NEW FAMILIES OF 3-TOTAL PRODUCT CORDIAL GRAPHS

R. Ponraj*, M. Sivakumar and M. Sundaram

1Department of Mathematics, Sri Paramakalyani College, Alwarkurchi-627412, India
2Department of Mathematics, Unnamalai Institute of Technology, Kovilpatti-628502, India
3Associate Professor (Rtd.), Department of Mathematics, Sri Paramakalyani College
Alwarkurchi-627412, India

(Received on: 14-05-12; Accepted on: 31-05-12)

ABSTRACT

Let \( f \) be a map from \( V(G) \) to \{0, 1, ..., \( k-1 \)\} where \( k \) is an integer, \( 2 \leq k \leq |V(G)| \). For each edge \( uv \) assign the label \( f(u)f(v) \mod k \). \( f \) is called a \( k \)-Total Product cordial labeling if \( |f(i) - f(j)| \leq 1 \), \( i, j \in \{0, 1, ..., k-1\} \), where \( f(x) \) denotes the total number of vertices and edges labelled with \( x \) \( (x=0, 1, 2, ..., k-1) \). A graph that admits a \( k \)-Total Product cordial labelling is called a \( k \)-Total Product cordial graph. In this paper we investigate 3-Total Product cordial labeling behaviour of some standard graphs like Wheels, Helms, Dragons, etc.

Keywords: Wheel, Helms, Dragon, \( C_n \Theta 2K_1 \).

Mathematics Subject Classification (2000): 05C78.

1. INTRODUCTION

The graphs considered here are finite, undirected and simple. The vertex set and edge set of a graph \( G \) are denoted by \( V(G) \) and \( X(G) \) respectively. The following definitions are used here.

- The corona \( G_1 \Theta G_2 \) of two graphs \( G_1 \) and \( G_2 \) is defined as the graph \( G \) obtained by taking one copy of \( G_1 \) (which has \( p_1 \) vertices) and \( p_1 \) copies of \( G_2 \) and then joining the \( i \)-th vertex of \( G_1 \) to every vertex in the \( i \)-th copy \( G_2 \).
- The graph \( W_n = C_n + K_1 \) is called a wheel.
- The Helms \( H_n \) is the graph obtained from wheel by attaching a pendant edge at each vertex of the cycle \( C_n \).
- A Dragon is formed by identifying the end vertex of the path to the vertex of a cycle.
- \( m \) copies of the graph \( G \) is denoted by \( mG \).

The notion of \( k \)-Product cordial labeling of graph was introduced in [2] where the \( k \)-Product cordial labeling behaviour of some standard graphs was studied. Also \( k \)-Total Product labeling of graphs was introduced in [4]. Obviously 2-Total Product cordial labeling is simply a Total Product cordial labeling[5]. Also 3-Total Product cordial labeling behaviour of some standard graphs was studied in [3]. In this paper we investigate 3-Total Product cordial labeling behaviour of Helms, Wheel, Dragon, \( C_n \Theta 2K_1 \) and some standard graphs. Terms not defined here are used in the sense of Harary[1].

2. \( k \)-TOTAL PRODUCT CORDIAL LABELING

Definition 2.1:

Let \( f \) be a function from \( V(G) \) to \{0, 1, ..., \( k-1 \)\} where \( k \) is an integer, \( 2 \leq k \leq |V(G)| \). For each edge \( uv \), assign the label \( f(u)f(v) \mod k \). \( f \) is called a \( k \)-Total Product cordial labeling of \( G \) if \( |f(i) - f(j)| \leq 1 \), \( i, j \in \{0, 1, ..., k-1\} \) where \( f(x) \) denotes the total number of vertices and edges labelled with \( x \) \( (x=0, 1, 2, ..., k-1) \). A graph with a \( k \)-Total Product cordial labelling is called a \( k \)-Total Product cordial graph.

Theorem 2.2: Let \( G \) be a \((p, p)\) graph. Then \( mG \) is 3-Total Product cordial where \( m \equiv 0 \mod 3 \).

Corresponding author: R. Ponraj

*Department of Mathematics, Sri Paramakalyani College, Alwarkurchi-627412, India
Proof: Let \( m=3t \). Clearly \( mG \) has 3pt vertices and 3pt edges. Assign the label 2 to all the vertices of first 2t copies of \( G \). Then assign 0 to all the vertices of remaining t copies of \( G \).

Then \( f(0) = f(1) = f(2) = 2pt \). Therefore \( f \) is a 3-Total Product cordial labeling.

