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On πgb - Closed Sets and Related Topics

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Abstract

In this paper, we study πgb- *closed sets due to Sreeja D. and Janaki C. [26] and investigate further properties of these sets. By means of* πgb - *closed sets we introduced a new class of functions called almost* πgb- *continuous functions which are generalizations both* πgb- *continuity and almost b-continuity. Moreover, the notions of* πgb - *compactness and quasi* -b- *normality in topological spaces are introduced and their some properties are studied.*

Mathematics Subject Classification: *54A05, 54C08, 54C10.*

Keywords: πgb - *closed set,* πgb - *closure operator, almost* πgb - *continuous function,* πgb - *compact space, quasi* -b - *normal space.*

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1. Introduction

Continuity on topological spaces, as significant and fundamental subject in the study of topology, has been researched by several mathematicians. Many investigations related to generalized closed sets have been published various forms of generalized continuity types have been introduced. The study of generalized closed sets in a topological space was initiated by Levine [18] and the concept of $T_{1/2}$ - space was introduced. gb - closed sets were defined and studied by Ekici [12] and Ganster - Steiner [14]. Recently, Benchalli and Bansali [4] introduced the notion of gb - compactness. In 1968, Zaitsev [28] defined the concept of π - closed sets and a class of topological spaces called quasi normal spaces. Later Dontchev and Noiri [8] introduced the notion of πg - closed sets and used this notion to obtain a characterization and some preservation theorems for quasi normal spaces. Park [23] defined πgp - closed sets. Next, Aslim, Caksu and Noiri [3] introduced the notion of πgs- closed sets. Caksu Guler and Aslım [6] obtained characterizations of quasi -s - normal spaces by using πgs - closed sets.

The aim of this paper is to investigate further properties of πgb - closed sets due to Sreeja D.and Janaki C. [26]. The paper consists of six sections. In section 3, we introduce the concept of πgb - closure and obtain some of its fundamental properties. Besides, in section 4, we present a new generalization of almost continuity called almost πgb - continuity. The notion of almost πgb - continuity is a weaker form of almost b- continuity [15]. Furthermore, in section 5, we introduce the concept of πgb - compactness and study their behaviour under πgb - continuous and almost πgb - continuous functions. In the last section, we introduce and characterize a new class of space, called quasi -b - normal spaces.

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) represent nonempty topological spaces on which no seperation axioms are assumed, unless otherwise mentioned. Also, in this paper spaces mean topological spaces and $f:(X,\tau)\to(Y,\sigma)$ (or simply $f:X\to Y$) denotes a function f of a space (X,τ) into a space (Y,σ).
Let A be a subset of a space X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

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Definition 2.1: A subset A of a topological space X is called:
(a) pre-open [20] if $A \subset \mathrm{int}(\mathrm{cl}(A))$,
(b) semi-open [19] if $A \subset \mathrm{cl}(\mathrm{int}(A))$,
(c) regular open [27] if $A = \mathrm{int}(\mathrm{cl}(A))$,
(d) b - open [1] or γ - open [13] if $A \subset \mathrm{int}(\mathrm{cl}(A)) \cup \mathrm{cl}(\mathrm{int}(A))$.

The finite union of regular open sets is said to be π - open. The complement of a π - open set is said to be π - closed. The complement of a b - open set is called b - closed [1]. The intersection of all b - closed sets containing A is called the b - closure [1] of A and is denoted by $\mathrm{bcl}(A)$. The b - interior [1] of A is defined to be the union of all b - open sets contained in A and is denoted by $\mathrm{bint}(A)$.

Lemma 2.2 [1]: Let A be a subset of a space X. Then
(a) $\mathrm{bcl}(A)=\mathrm{scl}(A) \cap \mathrm{pcl}(A)=A \cup ((\mathrm{int}(\mathrm{cl}(A)) \cap \mathrm{cl}(\mathrm{int}(A)))$,
(b) $\mathrm{bint}(A) = \mathrm{sint}(A) \cup \mathrm{pint}(A)=A \cap (\mathrm{int}(\mathrm{cl}(A)) \cup \mathrm{cl}(\mathrm{int}(A)))$.

Definition 2.3: A subset A of a topological space X is called:
(a) g - closed [18] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is open in X,
(b) gp - closed [19] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is open in X,
(c) gs - closed [2] if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is open in X,
(d) gb - closed [12,14] if $\mathrm{bcl}(A) \subset U$ whenever $A \subset U$ and U is open in X,
(e) πg - closed [8] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is π - open in X,
(f) πgp - closed [23] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is π - open in X,
(g) πgs- closed [3] if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is π - open in X,
(h) πgb - closed [26] if $\mathrm{bcl}(A) \subset U$ whenever $A \subset U$ and U is π - open in X,
(i) πgb - open (resp. g - open, gp - open, gs - open, gb - open, πg - open, πgp - open, πgs- open) if the complement of A is πgb - closed (resp. g - closed, gp - closed, gs - closed, gb - closed, πg - closed, πgp - closed, πgs- closed).
The family of all πgb - closed (resp. b - closed) sets in a topological space (X,τ) is denoted by $\pi\mathrm{GBC}(X)$ (resp. $\mathrm{BC}(X)$).

