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A SURVEY ON THE STABILITY OF SOME FUNCTIONAL EQUATIONS

Renu Chugh[1], Sushma[2] and Ashish Kumar[3*]

Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, India

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ABSTRACT

The theory of functional equations is a vast area of the Non-linear analysis which is rather hard to explore. The stability problem of functional equations arose from a question of Ulam in 1940. In the recent decades, the stability problems of several functional equations have been investigated by many mathematicians. This paper presents a survey of the stability of some different type of functional equations in various spaces such as normed space, Banach space, Fuzzy space, Random normed space etc.

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1. INTRODUCTION

In mathematics, and particularly in functional analysis, a function is traditionally a map from a vector space to the field underlying the vector space, which is usually the real numbers. In other words, it is a function which takes for its input-argument a vector and returns a scalar. Commonly the vector space is a space of functions, thus the functional takes a function for its input-argument, and then it is sometimes considered a function of a function. Its use originates in the calculus of variations where one searches for a function which minimizes a certain functional. A particularly important application in physics is searching for a state of a system which minimizes the energy functional. The traditional usage also applies when one talks about a functional equation meaning an equation between functionals: an equation $F = G$ between functional can be read as an 'equation to solve', with solutions being themselves functions. In such equations there may be several sets of variable unknowns, like when it is said that an additive function 'f' is one satisfying the functional equation $f(x+y)=f(x)+f(y)$.

In other words, a functional equation is an equation whose variables are ranging over functions. Functional equations arose from applications in several disciplines like Physics, biology, economics, statistics, information theory, taxation and geometry. Hence, we are seeking all possible functions satisfying the equation.

The present paper contains a detailed study of functional equations and stability of functional equations. In Section 2, we study about the stability problem introduced by Ulam and solution of his question given by many mathematicians for Cauchy functional equations. In Section 3, we deal with Jensen type functional equations and their stability in various spaces such as normed space, Banach space, Fuzzy space, Random normed space, 2-metric space etc. In Section 4, we discuss the Quadratic functional equations and stability of the quadratic functional equations. Section 5, contains a detailed study of Cubic functional equations and stability of Cubic functional equations in various spaces. In Section 6, we present the results on the stability of different types of Quartic functional equations in various spaces. In the last Section we discuss pexiderized functional equations and their stabilities.

2. STABILITY OF CAUCHY FUNCTIONAL EQUATIONS

Let $f: R \to R$, where R is the set of real numbers, be a function satisfying the functional equation $f(x+y) = f(x)+f(y)$ for all $x, y \in R$. This functional equation is known as Cauchy functional equation and the function 'f' is called additive function. The Cauchy functional equation was first treated by A.M. Legendre (1791) and C.F. Gauss (1809) but A.L. Cauchy (1821) first found its general continuous solution. In this section we study the stability of Cauchy functional equations obtained by various researchers.

A classical question in the theory of functional equations is the following "When is it true that a function which

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Corresponding author: Ashish Kumar[3*]
Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, India

approximately satisfies a functional equation $\in$ must be close to an exact solution $\in$?" If the problem accepts a solution, we say that the equation $\in$ is stable. In 1940, S.M. Ulam [97] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphism:

Let $(G_1, *)$ be a group and $(G_2, \circ, d)$ be a metric group with the metric 'd'. Given $\in >0$, does there exists a $\delta_\in >0$ such that if a mapping $h:G_1 \to G_2$ satisfies the inequality $D(h(x*y), h(x)\circ h(y))< \delta_\in$ $\forall\, x, y \in G_1$, then there is a mapping $H:G_1 \to G_2$ such that for each $x, y\in G_1$, $H(x*y)=H(x)\circ H(y)$ and $d(h(x), H(x))< \in$?

In the next year, D.H. Hyers [30] gave answer to the above question for additive groups under the assumption that groups are Banach spaces.

Theorem 2.1[30]: Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $T:E_1 \to E_2$ satisfies the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all $x, y\in E_1$ and $\in >0$ is a constant. Then the limit $T(x) = \text{Lim}\ 2^{-n}f(2^n x)$ exists for each $x\in E_1$ and T is unique additive mapping satisfying $\|f(x)-T(x)\| \le \varepsilon$ for all $x\in E_1$. Also, if for each x the function $t \to f(tx)$ from R to E_2 is continuous at a single point of E_1, then T is continuous everywhere in E_1.

It is possible to prove a stability result similar to Hyers functions that do not have bounded Cauchy difference. T. Aoki (1950) [107] first generalized the Hyers theorem for unbounded Cauchy difference. In generalizing the definition of Hyers he proved the following result, when $f:E \to E^1$ is a mapping and E and E^1 are normed spaces.

Theorem 2.2[107]: Let f(x) from E to E^1 be an approximately linear transformation, when there exists $K \ge 0$ and $0\le p<1$ such that $\|f(x+y)-f(x)-f(y)\| \le K(\|x\|^p + \|y\|^p)$ for any x and y in E. Let f(x) and $\phi(x)$ be transformations from E to E^1. These are called near when there exists $K\ge 0$ and $0\le p<1$ such that $\|f(x)-\phi(x)\| \le K\|x\|^p$ for any x in E.

The same result was rediscovered by Th. M. Rassias [105]. In 1978, Th.M. Rassias [105] proved a generalization of Hyers theorem for additive mappings as a special case in the form of following:

Theorem 2.3[105]: Suppose that E and F are real normed spaces with F a complete normed space, $f: E \to F$ is a mapping such that for each fixed $x\in E$ the mapping $t \to f(tx)$ is continuous on R, and let there exist $\in \ge 0$ and $p\in [0,1)$ such that $\|f(x+y)-f(x)-f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$ for all $x, y\in E$. Then there exists a unique linear mapping $T:E \to F$ such that $\|f(x)-T(x)\| \le \varepsilon\|x\|^p/(1-2^{p-1})$ for all $x\in E$.

The case of the existence of unique additive mapping had been obtained by T. Aoki [107]. In 1982 J.M. Rassias [52], followed the innovative approach of Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^q$ with $p+q \ne 1$.

Remark 2.4[115]: If p=0, then Theorem 2.3 implies Theorem 2.2.

In 1990, Th.M.Rassias during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for value of p greater or equal to 1. In 1991, Gajda [111] provided an affirmative solution to Th.M. Rassias's question for 'p' strictly greater than one. He established the following result:

Theorem 2.5[115]: Let E_1 and E_2 be two (real) normed linear spaces and assume that E_2 is complete. Let $f: E_1\to E_2$ be a mapping for which there exist two constants $\varepsilon\in[0,\infty)$ and $p \in R\backslash\{1\}$ such that $\|f(x+y)-f(x)-f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$ for all $x, y\in E_1$. Then there exists a unique additive mapping $T: E_1\to E_2$ such that $\|f(x)-T(x)\| \le \delta\|x\|^p$ for all $x\in E_1$, where
$$\delta = \begin{cases} 2\varepsilon/(2-2^p) & \text{for} \quad p<1 \\ 2\varepsilon/(2^p-2) & \text{for} \quad p>1 \end{cases}$$
, Moreover, for each $x\in E_1$, the transformation $t\to f(tx)$ is continuous, then the mapping T is linear.

However, Gajda [115] and Th.M.Rassias and P.Semrl [106] independently showed that a similar result can not beobtained for p=1. They presented the following:

Remark 2.6: Theorem 2.3 holds for all $p\in R\setminus\{1\}$. Gajda [115] in 1991 gave an example to show that the Theorem 2.3 fails if p=1. Gajda [115] succeeded in constructing an example of a bounded continuous function g: $R\to R$ satisfying $|g(x+y)-g(x)-g(y)| \le |x|+|y|$ for all x, $y\in R$, with $\lim_{x\to 0} g(x)/x = \infty$.

Gajda's function 'g' behaves badly near 0 and he constructed the following function g:R→R. The function g which Gajda [115] constructed is the following. For a fixed $\theta>0$, let g: $R\to R$ by $g(x)=\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x)$ for all $x\in R$,

where the function $\phi: R\to R$ is given by
$$\phi(x)=\begin{cases} \frac{1}{6}\theta & if\ 1\le x<\infty \\ \frac{1}{6}\theta x & if\ -1<x<1 \\ -\frac{1}{6}\theta & if\ -\infty<x\le -1. \end{cases}$$

This construction shows that Theorem (2.3) is false for p=1, as we see in the following result:

Theorem 2.7[115]: The function 'g' defined above satisfies the inequality $|g(x+y)-g(x)-g(y)|\le \theta(|x|+|y|)$ for all x, $y\in R$. But there is no constant $\delta\in[0,\infty)$ and no additive function a: $R\to R$ satisfying the inequality $|g(x)-a(x)|\le \delta|x|$ for all x, $y\in R$.

In 1991, Z. Gajda [115] proved that Hyer's theorem holds for the class of all complex-valued functions defined on a semi-group (S, +) (not necessarily commutative) if for any f: $S\to C$ such that the set $\{f(x+y)-f(x)-f(y): x, y\in S\}$ is bounded, there exists an additive function a: $S\to C$ for which the function f–a is bounded. In 1986, L. Szekelyhidi [67] has proved that the validity of Hyer's theorem for the class of complex-valued functions on S implies its validity for the functions mapping S into a semi-reflexive locally convex linear topological space X. Gajda [115] improved this result by assuming sequential completeness of the space X instead of its semi-reflexiveness. His assumption on X is essentially weaker than that of Szekelyhidi [67] and the result is as follows:

Theorem 2.8[67]: Suppose that Hyer's theorem holds for the class of all complex-valued functions on a semi-group (S, +) and let X be a sequentially complete locally convex linear topological (Hausdroff) space. If F: $S\to X$ is a function for which the mapping $(x, y)\to F(x+y)-F(x)-F(y)$ is bounded, then there exists an additive function A:$S\to X$ such that F-A is bounded.

In 1994, P. Gavruta [84] provided a further generalization of Th.M. Rassias [105] theorem in which he replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ by a general control function $\phi(x,y)$ for the existence of a unique linear mapping in the following way:

Theorem 2.9[84]: Let E be a abelian group, F be a Banach space and let $\phi: E\times E\to[0,+\infty)$ be a function satisfying $\phi(x,y)=\sum_{k=0}^{+\infty}\frac{1}{2^{k+1}}\phi(2^k x, 2^k y)<+\infty$ for all x, $y\in E$. If a function f: $E\to F$ satisfies the functional inequality $\|f(x+y)-f(x)-f(y)\|\le\phi(x,y)$ for all x, $y\in E$. Then there exists a unique additive mapping T: $E\to F$ which satisfies $\|f(x)-T(x)\|\le\phi(x,y)$ for all $x\in E$. If moreover f(tx) is continuous in t for fixed $x\in E$, then T is linear. □

B. Belaid and E. Elhoucien[14] generalized the results obtained by Hyers[30], Th.M Rassias[105] and P. Gavruta[84] for the Cauchy linear functional equation $f(x+y+a) = f(x) + f(y)$ for all x, $y\in E_1$ and 'a' is an arbitrary element in E_1 linear space. A particular case of this functional equation is: $f(x+y) = f(x) + f(y)$ for all x, $y\in E_1$ .They proved the stability problem in the sense Hyers-Ulam, Th. M. Rassias and P. Gavruta for the Cauchy linear functional equation. The following results obtained here extend the ones obtained by D. H. Hyers[30], Th.M. Rassias[105] and P. Gavruta[84].

Theorem 2.10[14]: Let G be an abelian Group and E be a Banach space. If a function f: G→E satisfies the functional inequality $\|f(x+y+a)-f(x)-f(y)\| \le \delta$, x, y∈G for some δ>0, then the limit

$T(x) = \lim_{n\to\infty} f(2^n x + (2^n - 1)a)/2^n$ exists for all x∈G and T: G→E is the unique function such that T(x+y+a)=T(x)+T(y) and $\|f(x)-T(x)\| \le \delta$ for any x, y∈G.

