On $\pi gb$ - Closed Sets and Related Topics

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Abstract

In this paper, we study $\pi gb$-closed sets due to Sreeja D. and Janaki C. [26] and investigate further properties of these sets. By means of $\pi gb$-closed sets we introduced a new class of functions called almost $\pi gb$-continuous functions which are generalizations both $\pi gb$-continuity and almost b-continuity. Moreover, the notions of $\pi gb$-compactness and quasi-b-normality in topological spaces are introduced and their some properties are studied.

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1. Introduction

Continuity on topological spaces, as significant and fundamental subject in the study of topology, has been researched by several mathematicians. Many investigations related to generalized closed sets have been published various forms of generalized continuity types have been introduced. The study of generalized closed sets in a topological space was initiated by Levine [18] and the concept of $T_{1/2}$-space was introduced, $gb$-closed sets were defined and studied by Ekici [12] and Ganster - Steiner [14]. Recently, Benchalli and Bansali [4] introduced the notion of $gb$-compactness. In 1968, Zaitsev [28] defined the concept of $\pi$-closed sets and a class of topological spaces called quasi normal spaces. Later Dontchev and Noiri [8] introduced the notion of $\pi g$-closed sets and used this notion to obtain a characterization and some preservation theorems for quasi normal spaces. Park [23] defined $\pi gp$-closed sets. Next, Aslim, Caksu and Noiri [3] introduced the notion of $\pi gs$-closed sets. Caksu Guler and Aslım [6] obtained characterizations of quasi-s-normal spaces by using $\pi gs$-closed sets.

The aim of this paper is to investigate further properties of $\pi gb$-closed sets due to Sreeja D.and Janaki C. [26]. The paper consists of six sections. In section 3, we introduce the concept of $\pi gb$-closure and obtain some of its fundamental properties. Besides, in section 4, we present a new generalization of almost continuity called almost $\pi gb$-continuity. The notion of almost $\pi gb$-continuity is a weaker form of almost b-continuity [15]. Furthermore, in section 5, we introduce the concept of $\pi gb$-compactness and study their behaviour under $\pi gb$-continuous and almost $\pi gb$-continuous functions. In the last section, we introduce and characterize a new class of space, called quasi-b-normal spaces.

2. Preliminaries

Throughout this paper, $(X,\tau)$ and $(Y,\sigma)$ (or simply X and Y) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Also, in this paper spaces mean topological spaces and $f : (X,\tau) \to (Y,\sigma)$ (or simply $f : X \to Y$ ) denotes a function f of a space $(X,\tau)$ into a space $(Y,\sigma)$. Let A be a subset of a space X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

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**Definition 2.1:** A subset \( A \) of a topological space \( X \) is called:
(a) pre-open [20] if \( A \subset \text{int}(\text{cl}(A)) \),
(b) semi-open [19] if \( A \subset \text{cl}(\text{int}(A)) \),
(c) regular open [27] if \( A = \text{int}(\text{cl}(A)) \),
(d) \( b \)-open [1] or \( \gamma \)-open [13] if \( A \subset \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A)) \).

The finite union of regular open sets is said to be \( \pi \)-open. The complement of a \( \pi \)-open set is said to be \( \pi \)-closed. The complement of a \( b \)-open set is called \( b \)-closed [1]. The intersection of all \( b \)-closed sets containing \( A \) is called the \( b \)-closure [1] of \( A \) and is denoted by \( b\text{cl}(A) \). The \( b \)-interior [1] of \( A \) is defined to be the union of all \( b \)-open sets contained in \( A \) and is denoted by \( b\text{int}(A) \).

**Lemma 2.2 [1]:** Let \( A \) be a subset of a space \( X \). Then
(a) \( b\text{cl}(A) = \text{sc}(A) \cap \text{pc}(A) = A \cup \text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) \),
(b) \( b\text{int}(A) = \text{int}(A) \cup \text{pint}(A) = A \cap \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A)) \).

**Definition 2.3:** A subset \( A \) of a topological space \( X \) is called:
(a) \( g \)-closed [18] if \( \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( X \),
(b) \( gp \)-closed [19] if \( \text{pcl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( X \),
(c) \( gs \)-closed [2] if \( \text{sc}(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( X \),
(d) \( gb \)-closed [12,14] if \( b\text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( X \),
(e) \( \pi g \)-closed [8] if \( \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( \pi \)-open in \( X \),
(f) \( \pi gp \)-closed [23] if \( \text{pcl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( \pi \)-open in \( X \),
(g) \( \pi gs \)-closed [3] if \( \text{sc}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( \pi \)-open in \( X \),
(h) \( \pi gb \)-closed [26] if \( b\text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( \pi \)-open in \( X \),
(i) \( \pi gb \)-open (resp. \( g \)-open, \( gp \)-open, \( gs \)-open, \( gb \)-open, \( \pi g \)-open, \( \pi gp \)-open, \( \pi gs \)-open) if the complement of \( A \) is \( \pi gb \)-closed (resp. \( g \)-closed, \( gp \)-closed, \( gs \)-closed, \( gb \)-closed, \( \pi g \)-closed, \( \pi gp \)-closed, \( \pi gs \)-closed).

The family of all \( \pi gb \)-closed (resp. \( b \)-closed) sets in a topological space \((X,\tau)\) is denoted by \( \pi GBC(X) \) (resp. \( BC(X) \)).