Definition 2.4: A function $f : X \to Y$ is said to be
(a) regular open [21] if $f(V)$ is regular open in Y for every open set V of X,
(b) b - closed [22] if $f(V)$ is b - closed in Y for every b - closed set V of X,
(c) m-π - closed [11] if $f(V)$ is π - closed in Y for every π - closed set V of X,
(d) π - continuous [8] (resp. πg - continuous [7], πgp - continuous [24], πgs- continuous [3]) if $f^{-1}(V)$ is π - closed (resp. πg - closed, πgp - closed, πgs- closed) in X for every closed set V of Y,
(e) b - continuous [13] (resp. g - continuous [18], gb - continuous [22]) if $f^{-1}(V)$ is b - closed (resp. g - closed, gb - closed) in X for every closed set V of Y,
(f) almost b - continuous [15] if $f^{-1}(V)$ is b - closed in X for every regular closed set V of Y,
(g) πgb - continuous [26] if $f^{-1}(V)$ is πgb - closed in X for every closed set V of Y,
(h) π - irresolute if $f^{-1}(V)$ is π - closed in X for every π - closed set V of Y,
(i) b - irresolute [10] if $f^{-1}(V)$ is b - closed in X for every b - closed set V of Y,
(j) πgb - irresolute [26] if $f^{-1}(V)$ is πgb - closed in X for every πgb - closed set V of Y.

3. Thefurtherproperties of πgb - closedsetsand πgb - closureoperatör

Theorem 3.1: [26] Every πgs- closed set is πgb - closed.

The following example show that above implication is not reversible.

Example 3.2: Let τ be the usual topology for $\mathbb{R}$ and $A = (0,2) \setminus \mathbb{Q} \subset \mathbb{R}$, where $\mathbb{Q}$ denotes the set of rational numbers. Then A is πgb - closed but it is not πgs- closed.

Theorem 3.3: For a subset A of X, the following statements are equivalent:
(1) A is π-open and πgb-closed.
(2) A is regular open.

Proof: (1) $\Rightarrow$ (2) Let A be a π-open and πgb-closed subset of X. Then $bcl(A) \subset A$ and so $int(cl(A)) \subset A$ holds. Since A is open then A is pre-open and thus $A \subset int(cl(A))$. Therefore, we have $int(cl(A))=A$, which shows that A is regular open.

(2) $\Rightarrow$ (1) Since every regular open set is π-open then $bcl(A)=A$ and $bcl(A) \subset A$. Hence A is πgb-closed.

A subset A of a topological space X is said to be Q-set [16] if $int(cl(A))= cl(int(A))$.

Theorem 3.4: For a subset A of X, the following statements are equivalent:
(1) A is π-clopen,
(2) A is π-open, Q-set and πgb-closed.

Proof: (1) $\Rightarrow$ (2) Let A be a π-clopensubset of X. Then A is π-closed and π-open. Thus A is closed and open. Therefore, A is Q-set. Since every π-closed is πgb-closed then A is πgb-closed.

(2) $\Rightarrow$ (1) By Theorem 3.3, A is regular open. Since A is Q-set, $A = int(cl(A)) = cl(int(A))$. Therefore, A is regular closed. Then A is π-closed. Hence A is π-clopen.

Proposition 3.5: [9] Let A be a subset of a topological space X. If A is semi-open then $pcl(A) = cl(A)$.

A topological space X is said to be extremely disconnected [5] if the closure of every open subset of X is open in X.

Theorem 3.6: A space X is extremely disconnected if and only if every πgb-closed subset of X is πgp-closed.

Proof: Suppose that X is extremely disconnected. Let A be πgb-closed and let U be an π-open set containing A. Then $bcl(A) = A \cup [int(cl(A) \cap cl(int(A))] \subset U$, i.e. $[int(cl(A)) \cap cl(int(A))] \subset U$. Since $int(cl(A))$ is closed, we have $cl(int(A)) \subset cl[int(cl(A)) \cap int(A)] \subset [cl(int(cl(A))) \cap cl(int(A))] \subset U$. It follows that $pcl(A) = A \cup cl(int(A)) \subset U$. Hence A is πgp-closed.

To prove the converse, let every πgb-closed subset of X be πgp-closed. Let A be a regular open subset of X. Then $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))] = A \cup [A \cap cl(int(A))] \subset A$. Then A is πgb-closed and so A is πgp-closed. Since every regular open is semi-open set and by Proposition 3.5, we have $cl(A) = pcl(A)$. Hence $cl(A) \subset A$. Therefore, A is closed. This shows that X is extremely disconnected.

A topological space X is said to be hyperconnected if the closure of every open subset is X.