**Corollary 2.3:** If \( m \equiv 0 \pmod{3} \), then \( mC_n \) is 3-Total Product cordial.

**Notation:** Let \( G \) be any graph. Then the graph obtained from \( G \) by identifying the central vertex of \( K_{1,p} \) to any vertex of \( G \) is denoted by \( G \ast K_{1,p} \).

**Theorem 2.4:** If \( G \) is a \((p, p)\) graph.

(i) If \( p \) is even then \( G \ast K_{\frac{p}{2}} \) is 3-Total Product cordial.

(ii) If \( p \) is odd then \( G \ast K_{\frac{p+1}{2}} \) and \( G \ast K_{\frac{p-1}{2}} \) are 3-Total Product cordial.

**Proof:** Case (i): \( p \) is even.

Assign the label 2 to all the vertices of \( G \) and 0 to all pendant vertices of \( G \). Clearly \( f(0) = f(1) = f(2) = p \).

Case (ii): \( p \) is odd.

Assign label as in case (i) Clearly \( f(0) = p - 1 \), Therefore \( f(1) = f(2) = p \) for the graph \( G \ast K_{\frac{p+1}{2}} \), and \( f(0) = p + 1 \), \( f(1) = p \) for the graph \( G \ast K_{\frac{p-1}{2}} \). Therefore \( f \) is a 3-Total Product cordial labeling.

**Theorem 2.5:** The Wheel \( W_n \) is 3-Total Product Cordial.

**Proof:** Let \( C_n \) be the cycle \( u_1u_2u_3 \ldots u_nu_1 \) and let \( V(W_n) = V(C_n) \cup \{u\} \quad E(W_n) = E(C_n) \cup \{uu_i : 1 \leq i \leq n\} \).

Define \( f(u) = 0 \),
\[
f(u_i) = 2, \quad 1 \leq i \leq n.
\]

Here \( f(0) = n + 1 \) and \( f(1) = f(2) = n \). Hence \( f \) is a 3-Total Product cordial labeling.

**Theorem 2.6:** The Helms \( H_n \) is 3-Total Product cordial.

**Proof:** Let the vertex set and edge set of the wheel \( W_n \) be defined as in theorem 2.5.

Let \( V(H_n) = V(W_n) \cup \{v_i, 1 \leq i \leq n\} \) and \( E(H_n) = E(W_n) \cup \{u_i, v_i : 1 \leq i \leq n\} \).

**Case (i):** \( n \equiv 0 \pmod{3} \)

Let \( n = 3t \)

Define \( f(u) = 0 \),
\[
f(u_i) = 2, \quad 1 \leq i \leq n
\]
\[
f(v_i) = 2, \quad 1 \leq i \leq 2t
\]
\[
f(v_{2t+i}) = 0, \quad 1 \leq i \leq t
\]

Here \( f(0) = 5t + 1 \) and \( f(1) = f(2) = 5t \). Hence \( f \) is a 3-Total Product cordial labeling.

**Case (ii):** \( n \equiv 1 \pmod{3} \)

Let \( n = 3t + 1 \). Assign labels to the vertices \( u_i, v_i \, (1 \leq i \leq n - 1) \) as in case(i). Then assign the labels 2, 2 to the vertices \( u_{n}, v_{n} \) respectively. Here \( f(0) = f(1) = f(2) = 5t + 2 \). Hence \( f \) is a 3-Total Product cordial labeling.
Let \( n = 3t + 2 \). Assign labels to the vertices \( u_i, v_i \) \( (1 \leq i \leq n - 1) \) as in case (ii). Then assign the labels 2, 0 to the vertices \( u_n, v_n \) respectively. Here \( f(0) = 5t + 3 \) and \( f(1) = f(2) = 5t + 4 \). Hence \( f \) is a 3-Total Product cordial labeling.

**Illustration 2.7:** A 3-Total Product cordial labeling of \( H_7 \) is

![Graph with vertex and edge labels from 0 to 2]

**Notation 2.8:** Let \( C_n \) be the cycle \( u_1, u_2, \ldots, u_n, u_1 \). Let \( G_n \) denotes the graph with \( V(G_n) = V(C_n) \cup \{v_i, w_i, 1 \leq i \leq n \} \) and \( E(G_n) = E(C_n) \cup \{u_i v_i, u_i w_i, v_i w_i : 1 \leq i \leq n \} \).