Theorem 2.11[14]: Let G be a normed space and E be a Banach space. If a function f: G→E satisfies the functional inequality $\|f(x+y+a)-f(x)-f(y)\| \le \theta(\|x\|^p + \|y\|^p)$ x, y∈G for some θ>0 and p∈[0,1), then there exists a unique function T: G→E such that T(x+y+a) = T(x)+T(y) and $\|f(x)-T(x)\| \le \theta\sum_{k=0}^{+\infty} 2^{k(p-1)} \|x+(1-1/2^k)a\|^p$ for all x, y∈G.

Theorem 2.12[14]: Let G be an abelian Group, E be a Banach space and let $\phi: G\times G \to [0,+\infty)$ be a function satisfying

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k y - 1)a, 2^k + (2^k - 1)a) < +\infty$$

for all x, y∈G. If a function f:G→E satisfies the functional inequality

$$\|f(x+y+a)-f(x)-f(y)\| \le \phi(x,y) \text{ for all x, y}\in G,$$

then there exists a unique function T: G→E such that T(x+y+a) = T(x) + T(y) and

$$\|f(x)-T(x)\| \le \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k - 1)a, 2^k x + (2^k - 1)a) \text{ for all x, y}\in G.$$

In 2009, C. Park and Th. M. Rassias [19] proved Hyers-Ulam stability of homomorphisms in Banach algebras for the mapping f:A→B where A and B are Complex Banach algebras which satisfies the functional equation μf(x+y) = f(μx)+f(μy) for all $\mu\in T^1 = \{v\in C:|v|=1\}$ for all x, y∈A and C – linear mapping (i.e. A C- linear mapping H:A→B is called a homomorphism in Banach algebra if H satisfies H(xy) = H(x) H(y) for all x, y∈A). They also obtained the Hyers-Ulam-Rassias stability of derivations on Banach algebra for the Cauchy functional equation. The results are as follows:

Theorem 2.13[19]: Let 'f' be a mapping from a complex Banach algebra 'A' with a norm $\|.\|_A$ into a Complex Banach algebra 'B' with norm $\|.\|_B$ for which there exists a function $\varphi: A^2 \to [0,\infty)$ such that $\lim_{j\to\infty} 2^{-j}\varphi(2^j x, 2^j y) = 0$, $\|\mu f(x+y) - f(\mu x) - f(\mu y)\|_B \le \varphi(x,y)$ and $\|f(xy) - f(x)f(y)\|_B \le \varphi(x,y)$ for all x, y∈A, $\mu\in T^1 = \{v\in C:|v|=1\}$ where C is a linear mapping. If there exists an L<1 such that $\varphi(x,x) \le 2L\varphi(x/2, x/2)$ for all x∈A, then there exists a unique homomorphism H:A→B such that $\|f(x)-H(x)\|_B \le \varphi(x,x)/(2-2L)$ for all x∈A.

Theorem 2.14[19]: Let f:A→A be a mapping for which there exists a function $\varphi: A^2 \to [0,\infty)$ such that

$\lim_{j\to\infty} 2^{-j}\varphi(2^j x, 2^j y) = 0$, $\|\mu f(x+y) - f(\mu x) - f(\mu y)\|_B \le \varphi(x,y)$

and $\|f(xy) - f(x)y - xf(y)\|_B \le \varphi(x,y)$ for all $\mu\in T^1$. If there exists an L<1 such that

$$\varphi(x,x) \le 2L\varphi(x/2, x/2) \text{ for all x}\in A,$$

then there exists a unique derivation δ: A→A such that $\|f(x)-\delta(x)\|_A \le \varphi(x,x)/(2-2L)$ for all x∈A.

J. R. Lee and C. Park [53], presented a further generalization of C. Park and Th. M. Rassais [19] result for the Cauchy functional equation $\mu f(x+y+z) = f(\mu x)+f(\mu y)+f(\mu z)$ and proved the stability in the spirit of Hyers, Ulam and Rassias using the fixed point alternative theorem. They obtained the following results of homomorphisms and of derivations on Banach algebras.

Theorem 2.15[53]: Let f: A→B, where A and B are Complex Banach algebras, be a mapping with norms $\|.\|_A$ and $\|.\|_B$ respectively for which there exists a function $\varphi : A^3 \to [0,\infty)$ such that

$$\|\mu f(x+y+z) - f(\mu x) - f(\mu y) - f(\mu z)\|_B \le \varphi(x,y,z),$$

$$\|f(xy) - f(x)f(y)\|_B \le \varphi(x,y,0)$$

for all x, y, z∈A, $\mu\in T^1 = \{v\in C: |v|=1\}$, where C is a linear mapping. If there exists an L<1 such that $\varphi(x,x,x) \le 3L\varphi(x/3, x/3, x/3)$ for all x∈A, then there exists a unique homomorphism H:A→B such that $\|f(x) - H(x)\|_B \le \varphi(x,x,x)/(3-3L)$ for all x∈A.

Theorem 2.16[53]: Let f: A→A, where A is a complex Banach Algebra, be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ such that

$$\|\mu f(x+y+z) - f(\mu x) - f(\mu y) - f(\mu z)\|_A \le \varphi(x,y,z),$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \varphi(x,y,0)$$

for all x, y, z∈A and $\mu\in T^1$. If there exists an L<1 such that $\varphi(x,x,x) \le 3L\varphi(x/3, y/3, z/3)$ for all x∈A, then there exists a unique derivation δ: A→A such that $\|f(x) - \delta(x)\|_A \le \varphi(x,x,x)/(3-3L)$ for all x∈A.

The functional equation $f(\alpha x+\beta y) = \alpha f(x) + \beta f(y)$ is called generalized Cauchy functional equation where α and β are fixed non-zero real numbers. Abbas Najati and Asghar Rahimi [7] introduced the above functional equation and proved the Hyers-Ulam-Rassias stability of A-linear mapping in Banach A-module. An additive mapping T: X→Y, where X and Y are left Banach A-modules is called A-linear if T(ax) = aT(x) for all 'a' in C^*-algebra and all x∈X. The result is as follows:

Theorem 2.17[7]: Let f: X→Y be a mapping with f(0)=0 for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that

$$\lim_{n\to\infty} 2^{-n}\varphi(x/2^n, y/2^n) = 0, \ \|f(\alpha x + \beta a y) - \alpha f(x) - \beta a f(y)\| \le \varphi(x,y),$$

for all x, y∈X and all 'a' in unitary group U(A), where A is C^*-algebra. If there exists an L<1 such that the function $x \mapsto \psi(x) = \varphi(x/2\alpha, x/2\beta) + \varphi(x/2\alpha, 0) + \varphi(0, x/2\beta)$ has the property $2\psi(x) \le L\psi(2x)$ for all x∈X, then there exists a unique A- linear mapping T: X→Y such that $\|f(x) - T(x)\| \le \psi(x)/(1-L)$ for all x∈X.

The first result on the stability of Cauchy functional equation f(x+y) = f(x)+f(y) in the setting of random normed spaces by using fixed point method has been proved by D. Mihet and V. Radu [29] in 2008 as follows:

Theorem 2.18[29]: Let X be a real linear space, 'f' be a mapping from X into a complete random normed space (Y, F, T_M) with f(0)=0 and let ϕ: $X^2 \to D_+$ be a symmetric mapping with the property $\alpha\in(0, 2)$ such that $\phi(2x, 2y)(\alpha t) \ge \phi(x, y)(t)$ for all x, y∈X and t>0. If $F_{f(x+y)-f(x)-f(y)} \ge \phi(x,y)$ for all x, y∈X, then there is a unique mapping g: X→Y such that $F_{g(x)-f(x)}(t) \ge \phi(x,x)((2-\alpha)t)$ ∀x∈X and t>0. Moreover $g(x) = \lim f(2^n x)/2^n$.

The Cauchy functional equation f(x+y) = f(x)+f(y) has numerous applications as to information theory and information measures, the problem of aggregated allocations, geometric objects, Hamel bases, harmonic analysis and stochastic processes etc. Later on many mathematicians such as Forti [46, 47], Hyers and Rassias [32] and Hyers, Isac and Rassias [31] developed the stability properties of the Cauchy functional equations. Further, the stability of Cauchy functional equations on various spaces such as Fuzzy normed spaces, Non−Archimedean normed spaces and Non−Archimedean fuzzy normed spaces have been obtained by various authors (see [2], [16], [34])

3. STABILITY OF JENSEN FUNCTIONAL EQUATIONS

In this section, first we introduce the convex function. In 1905, J.L. Jensen first introduced a function f: R→R which is said to be convex if and only if it satisfies the inequality f ((x+y)/2) ≤ (f(x) + f(y))/2 for all x, y∈R. The functions

satisfying the above inequality were established by Hadmard in 1893 and Holder in 1889. The functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2} \text{ for all x, y}\in R$$

is called the Jensen functional equation. Since the function f: R→R satisfies this above equation if and only if f(x) = A(x) + a, where A: R→R is a additive function and 'a' is an arbitrary constant. By replacing x and y with x+y and x–y respectively the above Jensen functional equation can be written as f(x+y)+f(x–y) = 2f(x). The first result on the stability of Jensen's equation was obtained by Z. Kominek [116] in 1989. He established the following:

Theorem 3.1[105]: Let D be a subset of R^n with non-empty interior. Assume that there exists an x_0 in the interior of D such that $D_0 = D - x_0$ satisfies the condition $(1/2)D_0 \subset D_0$. Let a mapping f: $D \to Y$ satisfy the inequality $\|2f((x+y)/2) - f(x) - f(y)\| \le \delta$ for some $\delta \ge 0$ and for all x, $y \in D$, then there exists a mapping F: $R^n \to Y$ and a constant K>0 such that $2F((x+y)/2) = F(x) + F(y)$ for all x, $y \in R^n$ and $\|f(x)-F(x)\| \le K$ for all $x \in D$.

J. C. Parnami and H. L. Vasudeva [54] further extended the result of Z. Komniek [116] and proved the following:

Theorem 3.2[54]: Let X, Y be a real normed space and a real Banach space, respectively. A mapping f: $X \to Y$ satisfying f(0) = 0 is a solution of the Jensen functional equation $2f((x+y)/2) = f(x) + f(y)$ if and only if it satisfies the additive Cauchy equation f(x+y) = f(x)+f(y). Hence, the most general continuous solution of Jensen's equation in R is f(x) = ax+b, where 'a' and 'b' are arbitrary constants.

In 1998, S.M. Jung [95] generalized the Z. Kominek's [116] result and using the ideas from the papers of Th. M. Rassais [105] and D.H. Hyers [30] investigated the Hyers–Ulam– Rassias stability of Jensen equation and its applications i.e. the stability of the Jensen functional equation by replacing 'δ' with $\delta + \theta(\|x\|^p + \|y\|^p)$ for the case p≥0(p≠1). Further, the result is applied to the study of an asymptotic behavior of the additive mappings: more precisely the following asymptotic property shall be proved: Let X , Y be a real normed space and a real Banach space respectively. A mapping f:X→Y satisfying f(0) = 0 is additive if and only if $\|2f((x+y)/2) - f(x) - f(y)\| \le \delta$ for all x, y∈R and the following results were obtained:

Theorem 3.3[95]: Let p>0 be given with p≠1. Suppose a mapping f: X→Y satisfies the inequality

$$\|2f((x+y)/2) - f(x) - f(y)\| \le \delta + \theta(\|x\|^p + \|y\|^p) \text{ for all x, y}\in X.$$

Further, assume f(0)=0 and δ=0 for the case of p>1. Then there exists a unique additive mapping F:X→Y such that

$$\|f(x) - F(x)\| \le \delta + \|f(0)\| + \theta\|x\|^p / (2^{1-p} - 1) \quad \text{for } p<1$$
$$\|f(x) - F(x)\| \le \theta\|x\|^p\, 2^{p-1} / (2^{1-p} - 1) \quad \text{for } p>1 \text{ for all } x\in X$$

S.M.Jung [95] proved in the following theorem that the mapping constructed by Rassias and Semrl [106] serves as a counterexample to above Theorem (3.3) for the case p = 1. □

Theorem 3.4[95]: The continuous real-valued mapping defined by

$$f(x) = \begin{cases} x\log_2(x+1) & \text{for} \quad x \ge 0 \\ x\log_2|x-1| & \text{for} \quad x < 0 \end{cases}$$

satisfies the inequality $\|2f((x+y)/2)-f(x)-f(y)\| \le 2(\|x\|+\|y\|)$, for all x, y∈R, and the range of $\|f(x)-a(x)\| \le \|x\|$ for x≠0 is unbounded for each additive mapping a:R→R.

