.

Let $F_\mu(G)$ be the set of all fuzzy subgroups $\mu$ of G such that $\mu(e) = 1$, and let $\sim_R$ be an equivalence relation on $F_\mu(G)$. We denote the set $\{\nu \in F_\mu(G) \mid \nu \sim_R \mu\}$ by $\mu / \sim_R$ and the set $\{\mu \in F_\mu(G) \mid \mu \sim_R \}$ by $F_\mu(G) / \sim_R$.

Definition 2.3: Let G be a group, and $\mu, \nu \in F_\mu(G)$. $\mu$ is equivalent to $\nu$, written as $\mu \sim \nu$ if
1. $\mu(x) > \nu(y) \iff \nu(x) > \nu(y)$ for all $x, y \in G$.
2. $\mu(x) = 0 \iff \nu(x) = 0$ for all $x \in G$.

The number of the equivalence classes $\sim$ on $F_\mu(G)$ is denoted by $s(G)$. We mean the number of distinct fuzzy subgroups of G is $s(G)$.

Theorem 2.4: Let G be a finite group. The number of distinct fuzzy subgroups of G such that their support is exactly equal to G is $s(G) + 1$.

Let G be a finite group. The number of distinct fuzzy subgroups of G such that their support is exactly equal to G is denoted by $s'(G)$.

Theorem 2.5: [3] Let G be a finite group. Then the number of distinct fuzzy subgroups of G such that their support is exactly a subgroup of G is $s(G) - 1$.

Theorem 2.6: [3] Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly equal to H is $s(H) + 1$.

Corollary 2.7: [3] Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly a subgroup of H is $s(H) - 1$.

Proposition 2.8: [6] Let $n \in N$. Then there are $2^{n-1} - 1$ distinct equivalence classes of fuzzy subgroups of $Z_{p^r}$.

3. THE NUMBER OF THE DISTINCT FUZZY SUBGROUPS OF THE ABELIANGROUP $Z_{p_1p_2...p_n}$

In this section, we characterize fuzzy subgroups of the abelian group $Z_{p_1p_2...p_n}$ such that $p_1, p_2, ..., p_n$ are distinct primes numbers ($n > 1$).

Proposition 3.1: Suppose that p and q are distinct primes. Then there are 11 distinct equivalence classes of fuzzy subgroups of $Z_{pq}$.

Proof: See Theorem 8.2.4 of [6].

Proposition 3.2: Suppose that p, q and r are distinct primes. Then there are 51 distinct equivalence classes of fuzzy subgroups of $Z_{pqr}$.

Proof: We know that $Z_{pqr}$ has the following maximal chains:

$$Z_{pqr} \supset Z_{pq} \supset Z_{pr} \supset \{0\}, Z_{pqr} \supset Z_{pq} \supset Z_{q} \supset \{0\}, Z_{pqr} \supset Z_{p} \supset Z_{q} \supset \{0\}, Z_{pqr} \supset Z_{p} \supset Z_{r} \supset \{0\}, Z_{pqr} \supset Z_{q} \supset Z_{r} \supset \{0\}, Z_{pqr} \supset Z_{q} \supset Z_{r} \supset \{0\}, Z_{pqr} \supset Z_{p} \supset Z_{r} \supset \{0\}, Z_{pqr} \supset Z_{p} \supset Z_{q} \supset \{0\}, Z_{pqr} \supset Z_{q} \supset Z_{r} \supset \{0\}, Z_{pqr} \supset Z_{p} \supset Z_{q} \supset \{0\}.$$

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All of subgroups of the group \( pqrZ \) are \( Z_{pq}, Z_{pr}, Z_{qr}, Z_p, Z_q, Z_r \) and \( \{0\} \). Thus

\[
s\left(\frac{G-1}{2}\right) = s^*\left(\{0\}\right) + s^*\left(Z_{pq}\right) + s^*\left(Z_{pr}\right) + s^*\left(Z_{qr}\right) + s^*\left(Z_p\right) + s^*\left(Z_q\right) + s^*\left(Z_r\right) ,
\]

therefore

\[
s\left(\frac{G-1}{2}\right) = 1 + 3s^*\left(Z_{pq}\right) + 3s^*\left(Z_p\right) = 1 + 3(6) + 3(2) = 2 .
\]

Hence

\[
s\left(G\right) = 51.
\]

**Theorem 3.3:** Suppose that \( p_1, p_2, \ldots, p_n \) are distinct primes. If \( G = Z_{p_1p_2\ldots p_n} \) and \( n > 1 \), then

\[
s\left(G\right) = \sum_{i=1}^{n} \binom{n}{i} s\left(\prod_{j=1}^{i} p_j\right) + 2^n + 1 .
\]

**Proof:** Denote \( \prod_{j=1}^{i} p_j \) as the number of subgroups of the group \( G \) as \( Z_n \). Therefore by theorem 2.5,

\[
s\left(\frac{G-1}{2}\right) = s^*\left(\{0\}\right) + \sum_{i=1}^{n} \binom{n}{i} s^*\left(Z_n\right) \]

and hence

\[
s\left(G\right) = 2\sum_{i=1}^{n} \binom{n}{i} s^*\left(Z_n\right) + 3 .
\]

By theorem 2.4,

\[
s\left(G\right) = \sum_{i=1}^{n} \binom{n}{i} s\left(Z_n\right) + 2^n + 1 .
\]

4. THE NUMBER OF DISTINCT FUZZY SUBGROUPS OF THE DIHEDRAL GROUP \( D_{2p_1p_2\ldots p_n} \)

In this section, we determine the number of distinct fuzzy subgroups of the dihedral group \( D_{2p_1p_2\ldots p_n} \) such that \( p_1, p_2, \ldots, p_n \) are odd distinct primes.