Theorem 3.7: Let X be a hyperconnected space. Then every πgb-closed subset of X is πgs-closed.

Proof: Assume that X is hyperconnected. Let A be πgb-closed and let U be an π-open set containing A. Then $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))] = A \cup int(cl(A)) = scl(A)$. Since $bcl(A) = scl(A)$, we have $scl(A) \subset U$. Hence, A is πgs-closed.

Theorem 3.8: Let A be a πgb-closed set such that $cl(A) = X$. Then A is πgp-closed.

Proof: Suppose that A be πgb-closed set such that $cl(A) = X$. Let U be an π-open set containing A. Since $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))]$ and $cl(A) = X$, we obtain $bcl(A) = A \cup cl(int(A)) = pcl(A) \subset U$. Therefore, A is πgp-closed.

Definition 3.9: A topological space X is said to be πgb-$T_{1/2}$ space [26] if every πgb-closed set is b-closed.

Theorem 3.10: For a space X, the following statements are equivalent:

(1) X is πgb - $T_{1/2}$,
(2) For every subset A of X, A is πgb - open if and only if A is b - open.

Proof: (1) ⇒ (2) Let the space X be πgb - $T_{1/2}$ and let A be a πgb - open subset of X. Then X \ A is πgb - closed and so X \ A is b - closed. Hence A is b - open.

Conversely, let A be a b - open subset of X. Thus X \ A is b - closed. Since every b - closed set is πgb - closed then X \ A is πgb - closed. Therefore, A is πgb - open.

(2) ⇒ (1) Let A be a πgb - open subset of X. Then X \ A is πgb - open. By the hypothesis X \ A is b - open. Thus A is b - closed. Since every πgb - closed set is b - closed, thus X is πgb - $T_{1/2}$.

Definition 3.11: The intersection of all πgb - closed sets, each containing a set A in a topological space X is called the πgb - closure of A and it is denoted by $\pi gb\text{-}cl(A)$.

Lemma 3.12: Let A be a subset of X and $x \in X$. Then $x \notin \pi gb\text{-}cl(A)$. if and only if $V \cap A \neq \emptyset$ for every πgb - open set V containing x.

Proof: Assume that there exists a πgb - open set V containing x such that $V \cap A = \emptyset$. Since $A \subset X \setminus V$, $\pi gb\text{-}cl(A) \subset X \setminus V$ and then $x \notin \pi gb\text{-}cl(A)$. a contradiction. To prove the converse, suppose that $x \notin \pi gb\text{-}cl(A)$. Then there exists a πgb - closed set F containing A such that $x \notin F$. Since $x \in X \setminus F$ and $X \setminus F$ is πgb - open, $(X \setminus F) \cap A = \emptyset$ a contradiction.

Lemma 3.13: Let A and B be subsets of X. Then we obtain

(a) $\pi gb\text{-}cl(\emptyset) = \emptyset$, $\pi gb\text{-}cl(X) = X$,
(b) $A\pi gb\text{-}cl(A)$,
(c) If A is πgb - closed then $\pi gb\text{-}cl(A) = A$,
(d) $\pi gb\text{-}cl(A) = \pi gb\text{-}cl(\pi gb\text{-}cl(A))$,
(e) If $A \subset B$ then $\pi gb\text{-}cl(A) \subset \pi gb\text{-}cl(B)$,
(f) $\pi gb\text{-}cl(A \cap B) \subset \pi gb\text{-}cl(A) \cap \pi gb\text{-}cl(B)$,
(g) $\pi gb\text{-}cl(A \cup B) \supset \pi gb\text{-}cl(A) \cup \pi gb\text{-}cl(B)$.

Proof: We are obtained by Definition 2.3 and Lemma 3.12.

Remark 3.14: The following examples show that the converses of Lemma 3.13 (c), (f) and (g) need not be true.

Example 3.15: Let $X=\{a,b,c,d,e,f\}$ and $\tau=\{X,\emptyset,\{a,b\},\{c,d\},\{a,b,c,d\}\}$. Let $A=\{a,b,c,d\}$. Then $\pi gb\text{-}cl(A) = A$ but A is not πgb - closed.

Example 3.16: Let $X=\{a,b,c,d,e\}$ and $\tau=\{X,\emptyset,\{a\},\{e\},\{a,e\},\{c,d\},\{a,c,d\},\{c,d,e\},\{a,c,d,e\}\}$. Let $A=\{a,c,d,e\}$ and $B=\{b,c,d\}$. Then A is not πgb - closed and B is πgb - closed. Since $\pi gb\text{-}cl(A) = X$ and $\pi gb\text{-}cl(B) = B$, we have $\pi gb\text{-}cl(A) \cap \pi gb\text{-}cl(B) = B = \{b,c,d\}$ but $\pi gb\text{-}cl(A \cap B) = \{c,d\}$.