Theorem 3.5[95]: Let d>0 and δ≥0 be given. Assume that a mapping f:X→Y satisfies the functional inequality $\|2f((x+y)/2)-f(x)-f(y)\| \le \delta$, for all x, y∈X with $\|x\|+\|y\| \ge d$. Then there exists a unique additive mapping F:X→Y such that $\|f(x)-F(x)\| \le 5\delta+\|f(0)\|$ for all x∈X. □

In 2003, L. Cadariu and V. Radu [65] extended the S.M. Jung [95] result and using the fixed point approach proved the Hyers–Ulam–Rassias stability of Jensen type functional equation. They proved the following result:

Theorem 3.6[69]: Let E be a (real or complex) linear space, F be a Banach space and $q_i = \begin{cases} 2 & i=0 \\ 1/2 & i=1 \end{cases}$. Suppose that the mapping f: E→F satisfies the condition f(0)=0 and an inequality of the form $\|2f((x+y)/2)-f(x)-f(y)\|_F \le \phi(x,y)$ for all x, y∈E, where φ: E×E→[0,∞) is a given function. If there exists L=L(i)<1 such that the mapping x→φ(x) = φ(x, 0) has the property $\phi(x) \le L.q_i.\phi(x/q_i)$ for all x∈E and the mapping φ has the property $\lim_{n\to\infty} \phi(2q_i^n x, 2q_i^n y)/2q_i^n = 0$ for all x, y∈E, then there exists a unique additive mapping j:E→F such that $\|f(x)-j(x)\|_F \le L^{1-i}\phi(x)/(1-L)$ for all x∈E. □

Several authors dealt about the stability of functional equations of various types. To cite some important references we refer to the work of Hyers [30], Hyers and Rassias [32], J.Parnami and H.L.Vasudeva [54], Rassias [105], T.M.Rassias and Semerl [106], S.M.Jung [95] and Ulam [97]. R.N.Mukherjee [88], in the setting of 2 – normed spaces extended the work of S.M. Jung [91] for Jensen's functional equations. In fact Mukherjee [88] investigated the Jensen's functional inequality of the following type:

$$\left\|\frac{1}{2}f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \le \delta+\theta\{\|x,z\|^p+\|y,z\|^p\} \quad (*)$$

where 'f' is a mapping between Banach spaces X and Y having 2–norm structure. Also z is a fixed element in X. In equation (*) p≥0 and p≠1. Moreover, a little modification of example in [30] shows that (*) is not stable for p=1. He obtained the following result:

Theorem 3.7[88]: Let p>0 and p≠1. Suppose 'f' is a mapping from X into Y such that X is a 2–normed space, Y is a Banach space. Let 'f' satisfy the inequality

$$\|2f((x+y)/2)-f(x)-f(y)\|_F \le \delta+\theta\{\|x,z\|^p+\|y,z\|^p\}$$

Also suppose that for p>1, δ>0 in above inequality. Further suppose that 'z' is not in the linear span of X. Then the following inequalities hold for an additive mapping F from X into Y.

$$\|f(x)-F(x)\| \le \delta+\|f(0)\|+\theta/(2^{1-p}-1)\{\|x,z\|^p\}, \qquad \text{for } p<1$$

or

$$\|f(x)-F(x)\| \le 2^{p-1}/(2^{1-p}-1)\{\|x,z\|^p\}, \quad \text{for } p>1, \text{ for all } x\in X.$$

Let X be an orthogonality space (see Rätz [58]) and Y be a real Banach space. A mapping f: X→Y is called orthogonally Jensen additive if it satisfies the so-called orthogonally Jensen additive functional equation

$$2(f(x+y)/2) = f(x) + f(y) \quad (**)$$

for all x, y∈X with x⊥y. Also the mapping f: X→Y is called orthogonally additive Jensen quadratic if it satisfies the so-called orthogonally Jensen quadratic functional equation

$$2(f((x+y)/2) + 2(f((x-y)/2) = f(x) + f(y) \quad (***)$$

for all x, y$\in$X and x$\perp$y. In 2008, Generalized Hyers–Ulam stability of Jensen functional equation (**) and Jensen quadratic functional equation (***) on orthogonal spaces were proved by C.Park and T.M. Rassias [18] as follows:

Theorem 3.8[18]: Let θ and p (p<1) be nonnegative real numbers. Suppose that f:X$\to$Y is a mapping with f(0)=0 fulfilling $\|2f((x+y)/2) - f(x) - f(y)\|_Y \le \theta(\|x\|_X^p + \|y\|_X^p)$ for all x, y$\in$X with x$\perp$y. Then there exists a unique orthogonally Jensen additive mapping T: X $\to$Y such that $\|f(x) - T(x)\| \le \theta\|x\|_X^p 2^p/(2-2^p)$ for all x$\in$X.

Theorem 3.9[18]: Let θ and p (p<2) be nonnegative real numbers. Suppose that f: X$\to$Y is a mapping with f(0)=0 fulfilling $\|2f((x+y)/2) + 2f((x-y)/2) - f(x) - f(y)\|_Y \le \theta(\|x\|_X^p + \|y\|_X^p)$ for all x, y$\in$X with x$\perp$y. Then there exists a unique orthogonally Jensen quadratic mapping T: X $\to$Y such that $\|f(x) - T(x)\| \le \theta\|x\|_X^p 2^p/(4-2^p)$ for all x$\in$X. □

V.A. Faiziev and P.K. Sahoo [108, 109] using new strategy established the stability of the Jensen type functional equation, namely $f(xy) + f(xy^{-1}) = 2f(y)$ on some classes of groups. They proved that any group 'A' can be embedded into some group G such that the functional equation is stable on G and also proved that the equation is also stable on some metabelian groups such as GL(n, C), SL(n, C) and T(n, C). Further, in 2007 they extend this result and proved the stability of Jensen functional equation, namely $f(xy) + f(xy^{-1}) = 2f(x)$ on some classes of non–commutative groups. The following results are established:

Theorem 3.10[108]: Let E_1 and E_2 be Banach spaces over reals. Then the equation $f(xy) + f(xy^{-1}) - 2f(y) = 0$ is stable for the pair (G; E_1) if and only if it is stable for the pair (G; E_2), where (G; E_1) and (G; E_2) are quasi-Jensen functions and G is a group.

Theorem 3.11[108]: The equation $f(xy) + f(xy^{-1}) - 2f(y) = 0$ is stable on any metabelian group.

Theorem 3.12[108]: Let G denote the group GL(n, C), SL(n, C) or T(n, C). Then the equation $f(xy) + f(xy^{-1}) - 2f(y) = 0$ is stable over G.

The stability result for Jensen type functional equations in the setting of intuitionistic fuzzy normed spaces was proved by S. Shakeri[96] in 2009. He gave the following result:

Theorem 3.13[96]: Let X be a linear space, $(Z, P_{\mu,\nu}^1, T)$ be an IFN-space, ϕ: X$\times$X$\to$Z be a function such that for some 0<α<2, $P_{\mu,\nu}^1(\phi(2x,2x),t) \ge_{L^*} P_{\mu,\nu}^1(\alpha\phi(x,x),t)$, for all x$\in$X, t>0 and $\lim_{n\to\infty} P_{\mu,\nu}^1(\phi(2^n x, 2^n x), 2^n t) = 1_{L^*}$ for all x, y$\in$X and t>0. Let $(Y, P_{\mu,\nu}, M)$ be a complete IFN-space. If f:X$\to$Y is a mapping such that

$$P_{\mu,\nu}(f(x+y) - f(x-y) - 2f(y), t) \ge_{L^*} P_{\mu,\nu}^1(\phi(x,y),t) \text{ for all x, y} \in \text{X, t>0 and f(0)=0.}$$

Then there exists a unique additive mapping A: X$\to$Y such that

$$P_{\mu,\nu}(f(x) - A(x), t) \ge_{L^*} P_{\mu,\nu}^1(\phi(x,y),(2-\alpha)t) \text{ for all x, y} \in \text{X.}$$

Further, a Generalized Hyers–Ulam–Rassias stability theorem in a Serstnev Probabilistic normed space (PN–space) was proved by M.E. Gardji, M. B. Ghaemi, H. Majanl and C. Park [77]. They introduced the notion of approximate Jensen mapping in PN-space and proved that if an approximate Jensen mapping in a Serstnezv PN-space is continuous at a point then we can approximate it by an everywhere continuous Jensen mapping. They also proved that if every approximate Jensen type mapping from the natural numbers into a Serstnev PN-space can be approximated by an additive mapping, then the norm of Serstnev PN-space is complete.

Theorem 3.14[77]: Let X be a real linear space and let 'f' be a mapping from X into a Serstnev Menger probabilistic Banach space (Y, ν, Π_M) such that f(0)=0. Suppose that ϕ is a mapping from X into a Serstnev Menger probabilistic normed space (Z, ω, Π_M) such that $\nu(2f((x+y)/2) - f(x) - f(y))(t) \geq \Pi_M\{\omega(\phi(x)), \omega(\phi(y))\}(t)$ for all x, $y \in X-\{0\}$ and positive real number t. If $\phi(3x) = \alpha\phi(x)$ for some real number α with $0<|\alpha|<3$, then there is a unique additive mapping $T:X\to Y$ such that $T(x) = \lim_{n\to\infty} 3^{-n} f(3^n)$ and $\nu(T(x) - f(x)) \geq \varphi_x(t)$, where $\varphi_x(t) = \Pi_M\{\Pi_M\{\omega(\phi(x)), \omega(\phi(-x))\}, \Pi_M\{\omega(\phi(3x)), \omega(\phi(-x))\}\}(3t)$.

Further, two natural extensions of Jensen's functional equation on the real line are the equations $f(xy)+f(xy^{-1}) = 2f(x)$ and $f(xy)+f(y^{-1}x) = 2f(x)$, where 'f' is a mapping from a multiplication group G in to a abelian group H. C.T. Ng [25, 26, 27] has solved these functional equations for the case where G is a free group and the linear group $G_n(R)$, a quadratically closed field or a finite field.

For the case where G is the symmetric group S_n, $n\geq1$, C.T. Ng mentioned in [27] and [28] that the above equations also holds. Recently, in 2011, C.T. Le and T.H. Thai [22] proved the elementary and direct proofs for these equations as follows:

Theorem 3.15[22]: $S_1(S_n, H) = Hom(S_n, H)$, where $S_1(S_n, H)$ is the set of all solutions of the functional equation $f(xy) + f(xy^{-1}) = 2f(x)$, for all x, $y\in G$ with the normalized condition f(e) = 0.

Theorem 3.16[22]: $S_2(S_n, H) = Hom(S_n, H)$, where $S_1(S_n, H)$ is the set of all solutions of the functional equation $f(xy) + f(y^{-1}x) = 2f(x)$, for all x, $y\in G$ with the normalized condition f(e) = 0.

The stability of various Jensen type functional equations on different spaces such as normed space, Banach space, fuzzy normed space, PN–space, RN–space etc. were proved by many authors (see [12], [35], [99])

4. STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS

In this section, we present the results concerning the stability of quadratic functional equations and the stability of generalized quadratic functional equations. The functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{4.1}$$

is the quadratic functional equation since the mapping $f(x) = x^2$ is a solution of this equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. Some other type of quadratic functional equations introduced by various authors are as follows:

$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(z+x), \tag{4.2}$$

$$f(x-y-z)+f(x)+f(y)+f(z) = f(x-y)+f(y+z)+f(z-x), \tag{4.3}$$

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z) = 4f(x)+4f(y)+4f(z) \quad (4.4)$$

a function 'f' between real spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that f(x)=B(x, x) for all x.