**Definition 2.4:** A function \( f : X \rightarrow Y \) is said to be
(a) regular open [21] if \( f(V) \) is regular open in \( Y \) for every open set \( V \) of \( X \),
(b) \( b \)-closed [22] if \( f(V) \) is \( b \)-closed in \( Y \) for every \( b \)-closed set \( V \) of \( X \),
(c) \( m \)-\( \pi \)-closed [11] if \( f(V) \) is \( \pi \)-closed in \( Y \) for every \( \pi \)-closed set \( V \) of \( X \),
(d) \( \pi \)-continuous [8] (resp. \( \pi g \)-continuous [7], \( \pi gp \)-continuous [24], \( \pi gs \)-continuous [3]) if \( f^{-1}(V) \) is \( \pi \)-closed (resp. \( \pi g \)-closed, \( \pi gp \)-closed, \( \pi gs \)-closed) in \( X \) for every closed set \( V \) of \( Y \),
(e) \( b \)-continuous [13] (resp. \( g \)-continuous [18], \( gb \)-continuous [22]) if \( f^{-1}(V) \) is \( b \)-closed (resp. \( g \)-closed, \( gb \)-closed) in \( X \) for every closed set \( V \) of \( Y \),
(f) almost \( b \)-continuous [15] if \( f^{-1}(V) \) is \( b \)-closed in \( X \) for every regular closed set \( V \) of \( Y \),
(g) \( \pi gb \)-continuous [26] if \( f^{-1}(V) \) is \( \pi gb \)-closed in \( X \) for every closed set \( V \) of \( Y \),
(h) \( \pi \)-irresolute if \( f^{-1}(V) \) is \( \pi \)-closed in \( X \) for every \( \pi \)-closed set \( V \) of \( Y \),
(i) \( b \)-irresolute [10] if \( f^{-1}(V) \) is \( b \)-closed in \( X \) for every \( b \)-closed set \( V \) of \( Y \),
(j) \( \pi gb \)-irresolute [26] if \( f^{-1}(V) \) is \( \pi gb \)-closed in \( X \) for every \( \pi gb \)-closed set \( V \) of \( Y \).

3. The further properties of \( \pi gb \)-closed sets and \( \pi gb \)-closure operator

**Theorem 3.1:** [26] Every \( \pi gs \)-closed set is \( \pi gb \)-closed.

The following example show that above implication is not reversible.

**Example 3.2:** Let \( \tau \) be the usual topology for \( \mathbb{R} \) and \( A = (0, 2) \setminus \mathbb{Q} \subset \mathbb{R} \), where \( \mathbb{Q} \) denotes the set of rational numbers. Then \( A \) is \( \pi gb \)-closed but it is not \( \pi gs \)-closed.
Theorem 3.3: For a subset $A$ of $X$, the following statements are equivalent:

1. $A$ is $\pi$-open and $\pi gb$-closed.
2. $A$ is regular open.

Proof: (1) $\Rightarrow$ (2) Let $A$ be a $\pi$-open and $\pi gb$-closed subset of $X$. Then $bcl(A) \subset A$ and so $int(cl(A)) \subset A$ holds. Since $A$ is open then $A$ is pre-open and thus $A \subset int(cl(A))$. Therefore, we have $int(cl(A))=A$, which shows that $A$ is regular open.

(2) $\Rightarrow$ (1) Since every regular open set is $\pi$-open then $bcl(A)=A$ and $bcl(A) \subset A$. Hence $A$ is $\pi gb$-closed.

A subset $A$ of a topological space $X$ is said to be $Q$-set [16] if $int(cl(A))=cl(int(A))$.

Theorem 3.4: For a subset $A$ of $X$, the following statements are equivalent:

1. $A$ is $\pi$-clopen,
2. $A$ is $\pi$-open, $Q$-set and $\pi gb$-closed.

Proof: (1) $\Rightarrow$ (2) Let $A$ be a $\pi$-clopen subset of $X$. Then $A$ is $\pi$-closed and $\pi$-open. Thus $A$ is closed and open. Hence, $A$ is $Q$-set. Since every $\pi$-closed is $\pi gb$-closed then $A$ is $\pi gb$-closed.

(2) $\Rightarrow$ (1) By Theorem 3.3, $A$ is regular open. Since $A$ is $Q$-set, $A = int(cl(A)) = cl(int(A))$. Therefore, $A$ is regular closed. Then $A$ is $\pi$-closed. Hence $A$ is $\pi$-clopen.

Proposition 3.5: [9] Let $A$ be a subset of a topological space $X$. If $A$ is semi-open then $pcl(A) = cl(A)$.

A topological space $X$ is said to be extremely disconnected [5] if the closure of every open subset of $X$ is open in $X$.

Theorem 3.6: A space $X$ is extremely disconnected if and only if every $\pi gb$-closed subset of $X$ is $\pi gp$-closed.

Proof: Suppose that $X$ is extremely disconnected. Let $A$ be $\pi gb$-closed and let $U$ be an $\pi$-open set containing $A$. Then $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))] \subset U$, i.e. $[int(cl(A)) \cap cl(int(A))] \subset U$. Since $int(cl(A))$ is closed, we have $cl(int(A)) \subset cl(int(cl(A))) \subset U$. It follows that $pcl(A) = A \cup cl(int(A)) \subset U$. Hence $A$ is $\pi gp$-closed.

To prove the converse, let every $\pi gb$-closed subset of $X$ be $\pi gp$-closed. Let $A$ be a regular open subset of $X$. Then $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))] = A \cup [A \cap cl(int(A))] \subset A$. Then $A$ is $\pi gb$-closed and so $A$ is $\pi gp$-closed. Since every regular open is semi-open set and by Proposition 3.5, we have $cl(A) = pcl(A)$. Hence $cl(A) \subset A$. Therefore, $A$ is closed. This shows that $X$ is extremely disconnected.

A topological space $X$ is said to be hyperconnected if the closure of every open subset is $X$.

Theorem 3.7: Let $X$ be a hyperconnected space. Then every $\pi gb$-closed subset of $X$ is $\pi gs$-closed.