Example 3.17: Let X be topological space in Example 3.16, let $A=\{a,c,e\}$ and $B=\{d\}$. Then A is not πgb - closed and B is πgb - closed. Since $\pi gb\text{-}cl(A) = A$ and $\pi gb\text{-}cl(B) = B$ and so $\pi gb\text{-}cl(A) \cup \pi gb\text{-}cl(B) = \{a,c,d,e\}$ but $\pi gb\text{-}cl(A \cup B) = X$.

4. Almost πgb - continuityandrelatedsomecontinuities

Remark 4.1: For a function $f : X \rightarrow Y$, the following implications hold:

$$\begin{array}{ccccccccc}
 & & & & b\text{-con} & \rightarrow & gb\text{-con} & & \\
 & & & \nearrow & & \nearrow & & \searrow & \\
 & & pre\text{-con} & \rightarrow & gp\text{-con} & \rightarrow & \pi gp\text{-con} & & \\
 & & \uparrow & & \uparrow & & \uparrow & \searrow & \\
\pi\text{-con} & \rightarrow & con & \rightarrow & g\text{-con} & \rightarrow & \pi g\text{-con} & \rightarrow & \pi gb\text{-con}\\
 & & \downarrow & & \downarrow & & \downarrow & \nearrow & \\
 & & semi\text{-con} & \rightarrow & gs\text{-con} & \rightarrow & \pi gs\text{-con} & & \\
 & & & \searrow & & \searrow & & \nearrow & \\
 & & & & b\text{-con} & \rightarrow & gb\text{-con} & &
\end{array}$$

Remark 4.2: The following examples show that:
(a) Every πgb - continuous function need not be gb - continuous or πg - continuous,
(b) Every πgb - continuous function need not be πgp - continuous,
(c) Every πgb - continuous function need not be πgs- continuous.

Example 4.3: Let $X=\{a,b,c,d,e\}$, $\tau =\{\varnothing, X,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\}$ and $Y=\{x,y,z,t\}$, $\sigma =\{Y,\varnothing,\{x,y,z\},\{t\}\}$. Define a function $f : X \rightarrow Y$ as follows: $f(a) = z$, $f(b) = f(e) = t$, $f(c) = y$ and $f(d) = x$. Then f is a πgb - continuous but it is not gb- continuous.

Example 4.4: Let $X=\{a,b,c,d\}$, $\tau =\{\varnothing, X,\{a\},\{b,c\},\{a,b,c\},\{a,b\},\{b\}\}$ and $Y=\{x,y,z\}$, $\sigma =\{Y,\varnothing,\{x,y\},\{x,z\},\{x\}\}$. Define a function $f : X \rightarrow Y$ as follows: $f(a) = y$, $f(b) = f(d) = x$, $f(c) = z$. Then f is πgb - continuous function which is neither πg - continuous nor πgp - continuous.

Example 4.5: Let X be the real numbers with the usual and $Y=\{0,1\}$ with the topology $\sigma =\{Y,\varnothing,\{1\}\}$. We define the function $f : X \rightarrow Y$ such as

$$f(x)=\begin{cases}0, & x \in (0,2)\setminus \mathbb{Q}\\ 1, & x \notin (0,2)\setminus \mathbb{Q}\end{cases}$$

Then f is πgb - continuous but it is not πgs- continuous.

Theorem 4.6: Let $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:
(1) f is πgb - continuous;
(2) The inverse image of every open set in Y is πgb - open in X.

Proof: (1)⇒(2) Let U be a open subset of X. Then $Y \setminus U$ is closed. Since f is πgb - continuous, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is πgb - closed in X. Hence $f^{-1}(U)$ is πgb - open in X.

(2)⇒(1) Let V be a closed subset of Y. Then $Y \setminus V$ is open and by the hypothesis (2) $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is πgb - open in X. So $f^{-1}(V)$ is πgb - closed. Therefore, f is πgb - continuous.

Theorem 4.7: If $f : X \rightarrow Y$ is πgb - continuous then $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$ for every subset A of X.

Proof: Let A be a subset of X. Since f is πgb - continuous and $A \subset f^{-1}(cl(f(A)))$, we obtain $\pi gb\text{-}cl(A) \subset f^{-1}(cl(f(A))$ and then $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$.

Remark 4.8: The converse of Theorem 4.7 need not be true as shown in the following example.

Example 4.9: Let X={a,b,c,d} and $\tau = \{X,\varnothing,\{a,b\},\{d\},\{a,b,d\}\}$, $\sigma = \{X,\varnothing,\{d\}\}$.We define the function $f:(X,\tau)\to(X,\sigma)$ such as $f(a) = c$, $f(b) = a$, $f(c) = d$, $f(d) = b$. Then $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$ for every subset A of X. Since {a,b,c} is closed in (X,σ) but $f^{-1}(\{a,b,c\}) = \{a,b,d\}$ is not πgb - closed in (X,τ) , f is not πgb - continuous.

