A NOTE ON NORMAL IDEALS IN REGULAR NEAR RINGS

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(Received on: 18-04-12; Accepted on: 09-05-12)

ABSTRACT

In this paper we study and characterize the cohen-macaulay property of the Rees algebras of a normal ideal of regular local near ring in the 3-dimensional case by assuming the ideal is 4-generated, has height2, is u mixed and is generally a complete intersection.

Keywords: Ring, Near-ring, Regular near ring, Regular -local –ring, Regular local near ring, Rees algebra.

1. INTRODUCTION

This paper studies the depth of Rees algebras associated to normal ideals. If I is an ideal in a commutative Near ring N, the integral closure of I, denoted \( \overline{I} \), is the set of elements \( x \in N \) such that \( x \) satisfies an equation of the form

\[
x^n + a_1 x^{n-1} + \ldots + a_n = 0 \quad \text{where} \quad a_j \in I \quad \text{for} \quad 1 \leq j \leq n.
\]

If \( x \in \overline{I} \) we say \( x \) is integral over I. It is not difficult to prove that \( \overline{I} \) is an ideal. An ideal I of a commutative Noetherian near ring N is said to be normal if all of its powers are integrally closed. If N is an integrally closed domain then the normality of I is equivalent to the normality of the Rees algebra of N with respect to I, \( N[I] = \bigoplus_{n=0}^{\infty} I^n t^n \). This paper was originally motivated by trying to prove that normality of the Rees algebra implied the Cohen-Macaulay property of the Rees algebra for a particular class of ideals in regular local Near rings.

2. PRELIMINARIES

In this section we shall give the definitions and required examples related to the next section topics.

2.1 Definition: A non empty set R with two binary operations ‘+’ and ‘.’ is said to be a ring if i) (R, +) is a commutative ring; ii) (R, .) is a semi group; iii) Distributive laws hold good.

2.2 Example: The set of all integers modulo m under addition and multiplication modulo m is a ring.

2.3 Definition: A near ring N is said to be Regular ring if for each element \( x \in N \) then there exists an element \( y \in N \) such that \( x = xyx \).

2.4 Example: (i) \( M(\Gamma) \) and \( M_2(\Gamma) \) are regular rings (Beidleman (10) NR Text)
(ii) Constant rings
(iii) Direct sum and product of fields.

2.5. Definition: A nonempty set N is said to be a Right near-ring with two binary operations ‘+’ and ‘.’ If i) \( (N, +) \) is a group (not necessarily abelian)
ii) \( (N, .) \) is a semi group and
iii) \( (x + y)z = xz + yz \) for all \( x, y, z \in N \)

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2.6 Example: Let Z be the set of positive, negative integers with ‘0’, then (Z, +, .) is a near ring with usual addition and multiplication.

2.7 Definition: A near ring N is said to be Regular near ring if for each element x ∈ N then there exists an element y ∈ N such that x = xyx.

2.8 Example: (i) M (Г) and are regular near rings (Beidleman (10) NR Text)

2.9 Definition: Let (R, M, k) be a Noetherian local ring. (The notation means that the maximal ideal is M and the residue field is k = R/M.) If d is the dimension of R, then by the dimension theorem every generating set of M has at least d elements. If M does in fact have a generating set S of d elements, we say that R is regular and that S is a regular system of parameters.

2.10 Example: If R has dimension 0, then R is regular iff {0} is a maximal ideal, in other words, iff R is a field.

2.11 Definition: A Local near ring which satisfy regular property then it is called a regular local near ring.

2.12 Definition: The Rees algebra of an ideal I of N is a graded algebra satisfying R[t] = R[t^n] and I^n t^i = t^j for some n and i and j.

3. NON COHEN – MACAULAY REES ALGEBRAS

Given a three dimensional regular local near ring (N, m) and normal four – generated height two unmixed ideal I ofN, it is natural to ask about the Cohen – Macaulayness of the Rees algebra N[t]. Assume further that I is generically a complete intersection (that is, I_p is a complete intersection for every prime ideal P minimal over I). Our main result of this section will characterize, in terms of a presentation matrix of I, when N[t] is Cohen – Macaulay for such an ideal.

Instead we need only the weaker condition that (mI^2 : m) = I^2. If I^2 is integrally closed then it satisfies this condition. For if w ∈ (mI^2 : m) then w m ⊂ mI^2, thus w ∈ I^2 = I^2 by the determinant trick.