In 1983, the first stability theorem for the quadratic functional equation $f(x+y)+f(x-y)-2f(x)-2f(y) = 0$ was proved by F. Skof [39] for the function 'f' from normed spaces into Banach spaces in the sense of Hyers–Ulam stability. The result is as follows:

Theorem 4.1[39]: If $f:X\to Y$ satisfies $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta$ for all x, $y\in R$ and $\delta>0$, then there exists a unique quadratic function $Q:X\to Y$ such that $\|f(x) - Q(x)\| \leq \delta/2$ for all $x\in X$.

In 1984, P.W.Cholewa's [85] extended X a Banach space to the case of an Abelian group G in the Skof's theorem and generalized the following result:

Theorem 4.2[85]: Let G and E be an abelian group and a Banach space respectively. If a function $f:G \to E$ satisfies the inequality $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \varepsilon$ for some $\varepsilon \ge 0$ and for all x, $y \in G$, then there exists a unique quadratic function $Q:G \to E$ such that $\|f(x)-Q(x)\| \le \varepsilon/2$ for all $x \in G$.

In 1992, S.Czerwik [93] gave a generalization of Skof-Cholewa's result. In this result Czerwik replaced the general condition $\phi(x, y)$ by $\theta(\|x\|^p + \|y\|^p)$. He established the following result:

Theorem 4.3[93]: Let $p \neq 2$, $\theta > 0$ be real numbers. Suppose that the function $f:V \to X$ satisfies $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \theta(\|x\|^p + \|y\|^p)$. Then there exists exactly one quadratic function $g:V \to X$ such that $\|f(x)-g(x)\| \le c + k\theta\|x\|^p$ for all x in V if $p \ge 0$ and for all $x \in V\backslash\{0\}$ if $p<2$ $c=\|f(0)\|/3$, $k=2/(4-2^p)$ and

$g(x)=\lim_{n\to\infty} 4^{-n} f(2^n x)$ for all x in V. When $p>2$, $c=0$ $k=2/(2^p-4)$ and $g(x)=\lim_{n\to\infty} 4^n f(2^{-n}x)$ for all x in V. Also, if the mapping $t \to f(tx)$ from R to X is continuous for each fixed x in V, then $g(tx)=t^2 g(x)$ for all t in R.

S. Czerwik [93] indicated that the quadratic equation is not stable in the sense of Hyers–Ulam–Rassias if p=2. He slightly modified Gajda's [115] construction and proved the following:

Theorem 4.4[93]: Suppose the function $f: R \to R$ is defined by $f(x)=\sum 4^{-n}\phi(2^n x)$ where the function $\phi:R \to R$ is given by

$$\phi(x) = \begin{cases} \theta & for\,|x| \ge 1, \\ \theta x^2 & for\,|x| < 1 \end{cases}$$

with a constant $\theta > 0$. Then the function 'f' satisfies the inequality

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le 32\theta(x^2+y^2)$$

for all x, $y \in R$. Moreover, there exists no quadratic function $q: R \to R$ such that the image set of $|f(x)-q(x)| \le x^2$ $(x \neq 0)$ is bounded.

In 1992, Ger. [86], provided the further generalization of S. Czerwik's [93] result and observed that at p=2 the stability can be proved if we replace $\phi(x, y) = 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 - \|x-y\|^2$. He gave the following more general result:

Theorem 4.5[86]: Let (G, +) be an abelian group and (Y, $\|.\|$) a real n-dimensional normed space.

Let $f: G \to Y$ be a function satisfying the inequality

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \phi(x,y) \text{ with } \phi(x, y) = 2F(x)+2F(y)-F(x+y)-F(x-y)$$

where $F:G \to R$ is given non-negative function with $2F(x)+2F(y)-F(x+y)-F(x-y) \ge 0$ for all x, $y \in G$.

Then there exists a quadratic function $Q: G \to Y$ such that

$\|f(x)-Q(x)\| \le nF(x)$ for all $x \in G$.

Borelli and G.L.Forti [23] extended the result of quadratic equation (4.1) and proved the following stability result for a wider class of functional equations which contains the quadratic equation as a special case:

Theorem 4.6[23]: Let G be an abelian group, E be a Banach space, and let $f: G \to E$ be a function with f(0)=0 and satisfying the inequality $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \phi(x,y)$ for all x, $y \in G$. Assume that one of the series

$$\sum_{i=1}^{\infty} 2^{-2i}\phi(2^{i-1}x, 2^{i-1}x) \text{ and } \sum_{i=1}^{\infty} 2^{2(i-1)}\varphi(2^{-i}x, 2^{-i}x)$$

converges for each $x \in G$ and denote by $\Phi(x)$ its sum. If $2^{-2i}\phi(2^{i-1}x, 2^{i-1}x) \to 0$ or $2^{2(i-1)}\phi(2^{-i}x, 2^{-i}x) \to 0$ as $i \to \infty$, then there exists a unique quadratic function $q: G \to E$ such that $\|f(x)-q(x)\| \le \Phi(x)$ for all $x \in G$.

F. Skof and S. Terracini [40] gave a further generalization of F. Skof's [39] result for finite intervals in R as follows:

Theorem 4.7[40]: Let E be a Banach space and let c, $\varepsilon > 0$ be given. If a function $f:[0, c) \to E$ satisfies the inequality $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \varepsilon$ for all $x \ge y \ge 0$ with $x+y<c$, then there exists a quadratic function $q: R \to E$ such that $\|f(x)-q(x)\| \le 79\varepsilon/2$ for all $x \in [0,c)$.

Theorem 4.8[40]: Let E be a Banach space and let c, $\varepsilon > 0$ be given. If a function $f:(-c, c) \to E$ satisfies the inequality $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \varepsilon$ for all x, $y \in R$ with $|x+y|<c$ and $|x-y|<c$, then there exists a quadratic function $q: R \to E$ such that $\|f(x)-q(x)\| \le 81\varepsilon/2$ for any $x \in (-c, c)$.

It is also possible to prove the Hyers–Ulam–Rassias stability of the quadratic functional equation (4.1) using the fixed point approach. Cadraiu and Radu [68, 70] applied the fixed point method to the investigation of Cauchy and Jensen functional equations. In 2006, S.M. Jung, T.S. Kim and K.S. Lee [94] using the ideas of Cadraiu and Radu [68, 70]

investigated the Hyers–Ulam stability using fixed point approach for large class of functions from a vector space E_1 into a complete β–normed space E_2.

Theorem 4.9[94]: Let E_1 and E_2 be vector spaces over K. In particular, let E_2 be a complete β-normed space where $0<\beta\le1$. Suppose $\phi: E_1 \times E_1 \to [0,\infty)$ is a given function and there exists a constant $0<L<1$ such that $\phi(2x, 2x) \le 4^{\beta} L\phi(x, x)$ for all $x \in E_1$. Furthermore, let $f: E_1 \to E_2$ be a function with $f(0)=0$ which satisfies

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|_{\beta} \le \phi(x, y) \text{ for all } x, y \in E_1.$$

If ϕ satisfies $\lim_{n\to\infty} \phi(2^n x, 2^n y)/4^{n\beta} = 0$ for any x, $y \in E_1$, then there exists a unique quadratic function $q: E_1 \to E_2$ such that

$$\|f(x)-q(x)\|_{\beta} \le \phi(x, x)/4^{\beta}(1-L),\ \forall\, x \in E_1.$$

The functional equation (4.2) was first solved by Kannappan in 1995. In fact, he proved that a functional on a real vector space is a solution of (4.2) if and only if there exists a symmetric biadditive function a such that f(x) = B(x,x)+A (x) for any x. The Hyers–Ulam stability of the functional equation (4.2) on restricted domains was introduced by S. Jung [86] and applied the result to the study of asymptotic behaviors of the quadratic functions. The functional equation (4.2) is different from other quadratic functional equations in a sense that every non-zero additive function is a solution of this, but it is not a solution of others. S. Jung [86] established the following result:

Theorem 4 10[86]: Suppose X is a real norm space and Y is a real Banach space. Let $f: X \to Y$ satisfies the inequality $\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(z+x)\| \le \delta$ and $\|f(x)-f(-x)\| \le \theta$ for some $\delta, \theta \ge 0$ and for all x , y, $z \in X$. Then there exists a unique quadratic mapping $Q: X \to Y$ which satisfies $\|f(x)-Q(x)\| \le 3\delta$.

C.G.Park and H.J. Lee [15] investigated further generalization of Skof's [39] result on normed spaces. It is shown that every almost quadratic mapping $h: X \to Y$ of a complex normed space X to a complex normed space Y is a quadratic mapping when $h(rx) = r^2h(x)$ $(r>0, r\neq1)$ holds for all x in X. They present a more general definition of C-quadratic and prove the following result:

Definition 4.1: A mapping $f: X \to Y$ is called C–quadratic if 'f' satisfies the functional equation (4.1) and $f(\lambda x) = \lambda^2 f(x)$ for all x, $y \in X$. (where $\lambda>0$, $\lambda\neq1$)

Theorem 4.11[15]: Let h: $X \to Y$ be a mapping satisfying $h(rx)=r^2 h(x)$ for all $x \in X$ for which there exists a function $\phi: X^2 \to [0,\infty)$ such that

(i) $\displaystyle\sum_{j=0}^{\infty}\frac{1}{r^{2j}}\phi(r^j x, r^j y) < \infty$,

(ii) $\left\|h(\lambda x+\lambda y)+h(\lambda x-\lambda y)-2\lambda^2 h(x)-2\lambda^2 h(y)\right\| \le \phi(x,y)$

for all $\lambda \in C$ and all x, y∈X. Then the mapping h:X →Y is a C–quadratic mapping.

M.S. Moslehian and Th.M. Rassias [74] proved that the Hyers–Ulam–Rassias stability holds for Non–Archimedean normed spaces. They consider that G is an additive group and X is a complete Non–Archimedean space and obtain the following result:

Theorem 4.12[74]: Let ϕ:G×G→[0,∞) be a function such that $\lim_{n\to\infty}\phi(2^n x, 2^n y)/|4|^j = 0$ for all x in G the limit $\lim_{n\to\infty}\max\{\phi(2^j x, 2^j y)/|4|^j; 0\le j<n\}$ denoted by $\varphi(x)$ exists. Suppose that f: G→X is a mapping satisfying f(0) = 0 and

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \phi(x,y)$$

for all x, y∈G. Then there exists a quadratic mapping Q: G→X such that

$$\|f(x)-Q(x)\| \le \varphi(x)/|4| \text{ for all } x\in G$$

The stability results regarding quadratic functional equations f(x+y)+f(x–y)=2f(x)+2f(y) and f(ax+by)+f(ax–by) = $2a^2f(x)+2b^2f(y)$ where a, b are nonzero real numbers with a≠±1 in fuzzy Banach spaces were proved by J.R Lee, S.Y. Jang, C. Park and D.Y. Shin [59] in 2010. They proved the following results for the mapping f: X→Y, where X is a vector space and Y is a fuzzy Banach space.

Theorem 4.13[59]: Let $\phi: X^2 \to [0,\infty)$ be a function such that $\varphi(x,y)=\sum_{n=0}^{\infty}4^{-n}\phi(2^n x, 2^n y)$ <∞, for all x, y∈X. Let f: X→Y be a mapping with f(0)=0 **s**uch that

$$\lim_{t\to\infty} N(f(x+y)+f(x-y)-2f(x)-2f(y), t\phi(x,y)) = 1$$

uniformly on X×X. Then Q(x) = N– $\lim_{n\to\infty}(f(2^n x)/4^n) = 0$ exists for each x∈X and defines a quadratic mapping Q:X→Y such that if for some δ>0,α>0 and

$$N(f(x+y)+f(x-y)-2f(x)-2f(y), \delta\phi(x,y)) \ge \alpha \quad \text{for all } x, y\in X,$$

then $N(f(x)-Q(x), \varphi(x,x)\delta/4) \ge \alpha$ for all x∈X. Furthermore, the quadratic mapping Q: X→Y is a unique mapping such that $\lim_{t\to\infty} N(f(x)-Q(x), t\varphi(x,x)) = 1$.