Theorem 4.10: Let $f : X \to Y$ be a function. Then the following statements are equivalent:
(1) For each $x \in X$ and each open set V containing $f(x)$ there exists a πgb - open set U containing x such that $f(U) \subset V$.
(2) $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$ for every subset A of X.

Proof: (1) ⇒ (2) Let $y \in f(\pi gb\text{-}cl(A))$ and let V be any open neighborhood of y. Then there exist a $x \in X$ and a πgb - open set U such that $f(x) = y$, $x \in U$, $x \in \pi gb\text{-}cl(A)$. and $f(U) \subset V$. By Lemma 3.12, $U \cap A \neq \varnothing$ and hence $f(A) \cap V \neq \varnothing$. Therefore, $y = f(x) \in cl(f(A))$.

(2) ⇒ (1) Let $x \in X$ and V be any open set containing $f(x)$. Let $A = f^{-1}(Y \setminus V)$. Since $f(\pi gb\text{-}cl(A)) \subset cl(f(A)) \subset Y \setminus V$ then $\pi gb\text{-}cl(A) = A$. Since $x \notin \pi gb\text{-}cl(A)$. there exists a πgb - open set U containing x such that $U \cap A = \varnothing$ and hence $f(U) \subset f(X \setminus A) \subset V$.

Theorem 4.11: Let X be an extremely disconnected space and $f : X \to Y$ be a function. If f is πgb - continuous and $m\text{-}\pi$ - closed then f is πgb - irresolute.

Proof: Let A be a πgb - closed subset of Y. Then $f^{-1}(A) \subset U$, where U is π - open in X. So $X \setminus U \subset f^{-1}(Y \setminus A)$. Hence $f(X \setminus U) \subset Y \setminus A$. Since f is $m\text{-}\pi$ - closed, $f(X \setminus U)$ is π - closed. Since Y \ A is πgb - open then $f(X \setminus U) \subset bint(Y \setminus A) = Y \setminus bcl(A)$. Thus $f^{-1}(bcl(A)) \subset U$. Since f is πgb - continuous and X is extremely disconnected, $f^{-1}(cl(A))$ is πgb - closed. Therefore, $bcl(f^{-1}(bcl(A))) \subset U$ and hence $bcl(f^{-1}(A)) \subset bcl(f^{-1}(bcl(A))) \subset U$. It follows that $f^{-1}(A)$ is πgb - closed. This shows that f is πgb - irresolute.

Definition 4.12: A function $f : X \to Y$ is said to be almost πgb - continuous if $f^{-1}(V)$ is πgb - closed in X for every regular closed set V of Y.

Theorem 4.13: For a function $f : X \to Y$, the following statements are equivalent:

(1) f is almost πgb - continuous;

(2) $f^{-1}(V)$ is πgb - open in X for every regular open set V of Y;

(3) $f^{-1}(int(cl(V)))$ is πgb - open in X for every open set V of Y;

(4) $f^{-1}(cl(int(V)))$ is πgb - closed in X for every closed set V of Y.

Proof: (1) ⇒ (2) Let V be a regular open subset of Y. Since Y \ V is regular closed and f is almost πgb - continuous then $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is πgb - closed in X. Thus $f^{-1}(V)$ is πgb - open in X.

(2) ⇒ (1) Let V be a regular closed subset of Y. Then Y \ V is regular open. By the hypothesis, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is πgb - open in X. Hence $f^{-1}(V)$ is πgb - closed. This shows that f is πgb - continuous.

(2) ⇒ (3) Let V be an open subset of Y. Then $int(cl(V))$ is regular open. By the hypothesis, $f^{-1}(int(cl(V)))$ is πgb - open in X.

(3) ⇒ (2) Let V be a regular open subset of Y. Since $V = int(cl(V))$ and every regular open set is open then $f^{-1}(V)$ is πgb - open in X.

(3) $\Rightarrow$ (4) Let V be a closed subset of Y. Then $Y \setminus V$ is open. By the hypothesis, $f^{-1}(int(cl(Y \setminus V))) = f^{-1}(int(cl(Y \setminus V))) = f^{-1}(Y \setminus cl(int(V))) = X \setminus f^{-1}(cl(int(V)))$ is πgb - open in X. Therefore, $f^{-1}(cl(int(V)))$ is πgb - closed in X.

(4) $\Rightarrow$ (3) Let V be a open subset of Y. Then $Y \setminus V$ is closed. By the hypothesis, $f^{-1}(cl(int(Y \setminus V))) = f^{-1}(Y \setminus int(cl(V)))$ $= X \setminus f^{-1}(int(cl(V)))$ is πgb - closed in X. Hence $f^{-1}(int(cl(V)))$ is πgb - open in X.