Theorem 3.1: Let (N, m) be a d-dimensional regular local near ring containing a field and I a height d – 1 unmixed ideal of N. Assume that I is generically a complete intersection, I has a d – generated reduction, µ(I) = d + 1, and I^2 m : m = I^2, but I^2 is not unmixed. Then there is a generating set {x_1, ..., x_d} for m, positive integers m ≥ 1, and a presentation matrix φ for I, such that I,φ = (x_1, ..., x_d, x^n) and

\[
\phi = \begin{pmatrix}
x_1 & x_2 & \cdots & x_d & x^m & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

Proof: By our assumption, m ∈ Ass(N/I^2). In particular, (I^2 : m) ≠ I^2. Choose an element e ∈ (I^2 : m) \ I^2. Because (mI^2 : m) = I^2, em ⊂ mI^2. Choose x ∈ m \ m^2 such that ex ∈ mI^2. Expand x = x_1 to a minimal generating set \{x_1, ..., x_d\} for m. Using that ex ∈ I^2 \ mI^2 we may assume that I = (p_1, ..., p_d) is a minimal generating set for I, and that ex = ap_1 + p for some p ∈ (p_2, ..., p_d)I and unit α. Set ex_1 = g_1 ∈ I^2. Consider the homomorphism

\[\psi : R[T_1, ..., T_{d+1}] → R[H]\]

Given by \[T_i → p_i t.\] Let F, G_i ∈ R[T_1, ..., T_{d+1}] be homogeneous of degree 2 (in the T_i’s) such that ψ(F) = pt^2, ψ(G_i) = g_i t^2. By our choice of p we can assume that F ∈ (T_2, ..., T_{d+1})R[T_1, ..., T_{d+1}]. Let Q be the kernel of ψ.
and let $Q_j$ denote the ideal generated by the homogeneous elements of $Q$ having degree at most $j$. The trivial relation 
$(ex)x_i = (ex)x$ forces $x_i(\alpha T_i^2 + P) - xG_i \in Q_2$. But $Q_2 = Q_1$ because $I$ is syzygetic [8], thus there are linear homogeneous polynomials $A_i \in R[T_1, \ldots, T_{d+1}]$ such that $x_i(\alpha T_i^2 + P) - xG_i = A_1L_1 + \ldots + A_dL_d$ where $Q_i = (L_1, \ldots, L_d)$.

The coefficient of $T_i^2$ must be $x_i - xb_i$ for some $b_i \in R$ and the coefficients of $L_i$ all lie in $m$, therefore $Q_i$ contains polynomials of the form $(x_i - xb_i)T_i + b_2T_2 + \ldots + b_dT_d$. The existence of these linear polynomials implies that $(x_i - xb_i, x) \in \langle (p_2, \ldots, p_{d+1}) : p_i \rangle$. Replacing $x_i$ with $x_i - xb_i$ yields that $x_i \in \langle (p_2, \ldots, p_{d+1}) : p_i \rangle$. Therefore invertible row and column operations yield that $\phi$ may be reduced to the form described in (1). In particular, $I_1(\phi) = (x_2, \ldots, x_d, x^m)$ for some $n \leq m$.

We are particularly interested in applying (3.1) to curves in $3$ – space, where several of our assumptions automatically are valid.

**Corollary 3.2.** Let $(N,m)$ be a $3$ – dimensional regular local near ring containing a field and $I$ a height $2$ unmixed ideal of $N$. Assume that $I$ is generically a complete intersection, $I$ has a $3$ – generated reduction, $\mu(I) = 4$, and $I^2m : m = I^2$. There is a generating set $\{x, y, z\}$ for $m$, positive integers $m \geq n$, and a presentation matrix $\phi$ for $I$,

$$I_1(\phi) = (y, z, x^n)$$

such that $I_1(\phi) = (y, z, x^n)$ and $\phi = \left[ \begin{array}{ccc} y & z & x^n \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \end{array} \right]$.

**Proof:** This follows at once from Proposition 3.1 as soon as we observe that the assumption that $I^2$ is not unmixed is automatic in this case we first analyze to have precisely three generators by the Hilbert – Burch theorem.

We recall Vasconcelos’ construction. That is, let $\phi$ be a $4 \times 3$ presentation matrix of $I = (p_1, p_2, p_3, p_4)$ and assume $I_1(\phi)$ the ideal generated by the entries of $\phi$ is a complete intersection, generated by $\{a, b, c\}$. Consider the homomorphism

$$\psi : \mathcal{N}[T_1, T_2, T_3] \rightarrow \mathcal{N}[I^r]$$

given by $T_i \rightarrow p_i$. The symmetric algebra of $I$ is a complete intersection whose defining ideal is generated by three elements $L_1, L_2, L_3$. These elements satisfy the matrix equation

$$(L_1, L_2, L_3) = (T_1, T_2, T_3) \phi$$

Build an associated matrix $B(\phi)$ via the matrix equation

$$3.3(T_1, T_2, T_3, T_4) \phi = (L_1, L_2, L_3) = (a, b, c) \cdot B(\phi)$$

We also define $C(\phi)$ to be the image of $B(\phi)$ in $k(T_1, T_2, T_3, T_4)$, i.e. the matrix $B(\phi)$ reduced modulo the maximal ideal of $N$.