Theorem 4.14[59]: Let $\phi: X^2 \to [0,\infty)$ be a function such that $\varphi(x,0)=\sum_{n=0}^{\infty}a^{-2n}\phi(ax,0)$ <∞ for all x, y∈X. Let f:X→Y be a mapping with f(0)=0 **s**uch that

$$\lim_{t\to\infty} N(f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y), t\phi(x,y)) = 1$$

uniformly on X×X. Then Q(x) = N– $\lim_{n\to\infty}(f(a^n x)/a^{2n}) = 0$ exists for each x∈X and defines a quadratic mapping Q:X→Y such that if for some δ>0, α>0 and

$$N(f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y), \delta\phi(x,y)) \ge \alpha \text{ for all } x, y\in X \text{ then}$$

$N(f(x)-Q(x), \varphi(x,0)\delta/a^2) \ge \alpha$,for all x∈X. Furthermore, the quadratic mapping Q: X→Y is a unique mapping such that $\lim_{t\to\infty} N(f(x)-Q(x), t\varphi(x,0)) = 1$ uniformly on X.

Later on the stability of several quadratic functional equations in various spaces such as normed spaces. Banach spaces, random normed spaces, Intuitinsitic normed space, orthogonal spaces etc have been investigated by a number of mathematicians (see [5, 20, 36, 38, 45, 59,99]). These results extends and improves some well known results.

5. STABILITY OF CUBIC FUNCTIONAL EQUATIONS

The functional equation

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x) \tag{5.1}$$

is said to be the cubic functional equation since $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be cubic mapping. The functional equations

$$f(x+2y)+f(x-2y)+f(2x)-2f(x)-4f(x+y)-4f(x-y) = 0, \tag{5.2}$$

$$f(x+3y)-3f(x+y)-3f(x-y)-f(x-3y)-48f(y)=0, \tag{5.3}$$

$$f(3x+y)-3f(x+y)-3f(x-y)-f(3x-y)-48f(x)=0, \tag{5.4}$$

$$3f(x+3y)+f(3x+y)-15f(x+y)-15f(x-y)-80f(y)=0, \tag{5.5}$$

$$f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)+7f(x)+7f(-x) = 2f(x+y)+4f(x+z)+f(x-z)+f(y+z)+f(y-z), \tag{5.6}$$

are cubic functional equations.

K.W. Jun and H.M. Kim [61], first introduced the cubic functional equation (5.1) and proved the Hyers–Ulam stability for the mappings $f: X \to Y$ where X is a real normed space and Y is a Banach space. In fact they investigated that kind of stability for the functional equation (5.1) in real vector spaces. The results are as follows:

Theorem 5.1[61]: A function $f: E_1 \to E_2$ where E_1 and E_2 are vector spaces satisfies (6.1) if and only if there exists a function $B: E_1\times E_1 \times E_1 \to E_2$ such that $f(x)=B(x, x, x)$ where B is symmetric for each fixed one variable and it is additive for fixed two variables.

Theorem 5.2[61]: Let X be a real vector space and Y be a Banach space, let

$$\| f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \phi(x,y) \quad (x, y\in X)$$

where $\phi: X^2 \to [0,\infty)$ satisfies $\sum_{i=0}^{\infty} \phi(2^i x,0)/8^i < \infty$ and $\lim_{n\to\infty} \phi(2^n x, 2^n y)/8^n = 0$ for all $x, y\in X$. Then $T(x) = \lim_{n\to\infty} f(2^n x)/8^n$ defines a unique cubic mapping from X to Y which satisfies the above inequality and the inequality $\|f(x)-T(x)\| \le 1/16 \sum_{i=0}^{\infty} \phi(2^i x,0)/8^i$ for all $x\in X$.

In 2008, E. Baktesh, Y.J. Cho, R. Saadati and S.M. Vaezpour [37] proved a further generalization of Jun and Kim [61] result in which they replaced random normed spaces instead of Banach spaces. They obtained the following result for the mapping $f:X\to Y$ where X is a linear space and Y is a complete Random normed space.

Theorem 5.3[37]: Let X be a linear space, $(Z,\mu^1,\min)$ a random normed space, and $\phi: X\times X\to Z$ a function such that for some $0<\alpha<8$, $\mu^1_{\phi(2x,0)}(t) \ge \mu^1_{\alpha\phi(x,0)}(t)$ for all $x\in X$, $t>0$, $f(0)=0$ and $\lim_{n\to\infty} \mu^1_{\phi(2^n x, 2^n y)}(8^n t) = 1$ for all $x, y\in X$ and all $t>0$. Let $(Y,\mu, \min)$ be a complete random normed space. If $f: X \to Y$ is a mapping such that

$$\mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) \ge \mu^1_{\phi(x,y)}(t) \quad \text{for all } x, y\in X,\ t>0,$$

Then there exists a unique cubic mapping $C: X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \ge \mu^1_{\phi(x,0)}(2(8-\alpha)t).$$

A.K. Mirmostafaee [3] extended the Jun and Kim [61] result and using a new approach (fixed point approach) proved the Hyers-Ulam stability of cubic functional equation (5.1) in Non-Archimedean normed spaces and established the following result:

Theorem 5.4[3]: Let X and Y be Non-Archimedean spaces over a Non-Archimedean field K. Suppose that $\phi: X^2 \to [0,\infty)$, then f: $X \to Y$ is said to be ϕ - approximately cubic if

$$\| f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \phi(x,y) \text{ for all } x, y \in X.$$

If Y is complete and for some integer $k \in K$ and $0<L<1$ such that $|k|^3 \phi(k^{-1}x, k^{-1}y) \le L\phi(x,y)$ for all $x, y \in X$, then there exist a unique cubic mapping C: $X \to Y$ such that $\|f(x)-f(0)-C(x)\| \le \psi_k(k^{-1}x)$ for all x in X.

where

$$\psi_2(x) = 1/|2| \max\{\phi(x,0), \phi(0,0)\}, \psi_3(x) = \max\{\phi(x,x), \phi(x,0), \phi(0,0)\} \text{ and for } j>3$$

$$\psi_j(x) = \max\{\phi(x,(j-2)x), ..., \phi(x,x), \phi(x,0), \phi(0,0), \phi(0,x), ..., \phi(0,(j-3)x)\}, x \in X.$$

In 2007, C. Baak and M.S. Moslehian [21] proved that Hyers–Ulam–Rassias stability of cubic functional equation (5.1) also holds for the orthogonality spaces. Suppose X is a real vector space either dim $X \ge 3$ and $\perp$ is a binary relation on X with the following properties:

(i) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$.
(ii) independence: if $x, y \in X-\{0\}$, $x \perp y$, then x, y are linearly independent.
(iii) homogeneity: if $x, y \in X$, $x \perp y$ then $\alpha x \perp \beta y$ for all $\alpha, \beta \in R$
(iv) the Thalesian property: if P is 2–dimensional subspace of X, $x \in P$ and $\lambda \in R$, then $y_0 \in P$ such that $x \perp y_0$ and $x+y_0 \perp \lambda x - y_0$.

Then $(X, \perp)$ is called an orthogonality space (see Rätz [58]) using this definition C. Baak and M.S. Moslehian [21] established the stability of the following results:

Theorem 5.5[21]: Let θ and p (p<3) be nonnegative real numbers. Suppose $f: X \to Y$ is a mapping with f(0)=0 fulfilling $\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\|_Y \le \theta(\|x\|_X^p + \|y\|_X^p)$ for all $x, y \in X$ with $x \perp y$. Then there exists unique orthogonally cubic mapping $T: X \to Y$ such that $\|f(x)-T(x)\|_Y \le \theta \|x\|_X^p / (16-2^{p+1})$ for all $x \in X$.

Theorem 5.6[21]: Let θ and p (p>3) be nonnegative real numbers. Suppose $f: X \to Y$ is a mapping with f(0)=0 fulfilling $\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\|_Y \le \theta(\|x\|_X^p + \|y\|_X^p)$ for all $x, y \in X$ with $x \perp y$. Then there exists unique orthogonally cubic mapping T: $X \to Y$ such that $\|f(x)-T(x)\|_Y \le \theta \|x\|_X^p / (2^{p+1}-16)$ for all $x \in X$.

Further, in 2004, K.H. Park and Y.S. Jung [60] introduced a new cubic functional equation (5.4) and obtained the Hyers–Ulam–Rassias stability for the mapping f: $G \to X$, where (G, +) is an abelian group and X is a vector space. Some corolleries were also obtained in the sense of Hyers–Ulam stability and Hyers–Ulam–Rassias stability. The following results were obtained:

Theorem 5.7[60]: A function $f: G \to X$ satisfies the functional equation (5.4) if and only if there exists a function $F: G \times G \to X$ such that f(x)=F(x, x) for all $x \in X$ and for fixed $y \in G$, the function $A: G \to X$ defined by A(x)=F(x, y) for all x in G is additive and for fixed x in G, the function $Q: G \to X$ defined by Q(y) = F(x, y) for all $y \in G$ is quadratic.

In the last two decades several cubic functional equations such as (5.1), (5.2), (5.3), (5.4), (5.5) were dealt in various spaces like random normed spaces, fuzzy normed spaces, orthogonal spces , normed spaces, Banach spaces etc. by various authors. In 2008 A.Wiwatwanich and P. Nakmahachalasiant [11] introduced a new cubic functional equation (5.3) and proved the Hyers-Ulam-Rassias stability for the mapping $f: E_1 \to E_2$ on Banach spaces as follows:

Theorem 5.8[11]: A function f: $E_1 \to E_2$, where E_1 and E_2 are Banach spaces, satisfies the functional equation (5.3) if and only if there exists a tri-additive symmetric function $A_3:E_1^3 \to E_2$ such that $f(x)=A_3(x, x, x)$ $\forall x \in E_1$.

Theorem 5.9[11]: Let an even function $\phi : E_1 \times E_2 \to [0,\infty)$ satisfies the following conditions

$$\sum_{k=0}^{\infty} \phi(0,3^k y)/27^k < \infty \text{ and } \lim_{n\to\infty} \phi(3^n x, 3^n y)/27^n = 0 \qquad \text{or} \qquad (*)$$

$$\sum_{k=1}^{\infty} 27^k \phi(0, y/3^k) \text{ and } \lim_{n\to\infty} 27^n \varphi\left(x/3^k, y/3^k\right) = 0 \qquad (**)$$

$\forall$ x, $y \in E_1$. Suppose that a function f: $E_1 \to E_2$ satisfies

$$\| f(x+3y)-3f(x+y)-3f(x-y)-f(x-3y)-48f(y)\| \le \phi(x,y).$$

Then there exist a unique cubic mapping C: $E_1 \to E_2$ which satisfies the inequality

$$\|f(y)-C(y)\| \le \begin{cases} \dfrac{1}{48}\sum_{k=0}^{\infty}\dfrac{1}{27^k}\left(\phi\left(0,3^k y\right)+\dfrac{\phi(0,3^{k+1}y)}{27}\right) & \text{if } (*) \text{holds} \\ \dfrac{1}{48}\sum_{k=1}^{n} 27^k\left(\phi\left(0,\dfrac{y}{3^k}\right)+\dfrac{1}{27}\phi(0,\dfrac{y}{3^{k+1}})\right) & \text{if } (**) \text{holds} \end{cases} \quad \forall\ y \in E_1.$$

A. and A. Alinejad [10] obtained a generalization of A. Wiwatwanich and P. Nakmahaachalasiant [11] result and proved the stability in the spirit of Hyers, Ulam and Rassias of the functional equation (5.3) in Fuzzy normed spaces as follows:

Theorem 5.10[10]: Let $\alpha \in (0, 27)\cup(27, \infty)$. Let X be a linear space and let (Z, N^1) be a fuzzy normed space. Suppose that an even function $Q:X\times X\to Z$ satisfies $Q(3^n x, 3^n y) = \alpha^n Q(x,y)$ for all x, $y\in X$ and for all $n\in N$. Suppose that (Y, N^1) is a fuzzy Banach space. If a function $f:X\to Y$ satisfies

$$N(f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)-48f(y),t) \ge N^1(Q(x,y),t) \text{ for all } x,y\in X \text{ and } t>0,$$

then there exists a unique cubic function $C:X\to Y$ which satisfies equation (5.3) and the inequality

$$N(f(x)-C(x),t) \ge \begin{cases} \min\{N^1(Q(0,x),(27-\alpha)t/3), N^1(Q(0,x),8(27-\alpha)t/\alpha)\}, & 0<\alpha<27 \\ \min\{N^1(Q(0,x),(\alpha-27)t/3), N^1(Q(0,x),8(\alpha-27)t/\alpha)\}, & \alpha>27 \end{cases}$$

holds for all x in X and t>0.

