AN INTEGRAL INEQUALITY FOR POLYNOMIALS

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ABSTRACT

In this paper, a compact generalization of certain known $L_p$ inequalities for polynomials is obtained, which refine some results due to De-Bruijn, Boas and Rahman and others.


Keywords and phrases: polynomials, $L_p$ inequalities, complex domain.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=1}^{n} a_j z^j$ of degree $n$. For $P \in P_n$, define

$$\|P(z)\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{1/p}, \quad p \geq 1$$

and

$$\|P(z)\|_\infty := \max_{|z|=1} |P(z)|.$$

If $P \in P_n$, then according to a famous result known as Bernstein’s inequality (for reference see [13, 16, 18]),

$$\|P'(z)\|_\infty \leq n \|P(z)\|_\infty,$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z|=R>1$, we have

$$\|P(Rz)\|_\infty \leq R^n \|P(z)\|_\infty$$

Inequality (2) is a simple deduction from maximum modulus principle (see [13, p.442] or [14, Vol I, p.137]).

Inequalities (1) and (2) can be obtained by letting $p \to \infty$ in the inequalities

$$\|P'(z)\|_p \leq n \|P(z)\|_p, \quad p \geq 1$$

and

$$\|P(Rz)\|_p \leq R^n \|P(z)\|_p, \quad R > 1, \quad p > 0$$

respectively. Inequality (3) was found by Zygmund [19], whereas inequality (4) is a simple consequence of a result of Hardy [10] (see also [16, Th.5.5]). Arestov [2] proved that (3) remains true for $0<p<1$ as well.

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z|<1$, then the inequalities (1) and (2) can be sharpened. In fact, if $P(z) \neq 0$ in $|z|<1$, then (1) and (2) can be respectively replaced by

$$\|P'(z)\|_\infty \leq \frac{n}{2} \|P(z)\|_\infty$$

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and
\[
\left\|P(Rz)\right\|_\infty \leq \frac{R^n + 1}{2} \left\|P(z)\right\|_\infty, \quad R > 1.
\]

(6)

Inequality (5) was conjectured by Erdös and later verified by Lax [12]. Ankiny and Rivilin [1] used inequality (5) to prove inequality (6).

Both the inequalities (5) and (6) can be obtained by letting \( p \to \infty \) in the inequalities
\[
\left\|P'(z)\right\|_p \leq n \left\|\frac{P(z)}{1 + z}\right\|_p, \quad p \geq 0,
\]
and
\[
\left\|P(Rz)\right\|_p \leq \frac{R^n + 1}{2} \left\|P(z)\right\|_p, \quad R > 1, \quad p > 0.
\]

(8)

Inequality (7) is due to De Bruijn [9] for \( p \geq 1 \). Rahman and Schmeisser [15] extended it for \( 0 < p < 1 \), whereas the inequality (8) was proved by Boas and Rahman [8] for \( p \geq 1 \) and later extended for \( 0 < p < 1 \) by Rahman and Schmeisser [15].

Aziz and Dawood [3] refined both the inequalities (5) and (6) by showing that if \( P \in P_n \) and \( P(z) \) does not vanish in \( |z| < 1 \) and \( m = \min_{{|z|=1}} |P(z)| \), then
\[
\left\|P'(z)\right\|_\infty \leq \frac{R}{2} \left\{\left\|\frac{P(z)}{1 + z}\right\|_\infty - m\right\},
\]
and
\[
\left\|P(Rz)\right\|_\infty \leq \frac{R^n + 1}{2} \left\{\left\|\frac{P(z)}{1 + z}\right\|_\infty - m\right\}.
\]

(10)

As a compact generalization of the inequalities (3), (4), (5), (6), recently Aziz and Rather [7] proved that if \( P \in P_n \) and \( P(z) \) does not vanish in \( |z| < 1 \), then for arbitrary real or complex numbers \( \alpha \) and \( \beta \) with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1 \) and \( p > 0 \),
\[
\left\|P(Rz) + \phi(R, r, \alpha, \beta) P(rz)\right\|_p \leq \frac{C_p}{p} \frac{\left\|P(z)\right\|_p}{\left\|1 + z\right\|_p}
\]

(11)

where
\[
C_p = \left\|\left(\frac{R^n + \phi(R, r, \alpha, \beta) r^n}{1 + \phi(R, r, \alpha, \beta)}\right) z + \left(1 + \phi(R, r, \alpha, \beta)\right)\right\|_p
\]

(12)

and
\[
\phi(R, r, \alpha, \beta) = \beta \left(\frac{R + 1}{r + 1}\right)^n - |\alpha| - \alpha
\]

(13)

In this paper, we prove the following interesting result which includes not only a generalization of the inequality (11) as a special case but also leads to some refinements and generalizations of certain known polynomial inequalities.