**Remark 3.4:** We will need to perform various changes on the generators of $I$, the choice of presentation $\phi$, and the matrix $B(\phi)$. It is convenient to record exactly what changes we will use and how they affect the rest of the data.

First, consider column operations upon $B(\phi)$. Let $\phi$ be a $3$ by $3$ invertible matrix with coefficients in $R$. Column operations upon $B(\phi)$ coming from the base ring $R$ are obtained by replacing $B(\phi)$ by $B(\phi) \phi$. To preserve equation (3.3) we must also multiply $\phi$ by $\theta$ and then the corresponding equation (2.3) is valid. This only changes the generators for the syzygies of $I$, and not the chosen generating set. Otherwise stated, $B(\phi \phi) = B(\phi) \phi$. 

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Row operations on $B(\phi)$ coming from – Correspond to multiplying $B(\phi)$ by an invertible 3 by 3 matrix $\theta$ with coefficients in $R$ on the left side of $B(\phi)$. To insure that (3.3) continues to hold, we must then multiply the matrix $(a, b, c)$ by $\theta^{-1}$ on the right, basically changing the choice of generators for the ideal $(a, b, c)$.

Finally, we are free to change the chosen generators of $I$. In this case we replace $\phi$ by $\theta \phi$, with $\theta$ in this case a 4 by 4 invertible matrix with coefficients in $N$. If we change the corresponding $T_i$ by multiplying on the right by $\theta^{-1}$, then (3.3) is still valid without change to the complete intersection $a,b,c$ or the matrix $B(\phi)$.

Vasconcelos proved that if $I$ is a prime ideal such that $\det (C(\phi)) \neq 0$ then $N[I]$ is Cohen – Macaulay. Implicit in the work of Aberbach and Huneke is an improvement of this statement.

**Proposition 3.5:** Let $(N, m)$ be three – dimensional regular local Near ring containing a field of characteristic not 2 and $I$ a four – generated height tow unmixed ideal of $N$ which is generically a complete intersection. Assume further that $I$ has a three – generated minimal reduction (automatic if $N/m$ is infinite) and $I_1(\phi)$ is a complete intersection. Then $N[I]$ is Cohen – Macaulay if and only if $\det (C(\phi)) \neq 0$.

**Proof:** The proof of the forward direction is contained in the proof of the (1) implies (3) part of the argument given in [2]. Conversely, if $\det (B(\phi)) \notin mN[T_1, T_2, T_3, T_4]$ then we may assume (after changing the generators of $I$ if necessary) that $\det (B(\phi)) = T_1^3 + A$ for some homogeneous (in the $T^i$s) degree 3 polynomial $A$ contained in $(T_2, T_3, T_4)N[T_1, T_2, T_3, T_4]$. Because the elements $\psi(f), \psi(g)$ and $\psi(h)$ vanish, $\psi(\det (B(\phi)) = 0$. Hence $p_1^3 \in (p_2^2, p_3, p_4)I^2$ which means I has reduction number two, therefore $N[I]$ is Cohen – Macaulay by [3].

The following theorem is the key theorem of this paper upon which the examples are based.

**Theorem 3.6:** Let $(N,m)$ be a 3 – dimensional regular local ring containing a field of characteristic not 2 and $I$ a height 2 unmixed ideal of $N$. Assume that $I$ is generically a complete intersection, $I$ has a 3 – generated reduction, $\mu(I) = 4$, and $I^2 m : I = I^2$. Further assume that a generating set $\{x,y,z\}$ for $m$ and a presentation matrix $\phi$ for $I$ have been chosen as in (3.2). Then $R[I]$ is not $C – M$ if and only if $I_1(\phi) = (y, z, x^n) = (u,v,w)$ and there is a presentation matrix $\theta$ for $I$ of the form

$$\theta = \begin{pmatrix} v & w & 0 \\ u & 0 & w \\ 0 & u & -v \\ 0 & 0 & 0 \end{pmatrix} \mod mI_1(\theta)$$

**Proof:** If $\theta$ has the form described in (3.6) then it easy to see that $\det (C(\theta)) = 0$. Therefore $R[I]$ is not Cohen – Macaulay by Proposition 3.5.