**Theorem 2.9:** \( G_n \) is 3-Total Product cordial.

**Proof:** Let the vertex set and edge set of the graph \( G_n \) be as defined above.

**Case (i):** \( n \equiv 0 \) (mod 3)

Let \( n = 3t \). Define \( f(v_i) = f(c_i) = 0, 1 \leq i \leq t \)
\( f(w_i) = 0, 1 \leq i \leq t - 1 \)
\( f(w_i) = 0 \).
\( f(u_{2t+i}) = f(v_{2t+i}) = f(w_{2t+i}) = 2, 1 \leq i \leq t \)
\( f(u_{2t+i}) = 2, 1 \leq i \leq t \)
\( f(v_{2t+i}) = 1, 1 \leq i \leq t \)
\( f(w_{2t+i}) = 2, 1 \leq i \leq t \)

Then \( f(0) = f(1) = f(2) = 7t \). Hence \( f \) is a 3-Total Product cordial labeling.

**Case (ii):** \( n \equiv 1 \) (mod 3)

Let \( n = 3t + 1 \). Assign labels to the vertices \( u_i, v_i, w_i, 1 \leq i \leq n - 1 \) as in case (i). Then assign the labels 2, 0 to the vertices \( u_n, v_n, w_n \) respectively. Here \( f(0) = 7t + 3 \) and \( f(1) = f(2) = 7t + 2 \). Hence \( f \) is 3-Total Product cordial labeling.

**Case (iii):** \( n \equiv 2 \) (mod 3)

Let \( n = 3t + 2 \). Assign labels to the vertices \( u_i, v_i, w_i, 1 \leq i \leq n - 2 \) as in case (i). Then assign the labels 0, 2, 2 and 2 to the vertices \( v_{n-1}, v_n, w_{n-1}, w_n, u_{n-1}, u_n \) respectively. Here \( f(0) = 7t + 5 \) and \( f(1) = f(2) = 7t + 4 \). Hence \( f \) is 3-Total Product cordial labeling.
Illustration 2.10: A 3-Total Product cordial labelling of $G_6$ is

![Diagram of $G_6$](image)

Figure (ii)

Theorem 2.11: $C_n \Theta K_1$ is 3-Total Product cordial.

Proof: Let $V(C_n \Theta K_1) = \{u_i, v_i, w_i, \; 1 \leq i \leq n\}$ and $E(C_n \Theta K_1) = \{u_i u_{i+1}, u_i v_1, 1 \leq i \leq n-1\} \cup \{v_1 v_i, w_i w_{i+1}, 1 \leq i \leq n\}$

Case (i): $n$ is even.

Define

$f(u_i) = 2, \; 1 \leq i \leq n$

$f(v_i) = f(w_i) = 0, \; 1 \leq i \leq \frac{n}{2}$

$f(v_{\frac{n}{2} + i}) = f(w_{\frac{n}{2} + i}) = 2, \; 1 \leq i \leq \frac{n}{2}$

Then $f(0) = f(1) = f(2) = 2n$. Hence $f$ is a 3-Total Product cordial labeling.

Case (ii): $n$ is odd.

Define $f(u_i) = 2, \; 1 \leq i \leq n$, $f(v_i) = 0, \; 1 \leq i \leq \frac{n+1}{2}$

$f(w_i) = 0, \; 1 \leq i \leq \frac{n-1}{2}$, $f(v_{\frac{n+1}{2} + i}) = 2, \; 1 \leq i \leq \frac{n-1}{2}$

$f(w_{\frac{n-1}{2} + i}) = 2, \; 1 \leq i \leq \frac{n+1}{2}$

Then $f(0) = f(1) = f(2) = 2n$. Hence $f$ is a 3-Total Product cordial labeling.
Illustration 2.12: A 3-Total Product cordial labeling of $C_7\Theta 2K_1$ is

![Diagram](image)

Figure (iii)

**Theorem 2.13:** The Dragon $C_m \oplus P_n$ is 3-Total Product cordial.

**Proof:** Let $C_m$ be the cycle $u_1, u_2, \ldots, u_m, u_1$ and $P_n$ be the path $v_1, v_2, \ldots, v_n$. Identify the vertex $u_1$ with $v_1$.

**Case (i):** $m \equiv 0 \pmod{3}$ and $n \equiv 0 \pmod{3}$

Let $m = 3t_1$ and $n = 3t_2$

Define

$f(u_i) = 0, \quad 1 \leq i \leq t_1$

$f(u_{i+t_1}) = 2, \quad 1 \leq i \leq 2t_1$

$f(v_i) = 0, \quad 2 \leq i \leq t_2 - 1$

$f(v_{i+t_2}) = f(v_{i+t_2+1}) = 1$,

$f(v_{i+t_2+2}) = 2, \quad 1 \leq i \leq 2t_2 - 1$

Then $f(0) = 2t_1 + 2t_2$ and $f(1) = f(2) = 2t_1 + 2t_2 - 1$. Hence $f$ is a 3-Total Product cordial labeling.