**Theorem 4.1:** Suppose that \( p \) is a prime and \( p \geq 3 \). If \( G \) is the dihedral group of order \( 2p \), then \( s(G) = 4p + 7 \).

**Proof:** We know that \( D_{2p} \) has the following maximal chains:

\( D_{2p} \supset Z_p \supset \{0\} \) and \( D_{2p} \supset Z_2 \supset \{0\} \) whose the number is \( p \). Now \( 2 \) is the number of distinct fuzzy subgroups whose support is \( Z_p \), \( 2p \) is the number of distinct fuzzy subgroups whose support is \( Z_2 \), and \( 2^p \) is the number of fuzzy subgroups whose support is \( \{0\} \). Therefore

\[
s\left(\frac{G-1}{2}\right) = 2p + 2 + 1 ,
\]

therefore

\[
s\left(G\right) = 4p + 7 .
\]

**Theorem 4.2:** Suppose that \( p \) and \( q \) are odd distinct primes. If \( G \) is the dihedral group of order \( 2pq \), then

\[
s\left(G\right) = 12pq + 8(p + q) + 23 .
\]

**Proof:** We know that \( D_{2pq} \) has the following maximal chains:

\( D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\} \) and \( D_{2pq} \supset D_{2p} \supset Z_p \supset \{0\} \) whose the number is \( pq \). Now \( 2pq \) is the number of the subgroups \( Z_q \) of the dihedral group \( D_{2pq} \) and the dihedral group \( D_{2pq} \) has just one subgroup as \( Z_p, Z_q, Z_p \). So that

\[
s\left(\frac{G-1}{2}\right) = s^*\left(\{0\}\right) + pq s^*\left(Z_p\right) + q s^*\left(D_{2q}\right) + p s^*\left(D_{2p}\right) + s^*\left(Z_p\right) + s^*\left(Z_q\right) .
\]

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Thus\[ s(G) = \frac{G - 1}{2} = 1 + 2pq + q^2(2q + 4) + p(q + 4) + 6 + 2 + 2, \]
therefore\[ s(G) = 12pq + 8(p + q) + 23. \]

**Table 1:** The number of distinct fuzzy subgroups of dihedral group $D_{2pq}$ for some selected primes.

<table>
<thead>
<tr>
<th>$G = D_{2pq}$</th>
<th>s(G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 3, q = 5$</td>
<td>267</td>
</tr>
<tr>
<td>$p = 3, q = 7$</td>
<td>355</td>
</tr>
<tr>
<td>$p = 3, q = 11$</td>
<td>531</td>
</tr>
<tr>
<td>$p = 3, q = 13$</td>
<td>619</td>
</tr>
<tr>
<td>$p = 13, q = 17$</td>
<td>2915</td>
</tr>
</tbody>
</table>

**Theorem 4.3:** Suppose that $p$, $q$, and $r$ are odd distinct primes. If $G$ is the dihedral group of order $2pqr$, then\[ s(G) = 52pq + 24(pq + pr + qr) + 24(p + q + r) + 103. \]

**Proof:** We have
\[ D_{2z} = \langle x, y \mid x^z = y^2 = 1, yxy = x^{-1} \rangle. \]

It is well known that for every divisor $r$ of $n$, $D_{2z}$ possesses a subgroup isomorphic to $Z_r$, namely $H_r^r = \langle x^\frac{n}{r} \rangle > x^\frac{1}{r}$ and $\frac{n}{r}$ subgroups isomorphic to $D_{2z}$, namely $H_i^r = \langle x^\frac{n}{r} \rangle > x^\frac{1}{r}$, $i = 1, 2, ..., \frac{n}{r}$. We know that $D_{2pq}$ has the following maximal chains each of which can be identified with the Chain,

\[
D_{2pq} \supset D_{2pq} \supset D_{2p} \supset Z_p \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2pr} \supset Z_p \supset \{0\},
\]

\[
D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\},
\]

\[
D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\},
\]

\[
D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\},
\]

\[
D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\},
\]

\[
D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}, D_{2pq} \supset D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}.
\]