**Theorem 1:** If \( P \in P_n \) does not vanish in \( |z| < 1 \) and \( m = \min_{{|z|=1}} |P(z)| \), then for arbitrary complex numbers \( \alpha, \beta, \delta \) with \( |\alpha| \leq 1, |\beta| \leq 1, |\delta| \leq 1, R > r \geq 1 \) and \( p > 0 \),
\[
\left\|P(Rz) + \phi(R, r, \alpha, \beta) P(rz) + \delta m\left[\frac{R^n + \phi(R, r, \alpha, \beta) r^n}{1 + \phi(R, r, \alpha, \beta)}\right] - \left[1 + \phi(R, r, \alpha, \beta)\right]\right\|_p \leq \frac{C_p}{p} \frac{\left\|P(z)\right\|_p}{\left\|1 + z\right\|_p},
\]

(14)

where \( C_p \) and \( \phi(R, r, \alpha, \beta) \) are defined by (12) and (13) respectively. The result is best possible and the equality in (14) holds for the polynomial \( P(z) = a z^n + b \), where \( |a| = |b| = 1 \).
Theorem 1 has various interesting consequences. Here we mention few of these. For \( \delta = 0 \), the inequality (14) reduces to inequality (11). Next, we mention the following compact generalization of inequalities (5), (6), (7), (8), (9) and (10), which follows from Theorem 1 by setting \( \beta = 0 \).

**Corollary 1:** If \( P \in P_n \) does not vanish in \( |z| < 1 \) and \( m = \min_{|z|=1} |P(z)| \), then for arbitrary complex numbers \( \alpha, \delta \) with \( |\alpha| \leq 1 \), \( |\delta| \leq 1 \), \( R > r \geq 1 \) and \( p > 0 \),

\[
\| P(Rz) - \alpha P(rz) + \frac{\delta}{2} \left( \left| R^n - \alpha r^n \right| - |1 - \alpha| \right) m \|_p \leq \left\| \frac{\left| R^n - \alpha r^n \right| (1 - \alpha)}{1 + |\alpha|} \right\|_p \| P \|_p.
\]

(15)

The result is best possible and equality in (15) holds for \( P(z) = z^n + 1 \).

**Remark 1:** Corollary 3 includes as a special case a result due to Rather [17, Theorem 1], which is obtained by taking \( \alpha = 0 \) in (15).

Next if we set \( \alpha = 1 \) and divide two sides of (15) by \( R - r \) and let \( R \to r \), we obtain,

**Corollary 2:** If \( P \in P_n \) does not vanish in \( |z| < 1 \) and \( m = \min_{|z|=1} |P(z)| \), then for each \( p > 0 \) and \( r \geq 1 \),

\[
\left\| P'(rz) + \frac{\delta}{2} n m r^{-n+1} \right\|_p \leq \frac{n r^{-n+1}}{1 + |z|} \| P(z) \|_p.
\]

The result is sharp.

Corollary 2 is an interesting generalization of inequality (7) due to De Bruijn [9]. Inequality (8) can also be obtained from inequality (15) by setting \( \alpha = \delta = 0 \).

Making \( p \to \infty \) in (14) and choosing the argument of \( \delta \) with \( |\delta| = 1 \) suitably, we obtain:

**Corollary 3:** If \( P \in P_n \) does not vanish in \( |z| < 1 \), then for \( \alpha, \beta \in C \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \) and \( R > r \geq 1 \),

\[
\left\| P(Rz) + \phi(R, r, \alpha, \beta) P(rz) \right\|_\infty \leq \left\{ \frac{\left| R^n + \phi(R, r, \alpha, \beta) r^n \right|}{2} + \left| 1 + \phi(R, r, \alpha, \beta) \right| \right\} \| P(z) \|_\infty

- \left\{ \frac{\left| R^n + \phi(R, r, \alpha, \beta) r^n \right|}{2} - \left| 1 + \phi(R, r, \alpha, \beta) \right| \right\} \min_{|z|=1} |P(z)|,
\]

(16)

where \( \phi(R, r, \alpha, \beta) \) is the same as defined in Theorem 1. The result is sharp and the equality holds for \( P(z) = az^n + b \), where \( |a| = |b| = 1 \).