Proof: Assume that $X$ is hyperconnected. Let $A$ be $\pi gb$-closed and let $U$ be an $\pi$-open set containing $A$. Then $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))] = A \cup int(cl(A)) = scl(A)$. Since $bcl(A) = scl(A)$, we have $scl(A) \subset U$. Hence, $A$ is $\pi gs$-closed.

Theorem 3.8: Let $A$ be a $\pi gb$-closed set such that $cl(A) = X$. Then $A$ is $\pi gp$-closed.

Proof: Suppose that $A$ be $\pi gb$-closed set such that $cl(A) = X$. Let $U$ be an $\pi$-open set containing $A$. Since $bcl(A) = A \cup [int(cl(A)) \cap cl(int(A))]$ and $cl(A) = X$, we obtain $bcl(A) = A \cup cl(int(A)) = pcl(A) \subset U$. Therefore, $A$ is $\pi gp$-closed.

Definition 3.9: A topological space $X$ is said to be $\pi gb$- $T_{1/2}$ space [26] if every $\pi gb$-closed set is $b$-closed.
Theorem 3.10: For a space $X$, the following statements are equivalent:

1. $X$ is $\pi gb$\text{-}T_{\frac{1}{2}}$,
2. For every subset $A$ of $X$, $A$ is $\pi gb$\text{-}open if and only if $A$ is $b$\text{-}open.

Proof: $(1) \Rightarrow (2)$ Let the space $X$ be $\pi gb$\text{-}T_{\frac{1}{2}}$ and let $A$ be a $\pi gb$\text{-}open subset of $X$. Then $X \setminus A$ is $\pi gb$\text{-}closed and so $X \setminus A$ is $b$\text{-}closed. Hence $A$ is $b$\text{-}open.

Conversely, let $A$ be a $b$\text{-}open subset of $X$. Thus $X \setminus A$ is $b$\text{-}closed. Since every $b$\text{-}closed set is $\pi gb$\text{-}closed then $X \setminus A$ is $\pi gb$\text{-}closed. Therefore, $A$ is $\pi gb$\text{-}open.

$(2) \Rightarrow (1)$ Let $A$ be a $\pi gb$\text{-}open subset of $X$. Then $X \setminus A$ is $\pi gb$\text{-}open. By the hypothesis $X \setminus A$ is $b$\text{-}open. Thus $A$ is $b$\text{-}closed. Since every $\pi gb$\text{-}closed set is $b$\text{-}closed, thus $X$ is $\pi gb$\text{-}T_{\frac{1}{2}}$.

Definition 3.11: The intersection of all $\pi gb$\text{-}closed sets, each containing a set $A$ in a topological space $X$ is called the $\pi gb$\text{-}closure of $A$ and it is denoted by $\pi gb$\text{-}cl$(A)$.

Lemma 3.12: Let $A$ be a subset of $X$ and $x \in X$. Then $x \notin \pi gb$\text{-}cl$(A)$ if and only if $V \cap A = \emptyset$ for every $\pi gb$\text{-}open set $V$ containing $x$.

Proof: Assume that there exists a $\pi gb$\text{-}open set $V$ containing $x$ such that $V \cap A = \emptyset$. Since $A \subset X \setminus V$, $\pi gb$\text{-}cl$(A)$ $\subset X \setminus V$ and then $x \notin \pi gb$\text{-}cl$(A)$, a contradiction. To prove the converse, suppose that $x \notin \pi gb$\text{-}cl$(A)$. Then there exists a $\pi gb$\text{-}closed set $F$ containing $A$ such that $x \notin F$. Since $x \in X \setminus F$ and $X \setminus F$ is $\pi gb$\text{-}open, $(X \setminus F) \cap A = \emptyset$ a contradiction.

Lemma 3.13: Let $A$ and $B$ be subsets of $X$. Then we obtain

(a) $\pi gb$\text{-}cl$(\emptyset) = \emptyset$, $\pi gb$\text{-}cl$(X) = X$,
(b) $A \subset \pi gb$\text{-}cl$(A)$,
(c) If $A$ is $\pi gb$\text{-}closed then $\pi gb$\text{-}cl$(A) = A$,
(d) $\pi gb$\text{-}cl$(A) = \pi gb$\text{-}cl$(\pi gb$\text{-}cl$(A))$,
(e) If $A \subset B$ then $\pi gb$\text{-}cl$(A) \subset \pi gb$\text{-}cl$(B)$,
(f) $\pi gb$\text{-}cl$(A \cup B) \supset \pi gb$\text{-}cl$(A) \cup \pi gb$\text{-}cl$(B)$,
(g) $\pi gb$\text{-}cl$(A \cap B) \supset \pi gb$\text{-}cl$(A) \cap \pi gb$\text{-}cl$(B)$.

Proof: We are obtained by Definition 2.3 and Lemma 3.12.

Remark 3.14: The following examples show that the converses of Lemma 3.13 (c), (f) and (g) need not be true.

Example 3.15: Let $X = \{a,b,c,d,e,f\}$ and $\tau = \{X,\emptyset,\{a,b\},\{c,d\},\{a,b,c,d\}\}$. Let $A = \{a,b,c,d\}$. Then $\pi gb$\text{-}cl$(A) = A$ but $A$ is not $\pi gb$\text{-}closed.

Example 3.16: Let $X = \{a,b,c,d,e\}$ and $\tau = \{X,\emptyset,\{a\},\{e\},\{a,e\},\{a,c,d\},\{c,d,e\},\{a,c,d,e\}\}$. Let $A = \{a,c,d,e\}$ and $B = \{b,c,d\}$. Then $A$ is not $\pi gb$\text{-}closed and $B$ is $\pi gb$\text{-}closed. Since $\pi gb$\text{-}cl$(A) = X$ and $\pi gb$\text{-}cl$(B) = B$, we have $\pi gb$\text{-}cl$(A) \cap \pi gb$\text{-}cl$(B) = B = \{b,c,d\}$ but $\pi gb$\text{-}cl$(A \cap B) = \{c,d\}$.