**Case (ii):** $m \equiv 0 \pmod{3}$ and $n \equiv 1 \pmod{3}$

Let $m = 3t_1$ and $n = 3t_2 + 1$. Assign label 1 to the vertex $v_n$ and assign labels to all the remaining vertices as in case (i). In this case $f(0) = f(1) = f(2) = 2t_1 + 2t_2$. Hence $f$ is a 3-Total Product cordial labeling.

**Case (iii):** $m \equiv 0 \pmod{3}$ and $n \equiv 2 \pmod{3}$

Let $m = 3t_1$ and $n = 3t_2 + 2$. Assign labels 2, 1 to the vertices $v_{n-1}$, $v_n$ respectively. Then assign the labels to all the remaining vertices as in case (i). Here $f(0) = 2t_1 + 2t_2$ and $f(1) = f(2) = 2t_1 + 2t_2 + 1$.

Hence $f$ is a 3-Total Product cordial labeling.

**Case (iv):** $m \equiv 1 \pmod{3}$ and $n \equiv 0 \pmod{3}$

Let $m = 3t_1 + 1$ and $n = 3t_2$. Assign label 1 to the vertex $u_m$. Then assign the labels to all the remaining vertices as in case (i). In this case $f(0) = f(1) = f(2) = 2t_1 + 2t_2$. Hence $f$ is a 3-Total Product cordial labeling.
Case (v): $m \equiv 1 (\text{mod } 3)$ and $n \equiv 1 (\text{mod } 3)$

Let $m = 3t_1 + 1$ and $n = 3t_2 + 1$. Assign label 1 to the vertex $v_n$. Then assign the labels to all the remaining vertices as in case (iv). Here $f(0) = 2t_1 + 2t_2$ and $f(1) = 2(2) = 2t_1 + 2t_2 + 1$. Hence $f$ is a 3-Total Product cordial labeling.

Case (vi): $m \equiv 1 (\text{mod } 3)$ and $n \equiv 2 (\text{mod } 3)$

Let $m = 3t_1 + 1$ and $n = 3t_2 + 2$. Assign label 0 to the vertex $v_n$. Then assign the labels to all the remaining vertices as in case (v). In this case $f(0) = 2t_1 + 2t_2 + 2$ and $f(1) = 2(2) = 2t_1 + 2t_2 + 1$. Hence $f$ is a 3-Total Product cordial labeling.

Case (vii): $m \equiv 2 (\text{mod } 3)$ and $n \equiv 0 (\text{mod } 3)$

Let $m = 3t_1 + 2$ and $n = 3t_2$. Assign labels 2, 0 to the vertices $u_{m-1}$, $u_m$ respectively. Then assign the labels to the all the remaining vertices as in case (i). In this case $f(0) = 2t_1 + 2t_2$ and $f(1) = f(2) = 2t_1 + 2t_2 + 1$. Hence $f$ is a 3-Total Product cordial labeling.

Case (viii): $m \equiv 2 (\text{mod } 3)$ and $n \equiv 1 (\text{mod } 3)$

Let $m = 3t_1 + 2$ and $n = 3t_2 + 1$. Assign label 0 to the vertex $v_n$. Then assign the labels to all the remaining vertices as in case (vii). Here $f(0) = 2t_1 + 2t_2 + 2$ and $f(1) = f(2) = 2t_1 + 2t_2 + 1$. Hence $f$ is a 3-Total Product cordial labeling.

Case (ix): $m \equiv 2 (\text{mod } 3)$ and $n \equiv 2 (\text{mod } 3)$

Let $m = 3t_1 + 2$ and $n = 3t_2 + 2$. Assign labels 2, 0 to the vertex $v_{m-1}$, $v_m$ respectively. Then assign the labels to the all the remaining vertices as in case (vii). Here $f(0) = f(1) = f(2) = 2t_1 + 2t_2 + 2$. Hence $f$ is a 3-Total Product cordial labeling.

Illustration 2.14: A 3-Total Product cordial labeling of the dragon $C_{12} \oplus P_8$ is

\[\text{Figure (iv)}\]

REFERENCES:


Source of support: Nil, Conflict of interest: None Declared

© 2012, IJMA. All Rights Reserved