Remark 4.14: For a function $f : X \to Y$, the following implications hold:

$$\begin{tikzcd}
\pi\text{-con} \arrow[r] & \text{con} \arrow[r] & g\text{-con} \arrow[r] \arrow[d] & \pi g\text{-con} \arrow[d] & \\
 & & gb\text{-con} \arrow[r] & \pi gb\text{-con} \arrow[r] & \text{almost-}\pi gb\text{-con} \\
 & & b\text{-con} \arrow[u] \arrow[r] & \text{almost-}b\text{-con} \arrow[ur] &
\end{tikzcd}$$

Remark 4.15:
(a) Every πgb - continuous function is almost πgb - continuous,
(b) Every almost b - continuous function is almost πgb - continuous.

However, none of these implications is reversible as shown by the following examples.

Example 4.16: In Example 4.9, f is almost πgb - continuous but it is not πgb - continuous since for the regular closed set $\{a,b,c\}$ of (X,σ), we have $f^{-1}(\{a,b,c\})=\{a,b,d\}$ is not πgb - closed in (X,τ).

Example 4.17: In Example 4.3, f is almost πgb - continuous but it is not almost b - continuous since for the regular closed set $\{x,y,z\}$ of (Y,σ), we have $f^{-1}(\{x,y,z\})=\{a,c,d\}$ is not b - closed in (X,τ).

Theorem 4.18: Let X be a πgb - $T_{1/2}$ topological space. Then $f : X \to Y$ is almost πgb - continuous if and only if f is almost b - continuous.

Proof: Necessity. Let A be a regular closed subset of Y and $f : X \to Y$ be an almost πgb - continuous function. Then $f^{-1}(A)$ is πgb - closed in X. Since X is πgb - $T_{1/2}$ space, $f^{-1}(A)$ is b - closed in X. Hence f is almost b - continuous.

Sufficiency. Suppose that f is almost b - continuous and A be a regular closed subset of Y. Then $f^{-1}(A)$ is b - closed in X. Since every b - closed set is πgb - closed then $f^{-1}(A)$ is πgb - closed. Therefore, f is almost πgb - continuous.

Theorem 4.19: Let X is a πgb - $T_{1/2}$ space and $f : X \to Y$ be a function. Then
(1) f is almost πgb - continuous if and only if f is almost b - continuous,
(2) f is πgb - continuous if and only if f is b - continuous.(or gb - continuous)

Proof: The proof is obvious.

5. πgb - compactness

Definition 5.1: A collection $\{G_i : i \in \Lambda\}$ of πgb - open sets in a topological space X is called a πgb - open cover of a subset A of X if $A \subset \{G_i : i \in \Lambda\}$ holds.

Definition 5.2: A topological space X is πgb - compact if every πgb - open cover of X has a finite subcover.

Definition 5.3: A subset A of a topological space X is said to be πgb-compact relative to X if, for every collection $\{U_i : i \in I\}$ of πgb-open subsets of X such that $A \subset \cup\{U_i : i \in I\}$ there exists a finite subset I_0 of I such that $A \subset \cup\{U_i : i \in I_0\}$.

Definition 5.4: A subset A of a topological space X is said to be πgb-compact if A is πgb-compact as a subspace of X.

Theorem 5.5: Every πgb-closed subset of a πgb-compact space is πgb-compact space relative to X.

Proof: Let A be a πgb-closed subset of a πgb-compact space X. Let $\{U_i : i \in I\}$ be a πgb-open cover of X. So $A \subset \cup_{i\in I} U_i$ and then $(X \setminus A) \cup (\cup_{i\in I} U_i) = X$. Since X is πgb-compact, there exists a finite subset I_0 of I such that $(X \setminus A) \cup (\cup_{i\in I_0} U_i) = X$. Then $A \subset \cup_{i\in I_0} U_i$ and hence A is πgb-compact relative to X**.**

A nearly compact space [25] is a topological space in which every cover by regular open sets has a finite subcover.

Theorem 5.6: The surjective πgb-continuous (resp. almost πgb-continuous) image of πgb-compact space is compact (resp. nearly compact).

Proof: Let $\{U_i : i \in I\}$ be any cover of Y by open (resp. regular open) subsets. Since f is πgb-continuous (resp. almost πgb-continuous), then $\{f^{-1}(U_i) : i \in I\}$ is πgb-open cover of X. By πgb-compactness of X, there exists a finite subset I_0 of I such that $X = \cup_{i\in I_0} f^{-1}(U_i)$. Since f is surjective, we obtain $Y = \cup_{i\in I_0} U_i$. This shows that Y is compact (resp. nearly compact).

Theorem 5.7: If $f : X \to Y$ is πgb-irresolute and a subset A of X is πgb-compact relative to X, then the image $f(A)$ is πgb-compact relative to Y.

Proof: Let $\{U_i : i \in I\}$ be any collection of πgb-open subsets of Y such that $f(A) \subset \cup\{U_i : i \in I\}$. Then $A \subset \cup \{f^{-1}(U_i) : i \in I\}$ holds. Since by hypothesis A is πgb-compact relative to X, there exists a finite subset I_0 of I such that $A \subset \cup\{f^{-1}(U_i) : i \in I_0\}$. Therefore, we have $f(A) \subset \cup\{U_i : i \in I_0\}$, which shows that $f(A)$ is πgb-compact relative to Y.