The cubic functional equation (5.5) is a new cubic functional equation introduced by Abbas Najati [8] which is some what different from (5.1), (5.3) and (5.4). Using the idea of Gavruta [84] he proved the stability in the spirit of Hyers, Ulam and Rassias. The following result and proved for the mapping $f:E\to X$, where E is a normed real linear space with norm $\|.\|_E$ and X is a real p-Banach space (i.e a quasi norm $\|.\|$ is called a p-norm($0<p\le1$) if $\|x+y\|^p \le \|x\|^p + \|y\|^p$ for all x, $y\in X$. In this case quasi- Banach space is called p-Banach space) with norm $\|.\|_X$.

Theorem 5.11[8]: Let $\phi : G\times G \to [0,\infty)$, where G is an abelian group, be a function such that

$\varphi(x) = \sum_{n=0}^{\infty}\phi^p(3^n x,0)/27^{np} < \infty$ and $\lim_{n\to\infty}\phi(3^n x,3^n y)/27^n = 0$ for all x, $y\in G$. Suppose that a mapping $f:G\to X$ satisfies the inequality $\|3f(x+3y)+f(3x-y)-15f(x+y)-15f(x-y)-80f(y)\|_X \le \phi(x,y)$ for all x, $y\in G$. Then there exists a unique cubic mapping $T:G\to X$ which satisfies the equation (5.5) and the inequality

$$\|T(x)-f(x)-40f(0)/13\|_X \le [\varphi(x)]^{1/p}/27 \text{ for all x in G. The mapping T: } G\to X \text{ is given}$$

$$T(x) = \lim_{n\to\infty} f(3^n x)/27^n \text{ for all x in G.}$$

Theorem 5.12[8]: Suppose that a mapping f: X→Y satisfies

$$\|3f(ax+3ay)+f(3ax-ay)- 15a^3f(x+y)-15a^3f(x-y)-80a^3f(y)\|_Y \le \phi(x,y)$$

and $\phi: X\times X \to [0,\infty)$ be a mapping satisfying the conditions $\varphi(x) = \sum_{n=0}^{\infty}\phi^p(3^n x, 0)/27^{np} < \infty$

and $\lim_{n\to\infty}\phi(3^n x, 3^n y)/27^n = 0$ for all x, y∈G and a∈B_1 = {u∈B:|u|=1},

where B is a unital p–Banach space. If f (tx) is continuous in t∈R for each fixed x∈X, then there exists a unique B–cubic mapping (i.e. A mapping T:X→Y is called B–cubic if T(ax) = a^3T(x) for all a∈B and all x∈X) T:X→Y which satisfies the equation (5.5) and the inequality $\|T(x)-f(x)-40f(0)/13\|_X \le [\varphi(x)]^{1/p}/27$ for all x in G.

Later on, in 2010, S. Zhang, J.M. Rassias and R. Saadati [102] extended the stability of cubic functional equation (5.5) in intuitionistic Random normed spaces (IRN–space) and proved the following result:

Theorem 5.13[102]: Let X be a linear space and $(Y, P_{\mu,\nu}, T)$ be a complete IRN –space. Let f: X→Y be a mapping with f(0)=0, for which there are $\xi,\zeta: X^2 \to D^+$, where ξ(x, y) is denoted by $\xi_{x,y}$, ς(x, y) denoted by $\varsigma_{x,y}$ and ($\xi_{x,y}$(t), $\varsigma_{x,y}$(t)) denoted by $Q_{\xi,\zeta}(x,y,t)$ with the property

$$P_{\mu,\nu}(3f(x+3y)+f(3x-y)-15f(x+y)-15f(x-y)-80f(y),t) \ge_{L^*} Q_{\xi,\zeta}(x,y,t). \text{ If}$$

$$T_{i=1}^{\infty}(Q_{\xi,\zeta}(3^{n+i-1}x, 0, 3^{3n+2i+1}t)) = 1_{L^*},\ \lim_{n\to\infty}(Q_{\xi,\zeta}(3^n x, 3^n y, 3^{3n}t) = 1_{L^*}$$

for every x, y in X and t>0, then there exists a unique cubic mapping C:X→Y such that

$$P_{\mu,\nu}(f(x)-C(x),t) \ge_{L^*} T_{i=1}^{\infty}(Q_{\xi,\zeta}(3^{i-1}x, 3^{2i+2}t).$$

Recently, Chang and Jung [50] extended their old results to the n-dimensional equation and proved the stability of above functional equation. He introduced the following cubic equation

$$f(mx+y)+f(mx-y) = mf(x+y)+mf(x-y)+2(m^3-m)f(x),\ m\ge 2 \text{ and prove as follows:}$$

Theorem 5.14[50]: Let δ be a real number and let f: X→Y be a mapping for which there exists a function $\phi: X\times X \to [-\delta, +\infty)$ such that

$$\phi(x) = \sum_{k=0}^{\infty} m^{-3k}\phi(m^k x, o) < \infty,\ \lim_{n\to\infty} m^{-3n}\phi(m^n x, m^n y) = 0$$

and
$$\| f(mx+y)+f(mx-y)-mf(x+y)-mf(x-y)-2(m^3-m)f(x)\| \le \delta + \phi(x,y)\ \forall\, x, y\in X,$$

where m is a positive integer with m>1, then there exists a unique cubic mapping T: X→Y such that

$$\| T(x)-f(x)\| \le \frac{\delta}{2(m^3-1)} + \frac{1}{2m^3}\phi(x), \quad \forall\, x\in X.$$

During the last few decades, the stability problems of several cubic functional equations in various spaces such as Menger Probabilistic Normed Spaces, Random normed spaces and Non–Archimedean Fuzzy normed spaces, Banach spaces, orthogonal spaces etc. have been extensively investigated by a number of mathematicians (see[100])

6. STABILITY OF QUARTIC FUNCTIONAL EQUATIONS

In this section, we present results concerning the stability of quartic functional equations and the stability of generalized quartic functional equations. The functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+24f(y)-6f(x) \quad (6.1)$$

is said to be a quartic functional equation, since the function $f(x)=cx^4$ is a solution of the functional equation. Some other type of quartic functional equations introduced by various authors are as follows:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \quad (6.2)$$

$$f(x+2y)+f(x-2y)=2f(x)+32f(y)+48f(\sqrt{xy}) \quad (6.3)$$

$$f(3x+y)+f(x+3y)=64f(x)+64f(y)+24f(x+y)-6f(x-y) \quad (6.4)$$

$$\begin{aligned} &f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+\left(\frac{x-y}{2}+z\right)+\left(\frac{x-y}{2}-z\right)+f(x)+f(y) \\ &-2f(z)=\frac{1}{8}\{f(x+y)+f(x-y)+4[f(y+z)+f(y-z)+f(x+z)+f(x-z)]\} \end{aligned} \quad (6.5)$$

$$f(ax+y)+f(ax-y)=a^2f(x+y)+a^2f(x-y)+2a^2(a^2-1)f(x)-2(a^2-1)f(y) \quad (6.6)$$

The stability problem for the quartic functional equation was first proved by J. M. Rassias [55] for the mappings $f: X \rightarrow Y$, where X is a real normed space and Y is a Banach space. He gave the following result:

Theorem 6.1[55]: Let X be a normed space and y be a real complete normed linear space. If $f: X \rightarrow Y$ satisfies the inequality $\| f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) - 24f(y) + 6f(x)\| \leq \varepsilon$, for all $x, y \in X$ and $\varepsilon>0$, then there exists a unique quartic function $F: X \rightarrow Y$ such that $\| F(x)-f(x)\| \leq 17\varepsilon/180$ for all x in X.

In 2003, J.K. Chung and P.K. Sahoo [57] first determined the general solution of the quartic functional equation (6.1) without assuming any regularity condition on unknown function 'f'. Using the elementary method they solved the quartic functional equation but it exploits an important result due to M. Hosszu [82]. The solution of this functional equation is also determined in certain commutative groups using two important results due to L. Szekelyhidi [72]. Results are as follows:

Theorem 6.2[57]: The function $f:R\rightarrow R$, where R is a real field satisfies the quartic functional equation (6.1) for all $x, y\in R$, if and only if 'f' is the form $f(x) = A^4(x)$ where $A^4(x)$ is the diagonal of a 4-additive symmetric function $A_4:R^4\rightarrow R$.

Theorem 6.3[57]: Let G and S be uniquely divisible abelian groups. The function $f: G\rightarrow S$ satisfies the quartic functional equation (6.1) for all $x, y\in G$ if and only if 'f' is the form $f(x) = A^4(x)$, where $A^4(x)$ is the diagonal of a 4-additive symmetric function $A_4:G^4\rightarrow S$.

In [91], S. Lee et al. introduced a new quartic functional equation (6.2) and established the stability. Later on, in 2008 Abbas Najati [9] gave a generalization of S.Lee et al's [91] result and using the ideas of Gavruta[84] in the spirit of Hyers, Ulam and Rassias. The following results were proved for the mapping $f: X\rightarrow Y$ where X is a linear space and Y is a Banach space.

Theorem 6.4[9]: Let $f: X\rightarrow Y$ be a mapping for which there exists a function $\phi : X\times X \rightarrow [0,+\infty)$ such that

$$\varphi(x)=\sum_{k=0}^{\infty} 2^{-4k}\phi(2^k x,0),\ \lim_{n\to\infty} 2^{-4n}\phi(2^{-n}x,2^{-n}y)=0$$

and $\left\|f(2x+y)+f(2x-y)-4f(x+y)-24f(x)+6f(y)\right\|_Y \leq \delta+\phi(x,y)$ for all $x, y\in X$, where $\delta\geq 0$.

Then there exists unique quartic mapping $T: X\rightarrow Y$ such that

$$\left\|T(x)-f(x)+f(0)/5\right\|_Y \leq \delta/30+\varphi(x)/32 \text{ for all x in X.}$$

F. Vajzovic [42], first investigated the orthogonally quadratic functional equation $f(x+y) +f(x-y) = 2f(x) +2f(y)$, $x\perp y$, where X is a Hilbert space, Y a scalar field, 'f' is continuous and $\perp$ means the Hilbert space is orthogonality space. Later on, M. Moslehian [75, 76], H. Drlizevic [41] and M. Fochi [83] generalized this result.

Let X be a orthogonality space and Y be a real Banach space. A mapping f: X→Y is called orthogonally quartic if it satisfies the so-called orthogonally quartic functional equation (6.2) for all x, y∈X with x⊥y. C.G. Park [17] introduced a further generalization of S. Lee et al [91] and Abbas Najati [9] results and obtained the orthogonal stability of the quartic functional equation (6.2) as follows in the sense of Rätz [58].