Example 3.17: Let $X$ be topological space in Example 3.16, let $A = \{a,c,e\}$ and $B = \{d\}$. Then $A$ is not $\pi gb$\text{-}closed and $B$ is $\pi gb$\text{-}closed. Since $\pi gb$\text{-}cl$(A) = A$ and $\pi gb$\text{-}cl$(B) = B$ and so $\pi gb$\text{-}cl$(A) \cup \pi gb$\text{-}cl$(B) = \{a,c,d,e\}$ but $\pi gb$\text{-}cl$(A \cup B) = X$.
4. Almost $\pi gb$ - continuity and related some continuities

**Remark 4.1:** For a function $f : X \rightarrow Y$, the following implications hold:

\[
\text{b- con.} \rightarrow \text{gb- con.} \quad \uparrow \quad \text{pre-con.} \rightarrow \text{gp- con.} \rightarrow \text{sgp- con.} \\
\quad \uparrow \quad \uparrow \\
\text{g- con.} \rightarrow \text{sg- con.} \rightarrow \text{sgb- con.} \quad \downarrow \\
\quad \downarrow \\
\text{semal-con.} \rightarrow \text{g- con.} \rightarrow \text{sg- con.} \\
\quad \downarrow \\
\text{b- con.} \rightarrow \text{gb- con.}
\]

**Remark 4.2:** The following examples show that:

(a) Every $\pi gb$ - continuous function need not be $gb$ - continuous or $\pi g$ - continuous,

(b) Every $\pi gb$ - continuous function need not be $\pi gp$ - continuous,

(c) Every $\pi gb$ - continuous function need not be $\pi gs$ - continuous.

**Example 4.3:** Let $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ and $Y = \{x, y, z, t\}$, $\sigma = \{Y, \emptyset, \{x, y, z\}, \{t\}\}$.

Define a function $f : X \rightarrow Y$ as follows: $f(a) = z$, $f(b) = f(e) = t$, $f(c) = y$ and $f(d) = x$. Then $f$ is a $\pi gb$ - continuous but it is not $gb$ - continuous.

**Example 4.4:** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b\}, \{b\}\}$ and $Y = \{x, y, z\}$, $\sigma = \{Y, \emptyset, \{x, y\}, \{x, z\}, \{x\}\}$.

Define a function $f : X \rightarrow Y$ as follows: $f(a) = y$, $f(b) = f(d) = x$, $f(c) = z$. Then $f$ is $\pi gb$ - continuous function which is neither $\pi g$ - continuous nor $\pi gp$ - continuous.

**Example 4.5:** Let $X$ be the real numbers with the usual and $Y = \{0, 1\}$ with the topology $\sigma = \{Y, \emptyset, \{1\}\}$. We define the function $f : X \rightarrow Y$ such as

\[
f(x) = \begin{cases} 
0, & x \in (0, 2) \setminus Q \\
1, & x \notin (0, 2) \setminus Q 
\end{cases}
\]

Then $f$ is $\pi gb$ - continuous but it is not $\pi gs$ - continuous.

**Theorem 4.6:** Let $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

(1) $f$ is $\pi gb$ - continuous;

(2) The inverse image of every open set in $Y$ is $\pi gb$ - open in $X$.

**Proof:**

(1) $\Rightarrow$ (2) Let $U$ be a open subset of $X$. Then $Y \setminus U$ is closed. Since $f$ is $\pi gb$ - continuous, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is $\pi gb$ - closed in $X$. Hence $f^{-1}(U)$ is $\pi gb$ - open in $X$.

(2) $\Rightarrow$ (1) Let $V$ be a closed subset of $Y$. Then $Y \setminus V$ is open and by the hypothesis (2) $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $\pi gb$ - open in $X$. So $f^{-1}(V)$ is $\pi gb$ - closed. Therefore, $f$ is $\pi gb$ - continuous.

**Theorem 4.7:** If $f : X \rightarrow Y$ is $\pi gb$ - continuous then $f(\pi gb-cl(A)) \subset cl(f(A))$ for every subset $A$ of $X$.

**Proof:** Let $A$ be a subset of $X$. Since $f$ is $\pi gb$ - continuous and $A \subset f^{-1}(cl(f(A)))$, we obtain $\pi gb-cl(A) \subset f^{-1}(cl(f(A)))$ and then $f(\pi gb-cl(A)) \subset cl(f(A))$.

**Remark 4.8:** The converse of Theorem 4.7 need not be true as shown in the following example.
Example 4.9: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a, b\}, \{d\}, \{a, b, d\}\}, \sigma = \{X, \emptyset, \{d\}\}$. We define the function $f : (X, \tau) \to (X, \sigma)$ such as $f(a) = c$, $f(b) = a$, $f(c) = d$, $f(d) = b$. Then $f^{\pi gb}(\text{cl}(A)) \subset \text{cl}(f(A))$ for every subset $A$ of $X$. Since $\{a, b, c\}$ is closed in $(X, \sigma)$ but $f^{-1}(\{a, b, c\}) = \{a, b, d\}$ is not $\pi gb$-closed in $(X, \tau)$, $f$ is not $\pi gb$-continuous.

Theorem 4.10: Let $f : X \to Y$ be a function. Then the following statements are equivalent:

(1) For each $x \in X$ and each open set $V$ containing $f(x)$ there exists a $\pi gb$-open set $U$ containing $x$ such that $f(U) \subset V$.