Assume $R[I]$ is not Cohen – Macaulay and that $\phi$ is a presentation matrix for $I$ having the form prescribed in (3.2), that is

$$\phi = \begin{pmatrix} y & z & x^m \\ \ldots & \ldots & \ldots \end{pmatrix}$$

Where $m \geq n$ and $I_1(\phi) = (y, z, x^n)$. By using invertible row operations we may also assume that $m$ is the least power of $x$ appearing in the last column of $\phi$. We will analyze the matrix $B(\phi) = \begin{pmatrix} T_1 + A & B & C \\ D & T_1 + E & F \\ G & H & x^m – x^n(T_1 + K) \end{pmatrix}$.
Where $A$, $B$, $C \ldots K \in N[T_2, T_3, T_4]$. If $m = n$ the $\phi$ satisfies the row condition, leading to the contradiction that $N[I]$ is Cohen – Macaulay. Therefore we assume $m > n$. Let $C = C(\phi)$ denote the image of $B$ modulo $mN[T_1, T_2, T_3, T_4]$, and use lower-case letters to denote images modulo $mN[T_1, T_2, T_3, T_4]$. Then

$$C(\phi) = \begin{pmatrix} t_1 + a & b & c \\ d & t_1 + e & f \\ g & h & 0 \end{pmatrix}.$$ 

Here $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$ are linear forms. In addition, the fact that $I_I(\phi) = (y, z, x^n)$ implies that either $g \neq 0$ or $h \neq 0$ (else the least pure power of $x$ in $I_I(\phi)$ would be greater than $n$).

Since $R[I]$ is not Cohen – Macaulay the determinant of $C(\phi)$ is zero by Proposition 3.5, therefore

$$g(bf - ct_1 - ce) - h(ft_1 + af - cd) = 0 \tag{3.8}$$

We consider two cases; either $g$ and $h$ are relatively prime or not. First suppose that $g$ and $h$ are relatively prime. Because $a, b, c, d, e, f, g, h \in \mathbb{k}[t_1, t_2, t_3, t_4]$, (2.8) implies that $gc + hf = 0$. Therefore there exists an $\alpha \in \mathbb{k}$ such that $c = -\alpha h$ and $f = \alpha g$. Substituting into (3.7) yields

$$C(\phi) = \begin{pmatrix} t_1 + a & b & -\alpha h \\ d & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}.$$ 

Thus,

$$0 = \det(C) = \alpha(bg^2 + ghe - gha - dh^2)$$

Using that $g$ and $h$ are relatively prime we obtain that $h$ divides $b$, hence $b = \beta h$ for some $\beta \in \mathbb{k}$.

Substituting above and factoring out $h$ implies that

$$\beta g^2 + ge - ga - dh = 0$$

Therefore $g$ divides $d$, hence $d = \gamma g$ for some $\gamma \in \mathbb{k}$.

Substituting and factoring our $g$ implies that $\beta g + e - a - \gamma h$, therefore

$$e - \gamma h = a - \beta g$$

Further, after making the substitutions $b = \beta h$ and $d = \gamma g$, $C(\phi)$ takes the form

$$C(\phi) = \begin{pmatrix} t_1 + a & \beta h & -\alpha h \\ \gamma g & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}.$$ 

By using row operations $C(\phi)$ may be reduced to

$$C(\phi) = \begin{pmatrix} t_1 + a - \beta g & 0 & -h \\ 0 & t_1 + e - \gamma h & g \\ g & h & 0 \end{pmatrix}.$$
As in Remark 2.4, these row operations will change the choice of generators of the ideal \((y, z, x^n)\). Let us call the new generations \(u, v, w\). Set \(q = a - \beta g = e - \gamma h\). By the above calculation

\[ C(\phi) = \begin{pmatrix} t_i + q & 0 & -h \\ 0 & t_i + q & g \\ g & h & 0 \end{pmatrix}. \]

Note that \(\{t_i + q, g, h\}\) are independent linear forms in \(k[t_i, t_2, t_3, t_4]\) because \(g\) and \(h\) are relatively prime linear forms and do not involve \(t_i\). Therefore by changing variables we may replace \(t_i + q, g\) and \(h\) with \(t_1, t_2, t_3\) (respectively replace \(T_1 + Q\), \(G\) and \(H\) with \(T_1, T_2\) and \(T_3\)). As in Remark 2.4 this change also will change our original choice of generators for \(I\). Substituting this variable – change yields that

\[ C(\phi) = \begin{pmatrix} t_1 & 0 & -t_3 \\ 0 & t_1 & t_2 \\ t_2 & t_3 & 0 \end{pmatrix}. \]

And lifting back to \(N[T_1, T_2, T_3, T_4]\)

\[ B(\phi) = \begin{pmatrix} T_1 & 0 & -T_3 \\ 0 & T_1 & T_2 \\ T_2 & T_3 & 0 \end{pmatrix} \mod mN[T_1, ..., T_4]. \]

Finally, to see that the matrix \(\phi\) may be chose to have the form described in the statement of the theorem, use the matrix equation.

\[ B(\phi) = (T_1^* T_2^* T_3^* T_4^*) \phi \]

to re build \(\emptyset\).

This completes the proof of theorem 3.6 in the case that \(g\) and \(h\) are relatively prime.

**REFERENCES**


Source of support: Nil, Conflict of interest: None Declared

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