ON THE PARTIAL ORDERING OF RANGE HERMITIAN MATRICES

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ABSTRACT
For a given range Hermitian matrix B, conditions are obtained for all matrices A that lie below B (above B) \( A \leq B \) \( (A \geq B) \) to be range Hermitian under a given partial ordering on matrices. As an application, it is shown that the monotonicity of the constitutive operators in linear electro-mechanical systems having the same structure operator is preserved for the corresponding transfer impedances.

Key words: Hermitian matrix, Almost definite matrix, quasi positive definite matrix, Positive semi definite matrix, Hermitian positive semi definite matrix, Range Hermitian matrix.

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1 INTRODUCTION
Let \( C_{nxn} \) be the set of all complex matrices of order \( n \) and \( C_n \) be the set of all complex vectors. For \( A \in C_{nxn} \), let \( R(A), N(A), A^*, A^+, A^- \) and \( rk(A) \) be the range space, null space, conjugate transpose, Moore – Penrose inverse, generalized inverse \( (A' \) is a solution of the matrix equation \( AXA = A) \) and rank of \( A \) respectively. \( A \in C_{nxn} \) is said to be almost definite \( (a.d \ [3]) \) if for \( x \in C_n \), \( x^*Ax = 0 \) \( \Rightarrow Ax = 0. \) \( A \in C_{nxn} \) is said to be positive semi definite \( (p.s.d \ [6]) \) if \( \Re(x^*Ax) \geq 0 \) for \( x \in C_n \). If \( A \) is also Hermitian, then \( A \) is Hermitian positive semi definite \( (h.p.s.d) \) and is denoted as \( A \geq 0. \) A matrix \( A \in C_{nxn} \) is said to be Almost positive definite \( (a.p.d) \) if it is both a.d and p.s.d. Mitra and Puri \( [9] \) have introduced and developed the concept of quasi positive definite \( (q.p.d) \) matrix. \( A \in C_{nxn} \) is said to be q.p.d if \( A \) is p.s.d and \( \Re(x^*Ax) = 0 \) \( \Rightarrow Ax = 0 \) and they have proved that a q.p.d matrix is always a.p.d. For properties of a.d, a.p.d and q.p.d matrices, one may refer \( [3, 9] \). These special types of matrices are widely used in the study of electrical networks and in linear electromechanical systems. It was pointed out by Duffin and Morley \( [3] \) that the unique transfer impedance in a general linear electromechanical system exists for every structure operator if and only if the constitutive operator is a.d. For terminology and representation of a general linear electromechanical system by a pair of equations, one may refer \( [3] \).

For \( A, B \in C_{nxn} \), \( A \geq B \iff A – B \geq 0 \iff A – B \) is h.p.s.d matrix. In this section; conditions for all those matrices that lie below (or) above a given EP matrix relative to h.p.s.d ordering to be EP are determined. First we shall prove certain lemmas, which will simplify the proof of the main result.
Lemma 2.1: If $A \in \mathbb{C}^{n \times n}$ is EP, then $N(A) \subseteq N(Sym A)$, where $Sym A = \frac{1}{2}(A + A^*)$ is the symmetric part of $A$.

Proof: Since $A$ is EP, $N(A) = N(A^*)$. For $x \in \mathbb{C}^n$, $Ax = 0 \iff A^*x = 0$. Hence, $(Sym A)x = 0$. Thus $N(A) \subseteq N(Sym A)$.

Lemma 2.2: Let $A \in \mathbb{C}^{n \times n}$. Then $A$ is EP and $rk(A) = rk(Sym A) \iff N(A) = N(Sym A)$.

Proof: ($\Rightarrow$) Since $A$ is EP, by Lemma (2.1) $N(A) \subseteq N(Sym A)$ and together with $rk(A) = rk(Sym A)$ it follows that $N(A) = N(Sym A)$.

Conversely, if $N(A) = N(Sym A)$, then $rk(A) = rk(Sym A)$ automatically holds. To prove $A$ is EP,

if possible, let us assume the contrary, that is, for $0 \neq x \in \mathbb{C}^n$, $Ax = 0$ and $A^*x \neq 0$. Then, $(Sym A)x = \frac{1}{2}(Ax + A^*x) \neq 0$.

Hence $x \notin N(Sym A)$. This contradicts that, $N(A) = N(Sym A)$. Hence $A$ is EP.

Remark 2.3: In particular, for a q.p.d matrix $A$, by Lemma (2.8) in [9], the condition $rk(A) = rk(Sym A)$ automatically holds. Further, by Lemma (2.1) in [9], the q.p.d matrix $A$ is also EP.

Hence Lemma (2.2) reduces to the following:

Lemma 2.4: Let $A \in \mathbb{C}^{n \times n}$ be q.p.d. Then, $N(A) = N(Sym A)$.