(2) $f^{\pi gb}(\text{cl}(A)) \subset \text{cl}(f(A))$ for every subset $A$ of $X$.

Proof: (1) $\Rightarrow$ (2) Let $y \in f^{\pi gb}(\text{cl}(A))$ and let $V$ be any open neighborhood of $y$. Then there exist a $x \in X$ and a $\pi gb$-open set $U$ such that $f(x) = y$, $x \in U$, $x \in \pi gb\text{-cl}(A)$, and $f(U) \subset V$. By Lemma 3.12, $U \cap A \neq \emptyset$ and hence $f(U) \cap V \neq \emptyset$. Therefore, $y = f(x) \in \text{cl}(f(A))$.

(2) $\Rightarrow$ (1) Let $x \in X$ and $V$ be any open set containing $f(x)$. Let $A = f^{-1}(Y \setminus V)$. Since $f^{\pi gb}(\text{cl}(A)) \subset \text{cl}(f(A)) \subset Y \setminus V$ then $\pi gb\text{-cl}(A) = A$. Since $x \in \pi gb\text{-cl}(A)$, there exists a $\pi gb$-open set $U$ containing $x$ such that $U \cap A \neq \emptyset$ and hence $f(U) \subset f(X \setminus A) \subset V$.

Theorem 4.11: Let $X$ be an extremely disconnected space and $f : X \to Y$ be a function. If $f$ is $\pi gb$-continuous and $m\pi$-closed then $f$ is $\pi gb$-irresolute.

Proof: Let $A$ be a $\pi gb$-closed subset of $Y$. Then $f^{\pi gb}(A) \subset U$, where $U$ is $\pi$-open in $X$. So $X \setminus U \subset f^{\pi gb}(Y \setminus A)$. Hence $f(X \setminus U) \subset Y \setminus A$. Since $f$ is $m\pi$-closed, $f(X \setminus U)$ is $\pi$-closed. Since $Y \setminus A$ is $\pi gb$-open then $f(X \setminus U) \subset \text{bint}(Y \setminus A) = Y \setminus \text{bcl}(A)$. Thus $f^{\pi gb}(\text{bcl}(A)) \subset U$. Since $f$ is $\pi gb$-continuous and $X$ is extremely disconnected, $f^{\pi gb}(\text{cl}(A))$ is $\pi gb$-closed. Therefore, $\text{bcl}(f^{\pi gb}(\text{bcl}(A))) \subset U$ and hence $\text{bcl}(f^{\pi gb}(\text{cl}(A))) \subset U$. It follows that $f^{\pi gb}(A)$ is $\pi gb$-closed. This shows that $f$ is $\pi gb$-irresolute.

Definition 4.12: A function $f : X \to Y$ is said to be almost $\pi gb$-continuous if $f^{\pi gb}(V)$ is $\pi gb$-closed in $X$ for every regular closed set $V$ of $Y$.

Theorem 4.13: For a function $f : X \to Y$, the following statements are equivalent:

(1) $f$ is almost $\pi gb$-continuous;
(2) $f^{\pi gb}(V)$ is $\pi gb$-open in $X$ for every regular open set $V$ of $Y$;
(3) $f^{\pi gb}(\text{int}(\text{cl}(V)))$ is $\pi gb$-open in $X$ for every open set $V$ of $Y$;
(4) $f^{\pi gb}(\text{cl}(\text{int}(V)))$ is $\pi gb$-closed in $X$ for every closed set $V$ of $Y$.

Proof: (1) $\Rightarrow$ (2) Let $V$ be a regular open subset of $Y$. Since $Y \setminus V$ is regular closed and $f$ is almost $\pi gb$-continuous then $f^{\pi gb}(Y \setminus V) = X \setminus f^{\pi gb}(V)$ is $\pi gb$-closed in $X$. Thus $f^{\pi gb}(V)$ is $\pi gb$-open in $X$.

(2) $\Rightarrow$ (1) Let $V$ be a regular closed subset of $Y$. Then $Y \setminus V$ is regular open. By the hypothesis, $f^{\pi gb}(Y \setminus V) = X \setminus f^{\pi gb}(V)$ is $\pi gb$-open in $X$. Hence $f^{\pi gb}(V)$ is $\pi gb$-closed. This shows that $f$ is $\pi gb$-continuous.

(2) $\Rightarrow$ (3) Let $V$ be an open subset of $Y$. Then $\text{int}(\text{cl}(V))$ is regular open. By the hypothesis, $f^{\pi gb}(\text{int}(\text{cl}(V)))$ is $\pi gb$-open in $X$.

(3) $\Rightarrow$ (2) Let $V$ be a regular open subset of $Y$. Since $V = \text{int}(\text{cl}(V))$ and every regular open set is open then $f^{\pi gb}(V)$ is $\pi gb$-open in $X$. 
(3) ⇒ (4) Let \( V \) be a closed subset of \( Y \). Then \( Y \setminus V \) is open. By the hypothesis, 
\[ f^{-1}(\text{int}(\text{cl}(Y \setminus V))) = f^{-1}(Y \setminus \text{cl}(\text{int}(V))) = X \setminus f^{-1}(\text{cl}(\text{int}(V))) \]
is \( \pi gb \)-open in \( X \). Therefore, 
\[ f^{-1}(\text{cl}(\text{int}(V))) \] is \( \pi gb \)-closed in \( X \).

(4) ⇒ (3) Let \( V \) be a open subset of \( Y \). Then \( Y \setminus V \) is closed. By the hypothesis, 
\[ f^{-1}(\text{cl}(\text{int}(Y \setminus V))) = f^{-1}(Y \setminus \text{int}(\text{cl}(V))) = X \setminus f^{-1}(\text{int}(\text{cl}(V))) \]
is \( \pi gb \)-closed in \( X \). Hence 
\[ f^{-1}(\text{int}(\text{cl}(V))) \] is \( \pi gb \)-open in \( X \).