Definition 5.8: A space X is said to be
(1) πgp-compact [24] if every πgp-open cover of X has a finite subcover.
(2) gb-compact [4] if every gb-open cover of X has a finite subcover.
(3) b-compact [13] if every b-open cover of X has a finite subcover.

Remark 5.9: Since every regular open set is open, b-open, gb-open, πgb-open and πgp-open set, for a space X, the following implications hold:

$$\begin{array}{ccccccccc} \pi gb\text{-compact} & \to & gb\text{-compact} & \to & b\text{-compact} & \to & \text{compact} & \to & \text{nearly-compact} \\ \downarrow & & & & & & & & \\ \pi gp\text{-compact} & & & & & & & & \end{array}$$

Definition 5.10: A function $f : X \to Y$ is said to be πgb-open if $f(U)$ is πgb-open in Y for every πgb-open set U of X.

Theorem 5.11: If $f : X \to Y$ is πgb-open bijection and Y is πgb-compact space then X is a πgb-compact space.

Proof: Let $\{U_i : i \in I\}$ be a πgb-open cover of X. So $X = \cup_{i\in I} U_i$ and then $Y = f(X) = f(\cup_{i\in I} U_i) = \cup_{i\in I} f(U_i)$**.**

Since f is πgb - open, for each $i \in I$, $f(U_i)$ is πgb - open set. By πgb - compactness of Y, there exists a finite subset I_0 of I such that $Y = \cup_{i\in I_0} f(U_i)$. Therefore, $X = f^{-1}(Y) = f^{-1}(\cup_{i\in I_0} f(U_i)) = \cup_{i\in I_0} f^{-1}(f(U_i)) = \cup_{i\in I_0} U_i$. This shows that X is πgb - compact.

Theorem 5.12: If $f : X \to Y$ is πgb - irresolute bijection and X is πgb - compact space then Y is a πgb - compact space.

Proof: Let $\{U_i : i \in I\}$ be a πgb - open cover of Y. So $Y = \cup_{i\in I} U_i$ and then $X = f^{-1}(Y) = f^{-1}(\cup_{i\in I} U_i) = \cup_{i\in I} f^{-1}(U_i)$. Since f is πgb - irresolute, it follows that for each $i \in I$, $f^{-1}(U_i)$ is πgb - open set. By πgb - compactness of X, there exists a finite subset I_0 of I such that $X = \cup_{i\in I_0} f^{-1}(U_i)$. Therefore, $Y = f(X) = f(\cup_{i\in I_0} f^{-1}(U_i)) = \cup_{i\in I_0} f(f^{-1}(U_i)) = \cup_{i\in I_0} U_i$. This shows that Y is πgb - compact.

Theorem 5.13: If $f : X \to Y$ is πgb - continuous bijection and X is πgb - compact space then Y is a πgb - compact space.

Proof: The proof is similar to that of Theorem 5.12.

6. Quasi - b - normal spaces

Definition 6.1: A space X is said to be quasi - b - normal if for every pair of disjoint π - closed subsets A,B of X, there exist disjoint b - open subsets U,V of X such that $A \subset U$ and $B \subset V$.

Definition 6.2: A space X is said to be quasi-normal [28] (resp. quasi - s - normal [6]) if for every pair of disjoint π - closed subsets A,B of X, there exist disjoint open (resp. semi-open) subsets U,V of X such that $A \subset U$ and $B \subset V$.

Definition 6.3: A space X is said to be b - normal (or γ - normal [12]) if for every pair of disjoint closed subsets A,B of X, there exist disjoint b - open subsets U,V of X such that $A \subset U$ and $B \subset V$.

Remark 6.4: For a topological space X, the following implications hold:

$$\begin{array}{ccc}
\textbf{normal} & \rightarrow & \textbf{quasi - normal} \\
\downarrow & & \downarrow \\
\textbf{s - normal} & \rightarrow & \textbf{quasi - s - normal} \\
\downarrow & & \downarrow \\
\textbf{b - normal} & \rightarrow & \textbf{quasi - b - normal}
\end{array}$$

In general, the converse of implications in the above diagram need not be true.

Example 6.5: Let X={a,b,c}, $\tau = \{X, \emptyset, \{a,b\}, \{a\}, \{a,c\}\}$. (X,τ) is quasi - b - normal space but it is not b - normal space.

Theorem 6.6: The following statements are equivalent for a space X;

(a) X is quasi - b - normal;
(b) For any disjoint π - closed sets A and B, there exist disjoint gb - open subsets U,V of X such that $A \subset U$ and $B \subset V$;
(c) For any closed set A and any π - open set B containing A, there exists a gb - open set U such that $A \subset U \subset bcl(U) \subset B$;
(d) For any disjoint π - closed sets A and B, there exist disjoint πgb - open subsets U,V of X such that $A \subset U$ and $B \subset V$;
(e) For any π - closed set A and any π - open set B containing A, there exists a πgb - open set U such that $A \subset U \subset bcl(U) \subset B$.