Theorem 6.5[17]: Let θ and p (p<4) be nonnegative real numbers. Suppose that f: X→Y is a mapping fulfilling $\|f(2x+y)+f(2x-y)-4f(x+y)-24f(x)+6f(y)\|_Y \le \theta(\|x\|_X^p+\|y\|_X^p)$ for all x, y∈X with x⊥y. Then there exists a unique orthogonally quartic mapping T: X→Y such that $\|f(x)-T(x)\| \le \theta\|x\|_X^p/(32-2^{p+1})$ for all x in X.

Note: For p>4 the above Theorem (6.5) shows that that $\|f(x)-T(x)\| \le \theta\|x\|_X^p/2^{p+1}-32$.

In 1994, Cheng and Mordeson [104] introduced the idea of a fuzzy norm on a linear space where metric is Kramosil and Michalet type [72]. Since then some mathematicians have defined fuzzy metrices and norm on a linear spaces from various point of views. The notion of fuzzy stability of the functional equation was initiated by Mirmostafaee and Moslehian in [1]. A.K. Mirmostafaee [4, 6] proved the stability of quartic functional equation (6.2) in fuzzy and quasi normed spaces and obtained the following results:

Theorem 6.6[4]: Let X be a linear space and let (Z, N^1) be a fuzzy normed space and ϕ:X×X→Z be a function. Let (Y, N) be a fuzzy Banach space and let f:X→Y be a ϕ- approximately quartic mapping in the sense that

$$N(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t) \ge N^1(\phi(x,y),t)$$

If for some α<16,

$N^1(\phi(2x,0),t) \ge N^1(\alpha\phi(x,0),t)$, f(0)=0 and $\lim_{n\to\infty} N^1(2^{-4n}\phi(2^n x,2^n y),t)=1$ for all x, y in X and t>0, then there exists a unique quartic mapping Q:X→Y such that

$$N(Q(x)-f(x),t) \ge N^1(\phi(x,0),2(16-\alpha)t).$$

Theorem 6.7[6]: Let $\phi: X\times X \to [0,\infty)$ be a mapping such that either

(i) For some α≠16, ϕ(2x,0)≤αϕ(x,0) for all x in X and for each x, y∈X, $\lim_{n\to\infty}(2^{-4n}\phi(2^n x,2^n y)=0$ or

(ii) For some α>16, αϕ(x,0)≤ϕ(2x,0) for all x in X and for each x, y∈X, $\lim_{n\to\infty}(2^{4n}\phi(2^{-n}x,2^{-n}y)=0$

Let f: X→Y satisfy the inequality

$$\|f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)\| \le \phi(x,y)$$

for each x, y∈X and f(0)=0, then there exists a unique quartic mapping Q:X→Y such that

$$\|f(x)-Q(x)\| \le \phi(x,0)/2\left|\alpha^p-16^p\right|^q \text{ for all } x\in X.$$

In 2009, D. Mihet, R. Saadati and S.M. Vaezapur [33] generalized the result of S. Lee et al [91] in random normed spaces. The notion of random normed space in which the values of the norms are probability distribution functions rather than numbers. They proved the following result for the mapping f: X→Y, where X is a real vector space and Y is a complete random normed space.

Theroem 6.8[33]: Let X be a real linear space, (Y,μ,T) be a complete random normed space and f:X→Y be a mapping with f(0)=0 for which there is $\xi: X\times X \to D^+$ (ξ(x, y) is denoted by $\xi_{x,y}$) with the property:

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)}(t) \ge \xi_{x,y}(t) \text{ for all } x, y\in X \text{ and } t>0.$$

If $\lim_{n\to\infty} T_{i=1}^{\infty}(\xi_{2^{n+i-1}x,0}(2^{4n+3i}t)) = 1$ and $\lim_{n\to\infty}\xi_{2^n x,2^n y}(2^{4n}t)) = 1$ for every x, y∈X and t>0, then there exists a unique quartic mapping Q:X→Y such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty}(\xi_{2^{i-1}x,0}(2^{3i+1}t)) \text{ for all } x\in X \text{ and } t>0.$$

Further, a new quartic functional equation (6.4) was investigated by M. Petapirak and P. Nakmahachalasint [80] in 2008. They proved the following generalized Hyers-Ulam-Rassias stability for the mapping f: X→Y, where X is a real normed space and Y a real Banach space.

Theorem 6.9[80]: Let $\phi : X^2 \to [0,\infty)$ be a function such that

$$\begin{cases} \sum_{i=0}^{\infty}\frac{\phi(3^i x,0)}{81^i}\text{converges} \quad \text{and} \\ \lim_{n\to\infty}\frac{\phi(3^n x,3^n y)}{81^n}=0 \end{cases} \quad (*) \quad \text{or} \quad \begin{cases} \sum_{i=1}^{\infty}81^i\phi\left(\frac{x}{3^i},0\right)\text{converges} \quad \text{and} \\ \lim_{n\to\infty}81^n\phi\left(\frac{x}{3^n},\frac{y}{3^n}\right)=0 \end{cases} \quad (**)$$

for all x, y∈X. If a function f: X→Y satisfies

$$\|f(3x+y)+f(x+3y)-64f(x)-64f(y)-24f(x+y)+6f(x-y)\| \leq \phi(x,y)$$

for all x, y∈X and f(0)=0, then there exists a unique function T:X→Y which satisfies the equation (6.4) and the inequality

$$\|f(x)-T(x)\| \leq \begin{cases} \frac{1}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i x,0)}{81^i} & \text{if (*)holds} \\ \frac{1}{81}\sum_{i=1}^{\infty}81^i\phi(\frac{x}{3^i},0) & \text{if (**)holds} \end{cases}$$

for all x∈X. Then the function T is given by

$$T(x) = \begin{cases} \lim_{n\to\infty}\frac{f(3^n x)}{81^n} & \text{if (*) holds} \\ \lim_{n\to\infty}81^n f(\frac{x}{3^n}) & \text{if (**) holds} \end{cases} \quad \text{for all x in X.}$$

K.Ravi and M.Arun Kumar [64] introduced a new quartic functional equation (6.3) and established the general solution and Hyers–Ulam–Rassias stability as follows:

Theorem 6.10[64]: Suppose f: X→Y satisfies

$$\|f(x+2y)+f(x-2y)-2f(x)-32f(y)-48f(\sqrt{xy})\| \leq \phi(x,y) \text{ for all } x, y\in X,$$

then there exists a unique quartic function Q:X→Y which satisfies the equation (6.3) and the inequality

$$\|f(y)-Q(y)\| \leq \frac{1}{32}\sum_{i=0}^{\infty}\frac{\phi(0,2^i y)}{16^i} \text{ for all y in X.}$$

The function Q is given by $Q(y) = \lim_{n\to\infty} f(2^n y)/16^n$, for all y in X.

The various types of functional equations and their stability on various spaces like metric space, Banach space, normed space, random normed spaces, intuitionistic normed space were discussed by many mathematicians [37,113] in the last few decades. In 2008, K. Ravi, R. Murli, E. Thandapani [63] motivated by M.S. Mosleian and T.M. Rassias [74] again introduced the new quartic functional equation (6.5) and proved the stability in Non-Archimedean normed spces. Recently, in 2011, M.A. Kumar, K. Ravi and M. J. Rassias [78] proved the stability of a quartic and orthogonally quartic functional equation in the sense of J. Rätz [58].

7. STABILITY OF PEXIDERIZED FUNCTIONAL EQUATIONS

One of the striking features of functional equations is that contrary to differential equations a single equation can determine more than one function. Pexiderized functional equations are generalization of the Cauchy functional equation. In 1903, J.V. Pexider considered the functional equation $f(x+y)=g(x)+h(y)$ in three functions which is the generalization of Cauchy functional equation $f(x+y)=f(x)+f(y)$. He also introduced other Cauchy Pexiderized functional equations such as Multiplicative, Exponential and Logaritmic functional equations. Lator on, many Pexiderized functional equations such as Quadratic, Sine, Cosine etc functional equations introduced by many Mathematicians such as Hyer, T.M.Rassias, M.S. Moslehian, Y. H. Lee etc . In this section we present the detailed study of stabilities of these functional equations.

In 1999, K.W. Jun, D.S. Shin and B.D. Kim [62] using the idea of Gavruta [84] established a generalization of the stability of approximately addtitive mapping in the spirit of Hyers, Ulam and Rassias. Let (X, +) and (Y, +) be abelian groups and f,g,h:$X\to Y$ be mappings. If f, g and h satisfies the functional equation

$$f(x+y)-g(x)-h(y) = 0 \tag{*}$$

for all x, $y\in X$, we call it a Pexider equation. Jun, Shin and Kim [57] investigated the following results for the stability of the generalized functional equation of Pexider type.

Theorem 7.1[62]: Let f, g, h: $G\to X$ be mappings satisfying the inequality

$$\|f(x+y)-g(x)-h(y)\| \le \phi(x,y) \text{ for all } x, y\in G.$$

Then there exists a unique additive mapping T: $G\to X$ such that

$$\|f(x)-T(x)\| \le \|g(0)\|+\|h(0)\|+\varepsilon(x),\quad \|g(x)-T(x)\| \le \|g(0)\|+2\|h(0)\|+\phi(x,0)+\varepsilon(x),$$

$$\|h(x)-T(x)\| \le 2\|g(0)\|+\|h(0)\|+\phi(x,0)+\varepsilon(x),\ \forall\ x\in G.$$

In 2000, Y.H. Lee and K.W. Jun [111] generalized the above result by replacing $\phi(x, y)$ with the condition $\theta(\|x\|^p+\|y\|^p)$ for the mappings f, g, h: $V\to X$ where V be a normed space, X be a Banach space, $\theta\ge 0$ and $p\ne 1$. The result is as follows:

Theorem 7.2[111]: Let V be a normed space and let f, g, h: $V\to X$ be mappings. Assume that there exists $\theta\ge 0$ and $p\ne 1$ such that $\|f(x+y)-g(x)-h(y)\| \le \theta(\|x\|^p+\|y\|^p)$ for all x, $y\in V\backslash\{0\}$. Then there exists a unique additive mapping T:$V\to X$ such that $\|f(x)-T(x)-f(0)\| \le \dfrac{4\theta(3+3^p)}{2^p\,|3-3^p|}\|x\|^p$ for all $x\in V\backslash\{0\}$.

In 1995, R. Ger and J. Sikorska [90] investigated the orthogonal stability of the Cauchy functional equation f(x+y) = f(x)+f(y), namely, they showed that if 'f' is a function from a orthogonality space X into a real Banach space Y, $\varepsilon>0$ is given for all x, $y\in X$ with $x\perp y$, f(x+y) = f(x)+f(y), then there exists one orthogonally additive mapping g:$X\to Y$ such that $x\in X$, $\|f(x)-g(x)\| \le 16\varepsilon/3$. M. S. Moslehian [76] proved the orthogonal stability of the Pexiderized Cauchy functional equation (*) for the mappings f_1, f_2, f_3: X $\to$Y, where (X, ⫫, ‖.‖) is an orthogonality space and (Y, ‖.‖) is a real Banach space. Thus Moslehian [76] generalized the main theorem of [5] as follows:

Theorem 7.3[76]: Suppose $(X,\perp)$ is an orthogonality module and $(Y, \|.\|)$ is a real Banach module. Let $f_1, f_2, f_3: X \to Y$ be mappings fulfilling $\|f_1(ax+ay) - abf_2(x) - abf_3(y)\| \le \varepsilon$ for some ε, all a, $b \in A_1$ and for all x, $y \in X$ with $x \perp y$. Then there exists exactly a quadratic mapping $Q: X \to Y$ and an additive mapping $T: X \to Y$ such that

$$\|f_1(x) - f_1(0) - Q(x) - T(x)\| \le 68\varepsilon/3,$$
$$\|f_2(x) - f_2(0) - Q(x) - T(x)\| \le 80\varepsilon/3,$$
$$\|f_3(x) - f_3(0) - Q(x) - T(x)\| \le 80\varepsilon/3,$$

for all $x \in X$. Furthermore, $T(ax) = aT(x)$ and $Q(ax) = a^2Q(x)$ for all $x \in X$, $a \in A_1$.