Remark 2.5: We observe that, in Lemma(2.2), both the conditions on $A$ are essential. This is illustrated in the following example:

Example 2.1: Let us consider $B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \in \mathbb{C}^{2 \times 2}$. $B$ is EP, being nonsingular, $N(B) = \{0\} = N(B^*)$. For $B$, $Sym B = \frac{1}{2}(B + B^*) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Here, $rk(Sym B) = 1 \neq rk B$ and $N(Sym B) \neq N(B)$. Hence, the Lemma (2.2) fails.

Theorem 2.6: Let $A, B \in \mathbb{C}^{n \times n}$ such that $A \geq B$, then the following hold:

(i) If $B$ is EP and $N(A) \subseteq N(B)$, then $A$ is EP.

(ii) If $A$ is EP and $N(B) \subseteq N(A)$, then $B$ is EP.

Proof:

(i) Since $A \geq B$, $A - B \geq 0$. Hence $A - B$ is Hermitian. For any $x \in N(A)$, since $B$ is EP and $N(A) \subseteq N(B)$; $Ax = 0 \Rightarrow Bx = B^*x = 0 \Rightarrow A^*x = Ax = Bx + B^*x = 0$. Hence $N(A) \subseteq N(A^*)$. Since $rk(A) = rk(A^*)$, $N(A) = N(A^*)$. Thus $A$ is EP.

(ii) Can be proved in a similar manner.

Hence the Theorem.

Corollary 2.7: Let $A, B \in \mathbb{C}^{n \times n}$ such that $A \geq B$. If $B$ is EP, then $N(A) \subseteq N(B) \iff R(B) \subseteq R(A)$.

Proof: If $B$ is EP and $N(A) \subseteq N(B)$, then By Theorem (2.6) (i) $A$ is EP. Hence, $N(A) \subseteq N(B) \iff N(A^*) \subseteq N(B^*) \iff R(B) \subseteq R(A)$. Hence the corollary.

Remark 2.8: In particular, if $A \geq B$ and $B$ is a. p.d then $N(A) \subseteq N(B)$ automatically holds and corollary (2.7) reduces to Lemma (2) in [8].

Theorem 2.9: Let $A, B \in \mathbb{C}^{n \times n}$ such that $A \geq B$. If $B$ is EP and $N(A) \subseteq N(B)$, then the following are equivalent:

(i) $N(A) = N(Sym B)$

(ii) $R(A) = R(Sym B)$

(iii) $rk(A) = rk(Sym B)$
Proof: Since $B$ is EP and $N(A) \subseteq N(B)$, by Theorem (2.6) (i) $A$ is EP. Further, by Lemma (2.1), $N(B) \subseteq N(\text{Sym } B)$. Hence $N(A) \subseteq N(B) \subseteq N(\text{Sym } B)$. Then by (iii), $N(A) = N(\text{Sym } B)$. Thus (i) holds. (i) $\implies$ (iii) is trivial. Thus (i) $\iff$ (iii). The equivalence of (i) and (ii) follows from the fact that $A$ and $\text{sym } B$ are EP matrices. Hence the Theorem.

Theorem 2.10: Let $A, B \in \mathbb{C}^{n \times n}$ such that $A \geq B$. If $B$ is EP and $\text{Sym } B \geq 0$ then the following are equivalent:

(i) $\text{R}(\text{Sym } A) = \text{R}(\text{Sym } B)$
(ii) $(\text{Sym } B)^+ \geq (\text{Sym } A)^+$

Proof: (i) $\implies$ (ii): $A \geq B \implies A^* \geq B^* \implies A + A^* \geq B + B^* \implies \text{Sym } A \geq \text{Sym } B$. Thus $\text{Sym } A \geq \text{Sym } B \geq 0$. Now by Theorem (1) of [5], $\text{R}(\text{Sym } A) = \text{R}(\text{Sym } B) \iff (\text{Sym } B)^+ \geq (\text{Sym } A)^+$. Hence the Theorem.

Remark 2.11: In particular if $B$ is q.p.d, then the condition $N(A) \subseteq N(B)$ in Theorem (2.9) and $\text{Sym } B \geq 0$ in Theorem (2.10) automatically hold. Further by Lemma (2.4), $N(\text{Sym } B) = N(B)$. Hence, Theorem (2.9) and Theorem (2.10), reduce the following:

Corollary 2.12 (Theorem 2 in [8]): Let $A, B \in \mathbb{C}^{n \times n}$ such that $A \geq B$ and $B$ is q.p.d. Then the following are equivalent:

(i) $\text{R}(A) = \text{R}(B)$
(ii) $\text{rk}(A) = \text{rk}(B)$
(iii) $(\text{Sym } B)^+ \geq (\text{Sym } A)^+$

Remark 2.13: In particular if $A \geq B \geq 0$, then $\text{Sym } A = A$ and $\text{Sym } B = B$. Theorem (2.9) and Theorem (2.10) reduce to the following known results:

Corollary 2.14 (Theorem 1 in [5]): Let $A, B \in \mathbb{C}^{n \times n}$ such that $A \geq B \geq 0$ then $B^+ \geq A^+ \iff \text{R}(A) = \text{R}(B)$.