Proof. (a) $\Rightarrow$ (b) The proof is obvious.

(b) $\Rightarrow$ (c) Let A be any π-closed subset of X and B any π-open subset of X such that $A \subset B$. Then A and $X \setminus B$ disjoint π-closed subset of X. Therefore, there exist disjoint gb-open sets U and V such that $A \subset U$ and $X \setminus B \subset V$.

By the definition of gb-open sets, we have $X \setminus B \subset \mathrm{bint}(V)$ and $U \cap \mathrm{bint}(V)=\varnothing$. Therefore, we obtain $\mathrm{bcl}(U) \cap \mathrm{bint}(V) = \varnothing$ and hence $A \subset U \subset \mathrm{bcl}(U) \subset B$.

(c) $\Rightarrow$ (d) Let A and B be any disjoint π-closed subsets of X. Then $A \subset X \setminus B$ and $X \setminus B$ is π-open and hence there exists a gb-open subset G of X such that $A \subset G \subset \mathrm{bcl}(G) \subset X \setminus B$. Since every gb-open set is πgb-open, G is πgb-open and $X \setminus \mathrm{bcl}(G)$ is b-open so $X \setminus \mathrm{bcl}(G)$ is πgb-open. Now put $V = X \setminus \mathrm{bcl}(G)$. Then G and V are disjoint πgb-open subsets of X such that $A \subset G$ and $B \subset V$.

(d) $\Rightarrow$ (e) The proof is similar to that of (b) $\Rightarrow$ (c)

(e) $\Rightarrow$ (a) Let A and B be any disjoint π-closed subsets of X. Then $A \subset X \setminus B$ and $X \setminus B$ is π-open and hence there exists a πgb-open subset G of X such that $A \subset G \subset \mathrm{bcl}(G) \subset X \setminus B$. Put $U = \mathrm{bint}(G)$ and $V = X \setminus \mathrm{bcl}(G)$. Then U and V are disjoint b-open subsets of X such that $A \subset G$ and $B \subset V$. Therefore, X is quasi-b-normal.

Definition 6.7: A function $f : X \to Y$ is said to be almost πgb-closed if for each regular closed subset F of X, $f(F)$ is πgb-closed subset of Y.

Proposition 6.8: A surjection $f : X \to Y$ almost πgb-closed if and only if for each subset G of Y and each $U \in RO(X)$ containing $f^{-1}(G)$, there exists a πgb-open subset V of Y such that $G \subset V$ and $f^{-1}(V) \subset U$.

Proof: Necessity. Suppose that f is almost πgb-closed. Let G be a subset of Y and $U \in RO(X)$ containing $f^{-1}(G)$. If $V = Y \setminus f(X \setminus U)$, then V is a πgb-open set of Y such that $G \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $X \setminus F \in RO(X)$ and $f^{-1}(Y \setminus f(F)) \subset X \setminus F$. There exists a πgb-open set V of Y such that $Y \setminus f(F) \subset V$ and $f^{-1}(V) \subset X \setminus F$. Therefore, we have $Y \setminus V \subset f(F)$ and $F \subset f^{-1}(Y \setminus V)$. Hence, we obtain $f(F) = Y \setminus V$ and $f(F)$ is πgb-closed in Y. This shows that f is almost πgb-closed.

Theorem 6.9: Let $f : X \to Y$ be a continuous, almost πgb-closed surjection. If X is normal, then Y is quasi-b-normal.

Proof: Let A and B be disjoint π-closed subsets of Y. Since f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of X. By the normality of X, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = \mathrm{int}(\mathrm{cl}(U))$ and $H = \mathrm{int}(\mathrm{cl}(V))$. Then G and H are disjoint regular open subsets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Proposition 6.8, there exist πgb-open subsets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since $G \cap H = \varnothing$ and f is surjective, $K \cap L = \varnothing$. It follows from Theorem 6.6 (d) that Y is quasi-b-normal.

Theorem 6.10: Let $f : X \to Y$ be a π-irresolute, almost πgb-closed surjection. If X is quasi-normal, then Y is quasi-b-normal.

Proof: Let A and B be disjoint π-closed subsets of Y. Since f is π-irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint π-closed subsets of X. By the quasi-normality of X, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = \mathrm{int}(\mathrm{cl}(U))$ and $H = \mathrm{int}(\mathrm{cl}(V))$. Then $f^{-1}(A) \subset U \subset G, f^{-1}(B) \subset V \subset H, G \cap H = \varnothing$ and $G,H \in RO(X)$. Since f is almost πgb-closed, by Proposition 6.8 there exist πgb-open subsets K and L of Y such that

$A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since f is surjective, we have $K \cap L = \varnothing$. This shows that Y is quasi-b-normal.

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