Remark 7.4[76]: If we replace the above inequality with $\|f_1(ax+ay) - af_2(x) - af_3(y)\| \le \varepsilon$, $x \perp y$, $a = a^2$, $\|a\| = 1$ then Theorem (7.3) is still true except that $T(ax) = aT(x)$ and $Q(ax) = a^2Q(x)$ hold merely for idempotents $a \in A_1$. This may be of special interest whenever we deal with the Banach algebras generated by their idempotents.

Theorem 7.5[76]: Suppose $(X,\perp)$ is an orthogonality module and $(Y, \|.\|)$ is a real Banach module. Let $f_1, f_2, f_3: X \to Y$ be mappings fulfilling

$$\|f_1(ax+ay) - abf_2(x) - abf_3(y)\| \le \varepsilon,$$

for some ε, all a, $b \in A_1$ and for all x, $y \in X$ with $x \perp y$. Then there exists exactly a quadratic mapping $Q: X \to Y$ and an additive mapping $T: X \to Y$ such that

$$\|f_1(x) - f_1(0) - Q(x) - T(x)\| \le 68\varepsilon/3,$$
$$\|f_2(x) - f_2(0) - Q(x) - T(x)\| \le 80\varepsilon/3,$$
$$\|f_3(x) - f_3(0) - Q(x) - T(x)\| \le 80\varepsilon/3,$$

for all $x \in X$. In addition, if the mapping $t \to f_1(tx)$ is continuous for each fixed $x \in X$, then T is A–linear and Q is A–quadratic.

S.M.Jung and P.K.Sahoo [92] and S.M. Jung [99] proved the stability of Pexiderized quadratic functional equation $f(x+y)+g(x-y)-2h(x)-2k(y) = 0$. Further, in 2005, M.S. Moselehian [75] using the ideas from his paper [76] investigated the Hyers–Ulam stability of the orthogonally quadratic functional equation of Pexider type $f(x+y)+f(x-y) = 2g(x)+2h(y)$, $x \perp y$ under certain conditions, where $(X, \perp)$ denotes the orthogonally normed space and $(Y\|.\|)$ is a real Banach space.

Theorem 7.6[75]: Suppose $\perp$ is symmetric on X and $f,g,h: X \to Y$ are mappings fulfilling

$$\|f(x+y) + f(x-y) - 2g(x) - 2h(y)\| \le \varepsilon$$

for some ε and for all x, $y \in X$ with $x \perp y$. Assume that f is odd. Then there exists exactly one additive mapping $T: X \to Y$ and exactly a quadratic mapping $Q: X \to Y$ such that

$$\|f(x) - T(x) - Q(x)\| \le 6\varepsilon$$
$$\|g(x) - T(x) - Q(x)\| \le 13\varepsilon/2 \text{ for all } x \in X.$$

Theorem 7.7[75]: Suppose $\perp$ is symmetric on X and f, g, $h: X \to Y$ are mappings fulfilling

$$\|f(x+y) + f(x-y) - 2g(x) - 2h(y)\| \le \varepsilon$$

for some ε and for all x, $y \in X$ with $x \perp y$. Assume that f is even. Then there exists exactly additive mapping $T: X \to Y$ and exactly a quadratic mapping $Q: X \to Y$ such that

$\|f(x)-Q(x)\| \le \alpha\varepsilon$, $\|g(x)-Q(x)\| \le \beta\varepsilon$ $\|h(x)-Q(x)\| \le \gamma\varepsilon$ for some scalars α, β, γ and for all $x \in X$.

M. Mirzavaziri and M.S. Moslehian [79] presented a further generalization of M.S. Moslehian [75] results and proved the Hyers-Ulam stability using fixed point alternative theorem. They obtained the following results for the mapping f, g, h, k: $X \to Y$, where X is a real ortgthogonality space with a symmetric orthogonal relation ⊥ and a Banach space Y.

Theorem 7.8[79]: Suppose that X is a real orthogonality space with a symmetric orthogonal relation ⊥ and Y is a Banach space. Let the mappings f, g, h, k:$X \to Y$ satisfy the following inequalities

$\|f(x+y)+g(x-y)-2h(x)-2k(y)\| \le \varepsilon$ for all x, $y \in X$ with $x \perp y$. Then there exists an orthogonally additive mapping T such that $\|f(x)-T(x)\| \le \varepsilon$ if and only if $\|f(2x)+f(-2x)-4f(x)-4f(-x)\| \le \varepsilon$, indeed if this inequality holds for all x in X, then there exists orthogonally additive mappings T, T^1, T^{11}:$X \to Y$ such that

$$\|f(x)-f(0)-T(x)\| \le 140\varepsilon/3,$$
$$\|g(x)-g(0)-T^1(x)\| \le 98\varepsilon/3,$$
$$\|h(x)+k(x)-h(0)-k(0)-T^{11}(x)\| \le 256\varepsilon/3 \text{ for all x in X.}$$

In 2008, A. K. Mirmostafee and M.S. Moslehian [1] proved the Fuzzy stability of the pexiderized quadratic functional equation $f(x+y)+f(x-y)=2g(x)+2h(x)$ using direct approach. Further, Z. Wang and W. Zhang [114] presented the generalization of A. K. Mirmostafee and M.S. Moslehian [1] and investigated the fuzzy stability by using fixed point approach and established the following results:

Theorem 7.9[114]: Let X be a linear space and let (Z, N^1) be a fuzzy normed space. Let $\phi: X \times X \to Z$ be a function such that $\phi(2x,2y)=\alpha\phi(x,y)$ for all x, $y \in X$ and $t>0$ for some real number α with $0<|\alpha|<2$. Let (Y, N) be a fuzzy Banach space and let f, g and h be odd function from X to Y such that $N(f(x+y)+f(x-y)-2g(x)-2h(y),t) \ge N^1(\phi(x,y),t)$ for all x, $y \in X$ and $t>0$. Then there exists a unique additive mapping $T: X \to Y$ such that

$$N(T(x)-f(x),t) \ge M_1(x,(2-|\alpha|)t/2),\ N(g(x)+h(x)-T(x),t) \ge M_1(x,(6-3|\alpha|)t/(10-2|\alpha|))$$

where $M_1(x,t)=\min\{N^1(\phi(x,x),(2/3)t), N^1(\phi(x,0),(2/3)t), N^1(\phi(0,x),(2/3)t),\}$.

Theorem 7.10[114]: Let X be a linear space and let (Z, N^1) be a fuzzy normed space. Let $\phi: X \times X \to Z$ be a function such that $\phi(2x,2y)=\alpha\phi(x,y)$ for all x, $y \in X$ and $t>0$ for some real number α with $0<|\alpha|<4$. Let (Y, N) be a fuzzy Banach space and let f, g and h be even functions from X to Y such that f(0) = g(0) = h(0) = 0 and

$$N(f(x+y)+f(x-y)-2g(x)-2h(y),t) \ge N^1(\phi(x,y),t) \text{ for all x, } y \in X \text{ and } t>0.$$

Then there exists a unique additive mapping $Q: X \to Y$ such that

$$N(Q(x)-f(x),t) \ge M_1(x,(4-|\alpha|)t/2),$$
$$N(Q(x)-g(x),t) \ge M_1(x,(12-3|\alpha|)t/(10-|\alpha|)),$$
$$N(Q(x)-h(x),t) \ge M_1(x,(12-3|\alpha|)t/(10-|\alpha|)),$$

where $M_1(x,t)=\min\{N^1(\phi(x,x),(2/3)t), N^1(\phi(x,0),(2/3)t), N^1(\phi(0,x),(2/3)t),\}$.

Saadati and Park [89] have recently introduced the concept of intuitionistic fuzzy normed space. Recently, S.A.Mohiuddine and H. Sevli [98] obtained the stability of the Hyers–Ulam–Rassias type of pexiderized quadratic functional equation $f(x+y)+f(x-y)= 2g(x)+2h(x)$ in intuitionistic fuzzy normed space. The results are as follows:

Theorem 7.11[98]: Let X be a linear space and let (Z, μ^1, ν^1) be a intuitionistic fuzzy normed space. Let $\phi:X\times X\to Z$ be a function such that $\phi(2x,2y)=\alpha\phi(x,y)$ for all x, y$\in$X and t>0 for some real number α with 0<|α|<2. Let (Y, μ, ν) be a intuitionistic fuzzy Banach space and let f, g and h be odd functions from X to Y such that

$$\mu(f(x+y)+f(x-y)-2g(x)-2h(y),t)\geq\mu^1(\phi(x,y),t) \text{ and}$$
$$\nu(f(x+y)+f(x-y)-2g(x)-2h(y),t)\leq\nu^1(\phi(x,y),t)$$

for all x, y$\in$X and t>0. Then there exists a unique additive mapping $T:X\to Y$ such that

$$\mu(f(x)-T(x),t)\geq\mu^{11}(x,(2-|\alpha|)t/4) \text{ and } \nu(f(x)-T(x),t)\leq\nu^{11}(x,(2-|\alpha|)t/4);$$
$$\mu(g(x)+h(x)-T(x),t)\geq\mu^{11}(x,(6-3|\alpha|)t/(14-|\alpha|)) \text{ and}$$
$$\nu(g(x)+h(x)-T(x),t)\geq\nu^{11}(x,(6-3|\alpha|)t/(14-|\alpha|))$$

where $\mu^{11}(x,t)=\mu^1(\phi(x,x),t/3)*\mu^1(\phi(x,0),t/3)*,\mu^1(\phi(0,x),t/3)$ and

$$\nu^{11}(x,t)=\nu^1(\phi(x,x),t/3)*\nu^1(\phi(x,0),t/3)*,\nu^1(\phi(0,x),t/3).$$

Theorem 7.12[98]: Let X be a linear space and let (Z, μ^1, ν^1) be a intuitionistic fuzzy normed space. Let $\phi:X\times X\to Z$ be a function such that $\phi(2x,2y)=\alpha\phi(x,y)$ for all x, y$\in$X and t>0 for some real number α with 0<|α|<4. Let (Y, μ, ν) be a intuitionistic fuzzy Banach space and let f, g and h be even functions from X to Y such that f(0)=g(0)=h(0)=0 and

$$\mu(f(x+y)+f(x-y)-2g(x)-2h(y),t)\geq\mu^1(\phi(x,y),t) \text{ and}$$
$$\nu(f(x+y)+f(x-y)-2g(x)-2h(y),t)\leq\nu^1(\phi(x,y),t)$$

for all x, y$\in$X and t>0. Then there exists a unique additive mapping Q: $X\to Y$ such that

$$\mu(Q(x)-f(x),t)\geq\mu^{11}(x,(4-|\alpha|)t/16),\ \mu(Q(x)-g(x),t)\geq\mu^{11}(x,(12-3|\alpha|)t/(52-|\alpha|)),$$

and $\mu(Q(x)-h(x),t)\geq\mu^{11}(x,(12-3|\alpha|)t/(52-|\alpha|))$

$$\nu(Q(x)-f(x),t)\leq\nu^{11}(x,(4-|\alpha|)t/16),\ \nu(Q(x)-g(x),t)\leq\nu^{11}(x,(12-3|\alpha|)t/(52-|\alpha|))$$

and $\nu(Q(x)-h(x),t)\leq\nu^{11}(x,(12-3|\alpha|)t/(52-|\alpha|))$.

Gawang Hui Kim [43, 44] proved the superstability and stability of the pexiderized Trigonometric functional equation. Further, the stability and superstability of several pexiderized functional equations such as Multiplicative,

Trigonometric, Sine, Cosine, Exponential etc. in various spaces have been investigated by a number of mathematicians. (see [9], [22], [30], [40], [45], [46]).

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