**Remark 4.14:** For a function \( f : X \to Y \), the following implications hold:

\[
\begin{align*}
\text{\( X \)-can} & \to \text{can} \to \text{\( g \)-can} \to \text{\( \pi gb \)-can} \\
\downarrow & & \downarrow & \\
\text{\( gb \)-can} & \to \text{\( \pi gb \)-can} \to \text{almost-\( \pi gb \)-can} & \uparrow & \\
\downarrow & & & \uparrow \\
\text{\( b \)-can} & \to \text{almost-\( b \)-can} & & \\
\end{align*}
\]

**Remark 4.15:**
(a) Every \( \pi gb \)-continuous function is almost \( \pi gb \)-continuous,
(b) Every almost \( b \)-continuous function is almost \( \pi gb \)-continuous.

However, none of these implications is reversible as shown by the following examples.

**Example 4.16:** In Example 4.9, \( f \) is almost \( \pi gb \)-continuous but it is not \( \pi gb \)-continuous since for the regular closed set \( \{a,b,c\} \) of \((X,\sigma)\), we have \( f^{-1}(\{a,b,c\})=\{a,b,d\} \) is not \( \pi gb \)-closed in \((X,\tau)\).

**Example 4.17:** In Example 4.3, \( f \) is almost \( \pi gb \)-continuous but it is not almost \( b \)-continuous since for the regular closed set \( \{x,y,z\} \) of \((Y,\sigma)\), we have \( f^{-1}(\{x,y,z\})=\{a,c,d\} \) is not \( b \)-closed in \((X,\tau)\).

**Theorem 4.18:** Let \( X \) be a \( \pi gb \)-T\(_{1/2}\) topological space. Then \( f : X \to Y \) is almost \( \pi gb \)-continuous if and only if \( f \) is almost \( b \)-continuous.

**Proof:** Necessity. Let \( A \) be a regular closed subset of \( Y \) and \( f : X \to Y \) be an almost \( \pi gb \)-continuous function. Then \( f^{-1}(A) \) is \( \pi gb \)-closed in \( X \). Since \( X \) is \( \pi gb \)-T\(_{1/2}\) space, \( f^{-1}(A) \) is \( b \)-closed in \( X \). Hence \( f \) is almost \( b \)-continuous.

Sufficiency. Suppose that \( f \) is almost \( b \)-continuous and \( A \) be a regular closed subset of \( Y \). Then \( f^{-1}(A) \) is \( b \)-closed in \( X \). Since every \( b \)-closed set is \( \pi gb \)-closed then \( f^{-1}(A) \) is \( \pi gb \)-closed. Therefore, \( f \) is almost \( \pi gb \)-continuous.

**Theorem 4.19:** Let \( X \) be a \( \pi gb \)-T\(_{1/2}\) space and \( f : X \to Y \) be a function. Then
(1) \( f \) is almost \( \pi gb \)-continuous if and only if \( f \) is almost \( b \)-continuous,
(2) \( f \) is \( \pi gb \)-continuous if and only if \( f \) is \( b \)-continuous.( or \( gb \)-continuous )

**Proof:** The proof is obvious.

5. \( \pi gb \)-compactness

**Definition 5.1:** A collection \( \{G_i : i \in \Lambda\} \) of \( \pi gb \)-open sets in a topological space \( X \) is called a \( \pi gb \)-open cover of a subset \( A \) of \( X \) if \( A \subset \bigcup \{G_i : i \in \Lambda\} \) holds.

**Definition 5.2:** A topological space \( X \) is \( \pi gb \)-compact if every \( \pi gb \)-open cover of \( X \) has a finite subcover.
Definition 5.3: A subset $A$ of a topological space $X$ is said to be $\pi gb$-compact relative to $X$ if, for every collection $\{U_i : i \in I\}$ of $\pi gb$-open subsets of $X$ such that $A \subseteq \bigcup \{U_i : i \in I\}$ there exists a finite subset $I_0$ of $I$ such that $A \subseteq \bigcup \{U_i : i \in I_0\}$.

Definition 5.4: A subset $A$ of a topological space $X$ is said to be $\pi gb$-compact if $A$ is $\pi gb$-compact as a subspace of $X$.

Theorem 5.5: Every $\pi gb$-closed subset of a $\pi gb$-compact space is $\pi gb$-compact relative to $X$.

Proof: Let $A$ be a $\pi gb$-closed subset of a $\pi gb$-compact space $X$. Let $\{U_i : i \in I\}$ be a $\pi gb$-open cover of $X$. So $A \subseteq \bigcup i U_i$ and then $(X \setminus A) \cup (\bigcup i U_i) = X$. Since $X$ is $\pi gb$-compact, there exists a finite subset $I_0$ of $I$ such that $(X \setminus A) \cup (\bigcup i U_i) = X$. Then $A \subseteq \bigcup i U_i$ and hence $A$ is $\pi gb$-compact relative to $X$.

A nearly compact space [25] is a topological space in which every cover by regular open sets has a finite subcover.

Theorem 5.6: The surjective $\pi gb$-continuous (resp. almost $\pi gb$-continuous) image of a $\pi gb$-compact space is compact (resp. nearly compact).

Proof: Let $\{U_i : i \in I\}$ be any cover of $Y$ by open (resp. regular open) subsets. Since $f$ is $\pi gb$-continuous (resp. almost $\pi gb$-continuous), then $\{f^{-1}(U_i) : i \in I\}$ is $\pi gb$-open cover of $X$. By $\pi gb$-compactness of $X$, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup i f^{-1}(U_i)$. Since $f$ is surjective, we obtain $Y = \bigcup i U_i$. This shows that $Y$ is compact (resp. nearly compact).