Corollary 2.15 (Theorem 1 in [4]): For $A, B \in \mathbb{C}^{n \times n}$ any two of the following conditions imply the other one.

(i) $A \geq B \geq 0$
(ii) $\text{rk}(A) = \text{rk}(B)$
(iii) $B^+ \geq A^+ \geq 0$.

Proof: (i) and (ii) $\implies$ (iii), (i) and (iii) $\implies$ (ii) follow from Theorem (2.10) using $\text{Sym } A = A$ and $\text{Sym } B = B$. The proof for (ii) and (iii) $\implies$ (i) runs as follows:

Since $\text{rk}A = \text{rk}A^+$ and $\text{rk}B = \text{rk}B^+$, $B^+ \geq A^+ \geq 0$ and $\text{rk}A^+ = \text{rk}B^+ \implies (A^+)^+ \geq (B^+)^+ \geq 0 \implies A \geq B \geq 0$. Thus (i) holds.

Hence the corollary.

Remark 2.16: We observe that in Theorem (2.9) the condition $N(A) \subseteq N(B)$ is essential. This is illustrated in the following:

Example 2.2: Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$. Here $A$ is not EP and $B$ is EP being nonsingular

$A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \implies A \geq B$

$N(A) = \{x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} / Ax = 0\}$. $N(A) \not\subseteq N(B)$.

$\text{Sym } B = \frac{1}{2}(B^* + B) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. $\text{rk}(\text{Sym } B) = 1 = \text{rk}A$

$N(\text{Sym } B) = \{x = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} / (\text{Sym } B)x = 0\}$.

Hence $N(A) \not= N(\text{Sym } B)$ but $\text{rk}(\text{Sym } B) = \text{rk}A$.

Thus in Theorem (2.9) statement (i) fails and statement (iii) holds. Thus the condition $N(A) \subseteq N(B)$ is essential in Theorem (2.9).
3. PARTIAL ORDERING ON BLOCK EP MATRICES

In this section, we shall discuss the h.p.s.d. orderings on EP block matrices, involving Schur complements. For a partitioned matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the matrix denoted as $M/A = D - CA^+$ B is called generalized Schur complement of $A$ in $M$ [2]. In our earlier work [7], we have determined conditions for a Schur complement in an EP matrix to be EP for the case when $\text{rk}(M) \neq \text{rk}(A)$. When $\text{rk}(M) = \text{rk}(A)$, $M/A = 0$.

**Lemma 3.1:** Let $H, K \in C_{n \times n}$ be p.s.d and EP such that $H \succeq -K$. Let $X$ and $Y$ be $n \times m$ matrices satisfying.

(3.1) $\text{N}(H) \subseteq \text{N}(X^*)$; $\text{N}(K) \subseteq \text{N}(Y^*)$ and

(3.2) $X^*H^+ = (H^+X)^*$; $Y^*K^+ = (K^+Y)^*$.

Then the following hold:

(i) There exist matrices $L$, $M \in C_{(n+m) \times (n+m)}$ such that both are p.s.d and EP.

(ii) $L + M \succeq 0$.

(iii) $L + M/H + K \succeq 0$.

**Proof:** Let us consider $L = \begin{bmatrix} H & X \\ X^* & X^*H^+X \end{bmatrix}$ and $M = \begin{bmatrix} K & Y \\ Y^* & Y^*K^+Y \end{bmatrix}$.

Since $H$ is EP, $\text{N}(H^*) = \text{N}(H) \subseteq \text{N}(X^*)$. Further the generalized Schur complement of $H$ in $L$, that is, $L/H = X^*H^+X - X^*H^+X = 0$. Hence, by Corollary under Theorem 1 of [2], $\text{rk}(L) = \text{rk}(H)$. By applying Theorem 3 of [7], using $H$ is EP and $X^*H^+ = (H^+X)^*$, we get $L$ is EP. Further, $L$ can be factorized as $L = P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P^*$, where $P = \begin{bmatrix} I \\ X^*H^+ \end{bmatrix}$.

Since $H$ is p.s.d, $L$ is also p.s.d. Thus $L$ is EP and p.s.d. Similarly we can see that $M$ is EP and p.s.d. Then (i) holds.