Corollary 3 is a refinement as well as a generalization of a result due to Aziz and Rather [4, Theorem 3]. For \( \alpha = \beta = 0 \), it reduces to (10). If we divide the two sides of inequality (16) by \( R - r \) with \( \alpha = 1 = r \) and let \( R \to r \), we get inequality (9).

Finally we mention the result which is a refinement as well as a generalization of a result due to Jain [11, Theorem 2], which follows from corollary 3 as a special case.

**Corollary 4:** If \( P \in P_n \), does not vanish in \( |z| < 1 \) and \( m = \min_{|z|=1} |P(z)| \), then for every \( \beta \in C \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \) and for \( |z| = 1 \),

\[
\left| P'(rz) + \frac{n \beta}{1 + r} P(rz) \right| \leq \frac{n}{2} \left( \left| r^{n+1} \beta + \frac{r^n}{1 + r} \right| + \left| \frac{\beta}{1 + r} \right| \right) \| P \|_\infty - \frac{n}{2} \left( \left| r^{n+1} \beta + \frac{r^n}{1 + r} \right| - \left| \frac{\beta}{1 + r} \right| \right) m
\]

(17)
and

\[ \left| P(Rz) + \beta \left( \frac{R + 1}{r + 1} \right)^n P(rz) \right| \leq \frac{1}{2} \left[ \left( R^n + \beta \left( \frac{R + 1}{r + 1} \right)^n \right) \|P\|_\infty \right. 
\left. + \left( R^n - \beta \left( \frac{R + 1}{r + 1} \right)^n \right) \|P\|_\infty \right] 
\]

The result is sharp and the extremal polynomial is \( P(z) = a z^n + b \) where \(|a| = |b| = 1\).

2. LEMMAS

For the proofs of these theorems, we need the following lemmas.

**Lemma 1:** If \( F(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq 1\) and \( f(z) \) is a polynomial of degree at most \( n \) such that

\[ |f(z)| \leq |F(z)| \quad \text{for } |z|=1, \]

then for every \( R > r \geq 1, \alpha, \beta \in \mathbb{C} \) with \(|\alpha| \leq 1, |\beta| \leq 1\) and \(|z| \geq 1\),

\[ |f(Rz) + \phi(R, r, \alpha, \beta) f(rz)| \leq |F(Rz) + \phi(R, r, \alpha, \beta) F(rz)| \]

where \( \phi(R, r, \alpha, \beta) \) is defined by (13).

Lemma 1 is due to A. Aziz and N.A. Rather [7].

**Lemma 2:** If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq 1\) and \( m = \min_{|z|=1} |P(z)| \), then for every \( R > r \geq 1, \alpha, \beta \in \mathbb{C} \) with \(|\alpha| \leq 1, |\beta| \leq 1\) and \(|z| \geq 1\),

\[ |P(Rz) + \phi(R, r, \alpha, \beta) P(rz)| \geq m \left| R^n + \phi(R, r, \alpha, \beta) r^n \right| \]

where \( \phi(R, r, \alpha, \beta) \) is defined by (13).

**Proof of Lemma 2:** For \( m = 0 \), there is nothing to prove. Assume \( m > 0 \), so that all the zeros of \( P(z) \) lie in \(|z| < 1\) and we have

\[ m \left| z^n \right| \leq |P(z)| \quad \text{for } |z|=1. \]

Applying Lemma 2 with \( F(z) \) replaced by \( P(z) \) and \( f(z) \) by \( m \left| z^n \right| \), we obtain for \(|z| \geq 1\),

\[ m \left| R^n z^n + \phi(R, r, \alpha, \beta) r^n z^n \right| \leq |P(Rz) + \phi(R, r, \alpha, \beta) P(rz)|. \]

That is,

\[ |P(Rz) + \phi(R, r, \alpha, \beta) P(rz)| \geq m \left| z^n \right| \left| R^n + \phi(R, r, \alpha, \beta) r^n \right| \]

for \( \alpha, \beta \in \mathbb{C} \) with \(|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1\) and \(|z| \geq 1\). That proves Lemma 2.

**Lemma 3:** If \( P \in \mathbb{P}_n \) does not vanish in \(|z| < 1\) and \( m = \min_{|z|=1} |P(z)| \), then for every \( R > r \geq 1, \alpha, \beta \in \mathbb{C} \) with \(|\alpha| \leq 1, |\beta| \leq 1\) and \(|z| = 1\),
\[
|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)|\leq|Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz)|-\left|R^n+\phi(R,r,\alpha,\beta)\right|\left|1+\phi(R,r,\alpha,\beta)\right|m
\]

where \( Q(z) = z^n \frac{P(1/z)}{z^n} \).