Theorem 5.7: If $f : X \rightarrow Y$ is $\pi gb$-irresolute and a subset $A$ of $X$ is $\pi gb$-compact relative to $X$, then the image $f(A)$ is $\pi gb$-compact relative to $Y$.

Proof: Let $\{U_i : i \in I\}$ be any collection of $\pi gb$-open subsets of $Y$ such that $f(A) \subseteq \bigcup U_i$. Then $A \subseteq \bigcup f^{-1}(U_i)$ holds. Since by hypothesis $A$ is $\pi gb$-compact relative to $X$, there exists a finite subset $I_0$ of $I$ such that $A \subseteq \bigcup f^{-1}(U_i)$. Therefore, we have $f(A) \subseteq \bigcup i U_i$, which shows that $f(A)$ is $\pi gb$-compact relative to $Y$.

Definition 5.8: A space $X$ is said to be
(1) $\pi gp$-compact [24] if every $\pi gp$-open cover of $X$ has a finite subcover.
(2) $gb$-compact [4] if every $gb$-open cover of $X$ has a finite subcover.
(3) $b$-compact [13] if every $b$-open cover of $X$ has a finite subcover.

Remark 5.9: Since every regular open set is open, $b$-open, $gb$-open, $\pi gb$-open and $\pi gp$-open set, for a space $X$, the following implications hold:

$$
\pi gb\text{-compact} \rightarrow gb\text{-compact} \rightarrow b\text{-compact} \rightarrow \text{compact} \rightarrow \text{nearly-compact}
$$

Definition 5.10: A function $f : X \rightarrow Y$ is said to be $\pi gb$-open if $f(U)$ is $\pi gb$-open in $Y$ for every $\pi gb$-open set $U$ of $X$.

Theorem 5.11: If $f : X \rightarrow Y$ is $\pi gb$-open bijection and $Y$ is $\pi gb$-compact space then $X$ is a $\pi gb$-compact space.

Proof: Let $\{U_i : i \in I\}$ be a $\pi gb$-open cover of $X$. So $X = \bigcup i U_i$ and then $Y = f(X) = f(\bigcup i U_i) = \bigcup i f(U_i)$.
Since $f$ is $\pi$gb-open, for each $i \in I$, $f(U_i)$ is $\pi$gb-open set. By $\pi$gb-compactness of $Y$, there exists a finite subset $I_0$ of $I$ such that $Y = \bigcup_{i \in I_0} f(U_i)$. Therefore, $X = f^{-1}(Y) = f^{-1}\left(\bigcup_{i \in I_0} f(U_i)\right) = \bigcup_{i \in I_0} f^{-1}(f(U_i)) = \bigcup_{i \in I_0} U_i$. This shows that $X$ is $\pi$gb-compact.

**Theorem 5.12:** If $f : X \to Y$ is $\pi$gb-irresolute bijection and $X$ is $\pi$gb-compact space then $Y$ is a $\pi$gb-compact space.

**Proof:** Let $\{U_i : i \in I\}$ be a $\pi$gb-open cover of $Y$. So $Y = \bigcup_{i \in I} U_i$ and then $X = f^{-1}(Y) = f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i)$. Since $f$ is $\pi$gb-irresolute, it follows that for each $i \in I$, $f^{-1}(U_i)$ is $\pi$gb-open set. By $\pi$gb-compactness of $X$, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup_{i \in I_0} f^{-1}(U_i)$. Therefore, $Y = f(X) = f\left(\bigcup_{i \in I_0} f^{-1}(U_i)\right) = \bigcup_{i \in I_0} f(f^{-1}(U_i)) = \bigcup_{i \in I_0} U_i$. This shows that $Y$ is $\pi$gb-compact.

**Theorem 5.13:** If $f : X \to Y$ is $\pi$gb-continuous bijection and $X$ is $\pi$gb-compact space then $Y$ is a $\pi$gb-compact space.

**Proof:** The proof is similar to that of Theorem 5.12.

6. **Quasi-b-normal spaces**

**Definition 6.1:** A space $X$ is said to be quasi-b-normal if for every pair of disjoint $\pi$-closed subsets $A, B$ of $X$, there exist disjoint $b$-open subsets $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

**Definition 6.2:** A space $X$ is said to be quasi-normal [28] (resp. quasi-s-normal [6]) if for every pair of disjoint $\pi$-closed subsets $A, B$ of $X$, there exist disjoint open (resp. semi-open) subsets $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

**Definition 6.3:** A space $X$ is said to be $b$-normal (or $\gamma$-normal [12]) if for every pair of disjoint closed subsets $A, B$ of $X$, there exist disjoint $b$-open subsets $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

**Remark 6.4:** For a topological space $X$, the following implications hold:

$$
\text{normal} \implies \text{quasi-normal} \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
\text{s-normal} \implies \text{quasi-s-normal} \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
\text{b-normal} \implies \text{quasi-b-normal}
$$

In general, the converse of implications in the above diagram need not be true.

**Example 6.5:** Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}, \{a\}, \{a, c\}\}$. $(X, \tau)$ is quasi-b-normal space but it is not $b$-normal space.

**Theorem 6.6:** The following statements are equivalent for a space $X$;

(a) $X$ is quasi-b-normal;
(b) For any disjoint $\pi$-closed sets $A$ and $B$, there exist disjoint $gb$-open subsets $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$;
(c) For any closed set $A$ and any $\pi$-open set $B$ containing $A$, there exists a $gb$-open set $U$ such that $A \subseteq U \subseteq \text{bcl}(U) \subseteq B$;
(d) For any disjoint $\pi$-closed sets $A$ and $B$, there exist disjoint $\pi$gb-open subsets $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$;
(e) For any $\pi$-closed set $A$ and any $\pi$-open set $B$ containing $A$, there exists a $\pi$gb-open set $U$ such that $A \subseteq U \subseteq \text{bcl}(U) \subseteq B$.
Proof. (a) ⇒ (b) The proof is obvious.