Clearly, $pq$ is the number of subgroups $Z_q$ of the dihedral group $D_{2pq}$, $qr$ is the number of the subgroups $D_{2r}$, pr is the number of the subgroups $D_{2p}$, $pq$ is the number of the subgroups $D_{2q}$ and $p$ is the number of the subgroups $D_{2q}$, q is the number of the subgroups $D_{2q}$, and $r$ is the number of the subgroups $D_{2q}$ of the dihedral group $D_{2pq}$ and the dihedral group $D_{2pq}$ has just one subgroup as $Z_{pq}$, $Z_{pq}$, $Z_{pq}$, $Z_{pq}$, $Z_{pq}$, $Z_q$, $Z_p$, $Z_r$. So that

\[
\frac{s(G)-1}{2} = s\left(\{0\}\right) + p\left(s\left(Z_q\right) + s\left(Z_p\right) + s\left(Z_r\right) + s\left(Z_{pq}\right) + s\left(Z_{pr}\right) + s\left(Z_{qr}\right) + s\left(Z_{pq}\right)\right) + q\left(s\left(D_{2p}\right) + pqs\left(D_{2q}\right) + s\left(D_{2q}\right) + s\left(D_{2pr}\right) + s\left(D_{2qr}\right)\right) + r\left(s\left(D_{2pq}\right) + s\left(D_{2pr}\right) + s\left(D_{2qr}\right)\right),
\]

Therefore

\[
\frac{s(G)-1}{2} = 1 + 2pq + 3\left(2Z_{pq}\right) + 3\left(6\right) + 26 + qr\left(2p + 4\right) + pr\left(2q + 4\right) + pq\left(2r + 4\right)
\]

\[
\frac{12pr + 8\left(p + r\right)}{2} + \frac{12pq + 8\left(p + q\right)}{2} + \frac{12qr + 8\left(q + r\right)}{2}.
\]

Thus

\[
s(G) = 52pq + 24\left(p + pr + qr\right) + 24\left(p + q + r\right) + 103.
\]
Table 2: The number of distinct fuzzy subgroups the dihedral group $D_{2pqr}$ for some selected primes.

<table>
<thead>
<tr>
<th>$G = D_{2pqr}$</th>
<th>$s(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 3, q = 5, r = 7$</td>
<td>7627</td>
</tr>
<tr>
<td>$p = 3, q = 7, r = 11$</td>
<td>15763</td>
</tr>
<tr>
<td>$p = 5, q = 7, r = 13$</td>
<td>28947</td>
</tr>
<tr>
<td>$p = 7, q = 13, r = 17$</td>
<td>91779</td>
</tr>
<tr>
<td>$p = 13, q = 17, r = 19$</td>
<td>238611</td>
</tr>
</tbody>
</table>

Theorem 4.4: Suppose that $p_1, p_2, \ldots, p_n$ are odd distinct primes and $P = 2 \times p_1 \times p_2 \times \ldots \times p_n$. If $G = D_2$, and $n > 1$, then

$$s(G) = 2P + \sum_{i=1}^{n} \left( \binom{n}{i} s(Z_{\prod p_i}) \right) + \frac{P}{2} \sum_{i|p, 2 \nmid p} s(D_{2p}) + \frac{P}{2} (2^{n-1} - 3) + 2^n + 2.$$

Proof: We have

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, yxy = x^{-1} \rangle.$$  

It is well known that for every divisor $r$ of $n$, $D_{2n}$ possesses a subgroup isomorphic to $Z_r$, namely $H_{2n}^r = \langle x^{\frac{n}{r}} \rangle$ and $\frac{n}{r}$ subgroups isomorphic to $D_{2r}$, namely $H_{2n}^r = \langle x^\frac{n}{r}, x^{-1} y \rangle, i = 1, 2, \ldots, \frac{n}{r}$. Let

$$\Pi = \left\{ p_1 \times \ldots \times p_i, i_1, \ldots, i_k \in \{1, \ldots, n\}, i_1 < \ldots < i_k \right\}, k = 1, 2, \ldots, n.$$

We know that $G = D_2$ has the following maximal chains each be identified with the chain

$$D_{2n} \supset D_{2r_1} \supset \ldots \supset D_{2r_k} \supset Z_{\Pi} \supset \{0\},$$
$$D_{2n} \supset Z_{\Pi} \supset \ldots \supset Z_{\Pi} \supset \{0\},$$
$$D_{2n} \supset D_{2r_1} \supset \ldots \supset D_{2r_k} \supset Z_2 \supset \{0\},$$

such that $\Pi \subseteq \Pi_i$ for all $i \in \{1, \ldots, n\}$. Now $\frac{P}{2}$ is the number of subgroups of the group $G$ as $Z_2$, and for all $i = 1, \ldots, n$, \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) is the number of subgroups of the group $G$ as $Z_2$. Also $\frac{P}{2t}$ is the number of subgroups of the group $G$ as $D_{2t}$, for every divisor $t$ of $\frac{P}{2}$. Therefore by theorem 2.5,

$$s(G) = 2P + \sum_{i=1}^{n} \left( \binom{n}{i} s(Z_{\Pi}) \right) + \frac{P}{2} \sum_{i|p, 2 \nmid p} s(D_{2p}) + \frac{P}{2} (2^{n-1} - 3) + 2^n + 2.$$

REFERENCES


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