Since $L$ and $M$ are p.s.d, $L + M$ is p.s.d. Similarly we can see that $M$ is EP and p.s.d. Then (i) holds.

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**Theorem 3.2:** Let $H$ and $K$ be EP as well as p.s.d matrices of order $n$ such that $H \succeq -K$, partitioned in the form

$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ and $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ satisfying

(3.3) $\text{N}(H_{11}) \subseteq \text{N}(H_{21})$; $\text{N}(H_{11}/H_{21}) \subseteq \text{N}(H_{12})$

(3.4) $\text{N}(K_{11}) \subseteq \text{N}(K_{21})$; $\text{N}(K_{11}/K_{21}) \subseteq \text{N}(K_{12})$

(3.5) $H_{21}H_{11}^+ = (H_{11}H_{21})^* = (H_{11}H_{21}^+)^*$

(3.6) $K_{21}K_{11}^+ = (K_{11}K_{21})^* = (K_{11}K_{21}^+)^*$.

Then $H + K/ H_{11} + K_{11} \succeq H_{11}/H_{11} + K_{11}/K_{11} \geq 0$.

**Proof:** Since $H$ is EP satisfying (3.3), by Theorem 1 and Remark 2 of [7], $H_{11}$ and $H_{11}/H_{21}$ are EP. Similarly $K$ is EP satisfying (3.4) implies $K_{11}$ and $K_{11}/K_{21}$ are EP. Since $H \succeq -K$, $H_{11} \succeq -K_{11}$ and $H + K/H_{11} + K_{11} \succeq 0$ by result Albert[1]. By definition of generalized Schur complement [2], we have $H + K/H_{11} + K_{11} = H_{22} + K_{22} - (H_{21} + K_{21}) (H_{11} + K_{11})^+ (H_{12} + K_{12})$. By using (3.5), (3.6) and $H + K/H_{11} + K_{11}$ is Hermitian we get

$H + K/H_{11} + K_{11} \succeq H_{22} + K_{22} - (H_{21}H_{11} + K_{21}K_{11})^+$

$= (H_{22} - H_{21}H_{11} - K_{21}K_{11})^+$

$= (H_{22} - H_{21}H_{11} - K_{21}K_{11})^+$

$= H/H_{11} + K/K_{11}$. 

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Thus $H+K/H_{11}+K_{11} \geq H/H_{11} + K/K_{11}$. Since $H_{21}H_{11}^* = (H_{11}^*H_{12})^*$ and $K_{21}K_{11}^* = (K_{11}^*K_{12})^*$ by applying Theorem 3 of [7] for the p.s.d and EP matrices $H$ and $K$, we see that $H/H_{11}$ and $K/K_{11}$ are both EP and p.s.d. Since $H \geq -K$, $H+K$ is Hermitian. By using (3.5), $H_{21}H_{11}^*H_{12}$ is Hermitian and by using (3.6), $K_{21}K_{11}^*K_{12}$ is Hermitian. Hence, $H/H_{11} + K/K_{11} = H_{22}+K_{22} - H_{21}H_{11}^*H_{12} - K_{21}K_{11}^*K_{12}$ is Hermitian. Since $H/H_{11}$ and $K/K_{11}$ are p.s.d. $H/H_{11} + K/K_{11}$ is p.s.d. Hence, $H/H_{11} + K/K_{11}$ is Hermitian and p.s.d. Therefore $H/H_{11} + K/K_{11} \geq 0$. Thus $H+K/H_{11} + K_{11} \geq H/H_{11} + K/K_{11} \geq 0$. Hence the Theorem.

4. APPLICATION TO LINEAR ELECTROMECHANICAL SYSTEMS:

Let us consider two linear electromechanical system with constitutive operators $H$ and $K$ having the same structure operator $A$. For terminology and notation one may refer [3]. The transfer impedance $\psi(H)$ and $\psi(K)$ exist by Theorem 7 of [3].

$$\psi(H) = (A^+)^*(H_{22} - H_{21}H_{11}^*H_{12})A^+ = (A^+)^*(H/H_{11})A^+$$

$$\psi(K) = (A^+)^*(K_{22} - K_{21}K_{11}^*K_{12})A^+ = (A^+)^*(K/K_{11})A^+.$$

If we assume that the constitutive operators $H$ and $K$ satisfies (3.5) and (3.6) respectively and $H \geq -K$, then by Therorem (3.2), $H/H_{11} \geq -K/K_{11} \Rightarrow \psi(H) \geq -\psi(K)$. Thus the monotonicity of the constitutive operators is preserved for the corresponding transfer impedances.

CONCLUSION

We have extended matrix inequalities on a pair of h.p.s.d matrices in the references [1, 2, 4, 5] and on a.p.d matrices in [8,9] for a wider class of range Hermitian matrices.

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