(b) ⇒ (c) Let A be any \( \pi \)-closed subset of X and B any \( \pi \)-open subset of X such that \( A \subset B \). Then A and \( X \setminus B \) disjoint \( \pi \)-closed subset of X. Therefore, there exist disjoint gb-open sets U and V such that \( A \subset U \) and \( X \setminus B \subset V \).

By the definition of gb-open sets, we have \( X \setminus B \subset b \text{int}(V) \) and \( U \cap b \text{int}(V) = \emptyset \). Therefore, we obtain \( b \text{cl}(U) \cap b \text{int}(V) = \emptyset \) and hence \( A \subset U \subset b \text{cl}(U) \subset B \).

(c) ⇒ (d) Let A and B be any disjoint \( \pi \)-closed subsets of X. Then \( A \subset X \setminus B \) and \( X \setminus B \) is \( \pi \)-open and hence there exists a gb-open subset G of X such that \( A \subset G \subset b \text{cl}(G) \subset X \setminus B \). Since every gb-open set is gb-open, G is gb-open and \( X \setminus b \text{cl}(G) \) is gb-open. Now put \( V = X \setminus b \text{cl}(G) \). Then G and V are disjoint gb-open subsets of X such that \( A \subset G \) and \( B \subset V \).

(d) ⇒ (e) The proof is similar to that of (b) ⇒ (c)

(e) ⇒ (a) Let A and B be any disjoint \( \pi \)-closed subsets of X. Then \( A \subset X \setminus B \) and \( X \setminus B \) is \( \pi \)-open and hence there exists a gb-open subset G of X such that \( A \subset G \subset b \text{cl}(G) \subset X \setminus B \). Put \( U = b \text{int}(G) \) and \( V = X \setminus b \text{cl}(G) \). Then U and V are disjoint gb-open subsets of X such that \( A \subset G \) and \( B \subset V \). Therefore, X is quasi-b-normal.

Definition 6.7: A function \( f : X \rightarrow Y \) is said to be almost \( \pi_{gb} \)-closed if for each regular closed subset F of X, \( f(F) \) is \( \pi_{gb} \)-closed subset of Y.

Proposition 6.8: A surjection \( f : X \rightarrow Y \) almost \( \pi_{gb} \)-closed if and only if for each subset G of Y and each \( U \in RO(X) \) containing \( f^{-1}(G) \), there exists a \( \pi_{gb} \)-open subset V of Y such that \( G \subset V \) and \( f^{-1}(V) \subset U \).

Proof: Necessity. Suppose that f is almost \( \pi_{gb} \)-closed. Let G be a subset of Y and \( U \in RO(X) \) containing \( f^{-1}(G) \). If \( V = Y \setminus f(X \setminus U) \), then V is a \( \pi_{gb} \)-open set of Y such that \( G \subset V \) and \( f^{-1}(V) \subset U \).

Sufficiency. Let F be any regular closed set of X. Then \( X \setminus F \in RO(X) \) and \( f^{-1}(Y \setminus f(F)) \subset X \setminus F \). There exists a gb-open set V of Y such that \( Y \setminus f(F) \subset V \) and \( f^{-1}(V) \subset X \setminus F \). Therefore, we have \( Y \setminus V \subset f(F) \) and \( F \subset f^{-1}(Y \setminus V) \). Hence, we obtain \( f(F) = Y \setminus V \) and \( f(F) \) is gb-closed in Y. This shows that f is almost gb-closed.

Theorem 6.9: Let \( f : X \rightarrow Y \) be a continuous, almost gb-closed surjection. If X is normal, then Y is quasi-b-normal.

Proof: Let A and B be disjoint \( \pi \)-closed subsets of Y. Since f is continuous, \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint closed subsets of X. By the normality of X, there exist disjoint open sets U and V such that \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). Put \( G = \text{int}(\text{cl}(U)) \) and \( H = \text{int}(\text{cl}(V)) \). Then G and H are disjoint regular open subsets of X such that \( f^{-1}(A) \subset G \) and \( f^{-1}(B) \subset H \). By Proposition 6.8, there exist gb-open subsets K and L of Y such that \( A \subset K \), \( B \subset L \), \( f^{-1}(K) \subset G \) and \( f^{-1}(L) \subset H \). Since \( G \cap H = \emptyset \) and f is surjective, \( K \cap L = \emptyset \). It follows from Theorem 6.6 (d) that Y is quasi-b-normal.

Theorem 6.10: Let \( f : X \rightarrow Y \) be a \( \pi \)-irresolute, almost gb-closed surjection. If X is quasi-normal, then Y is quasi-b-normal.

Proof: Let A and B be disjoint \( \pi \)-closed subsets of Y. Since f is \( \pi \)-irresolute, \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint \( \pi \)-closed subsets of X. By the quasi-normality of X, there exist disjoint open sets U and V such that \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). Put \( G = \text{int}(\text{cl}(U)) \) and \( H = \text{int}(\text{cl}(V)) \). Then \( f^{-1}(A) \subset U \subset G \), \( f^{-1}(B) \subset V \subset H \), \( G \cap H = \emptyset \) and \( G,H \in RO(X) \). Since f is almost gb-closed, by Proposition 6.8 there exist gb-open subsets K and L of Y such that...
A ⊂ K, B ⊂ L, f⁻¹(K) ⊂ G and f⁻¹(L) ⊂ H. Since f is surjective, we have K ∩ L = ∅. This shows that Y is quasi-b-normal.

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