**Proof of Lemma 3:** Since \( m = \min_{|z|=1} |P(z)| \), we have
\[
m \leq |P(z)| \text{ for } |z|=1.
\]

Therefore, for every complex number \( \lambda \) with \( |\lambda| < 1 \), the polynomial \( H(z) = P(z) - \lambda m \) of degree \( n \) does not vanish in \( |z| < 1 \). If
\[
G(z) = z^n H(1/z) = Q(z) - \lambda m z^n,
\]
then all the zeros of polynomial \( G(z) \) of degree \( n \) lie \( |z| \leq 1 \) and
\[
|H(z)| = |G(z)| \text{ for } |z|=1.
\]

Applying Lemma 1 with \( f(z) \) replaced by \( H(z) \) and \( F(z) \) by \( G(z) \), we get for every \( R > r \geq 1 \), \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \) and \( |z| \geq 1 \),
\[
|H(Rz)+\phi(R,r,\alpha,\beta)H(rz)| \leq |G(Rz)+\phi(R,r,\alpha,\beta)G(rz)|.
\]

That is,
\[
|P(Rz)+\phi(R,r,\alpha,\beta)P(rz) - \lambda m \{1+\phi(R,r,\alpha,\beta)\}| \leq |Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz) - \lambda m z^n \{R^n+\phi(R,r,\alpha,\beta)r^n\}|
\]

for every \( R > r \geq 1 \), \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \) and \( |z| \geq 1 \). Since all the zeros of \( Q(z) \) lie in \( |z| \leq 1 \), we choose argument of \( \lambda \) with \( |\lambda| < 1 \), such that
\[
|Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz) - \lambda m z^n \{R^n+\phi(R,r,\alpha,\beta)r^n\}| = |Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz) - |\lambda| m \{R^n+\phi(R,r,\alpha,\beta)r^n\}|z^n|
\]

for \( |z| \geq 1 \), which is possible by Lemma 3, we have from (22),
\[
|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)| - |\lambda| m \{R^n+\phi(R,r,\alpha,\beta)\} \leq |Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz) - |\lambda| m \{R^n+\phi(R,r,\alpha,\beta)r^n\}|z^n|
\]

for \( |z| \geq 1 \). Equivalently for every \( R > r \geq 1 \), \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \) and \( |z| \geq 1 \),
\[
|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)| \leq |Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz)| - |\lambda|^p \{R^n+\phi(R,r,\alpha,\beta)r^n\}|z^n| \leq |1+\phi(R,r,\alpha,\beta)|m.
\]

Letting \( |\lambda| \to 1 \), we get the conclusion of lemma 4.

We also need the following two Lemmas due to A.Aziz and N.A.Rather [8, 9].

**Lemma 4 [7]:** If \( P \in \mathbb{P}_n \), then for arbitrary real or complex numbers \( \alpha, \beta \), with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \), \( R > r \geq 1 \), \( p > 0 \) and \( \gamma \) real,
\[
\begin{align*}
&\int_{0}^{2\pi} |P(Re^{i\theta}) + \phi(R, r, \alpha, \beta)P(re^{i\theta}) + e^{i\gamma}[R^n P(e^{i\theta}/R) + \phi(R, r, \alpha, \beta)r^n P(e^{i\theta}/r)]|^p \, d\theta \\
\leq |R^n + \phi(R, r, \alpha, \beta)r^n + e^{i\gamma}(1 + \phi(R, r, \alpha, \beta)) P(e^{i\theta}/r)| \int_{0}^{2\pi} |P(e^{i\theta})| \, d\theta. \tag{24}
\end{align*}
\]

The result is sharp and the extremal polynomial is \( P(z) = \lambda z^n \), \( \lambda \neq 0 \).

**Lemma 5 [6]:** If \( A, B, C \) are non-negative real numbers with \( B+C \leq A \), then for every real number \( \alpha \),

\[
|A-C| e^{i\alpha} + |B+C| \leq |Ae^{i\alpha} + B|.
\]

### 3. PROOF OF THE THEOREM

**Proof of Theorem 1:** By hypothesis \( P \in \mathbb{P}_n \) and \( P(z) \neq 0 \) in \(|z| < 1\), therefore by Lemma 3, we have for arbitrary real or complex numbers \( \alpha, \beta \), with \(|\alpha| \leq 1, |\beta| \leq 1, R>r \geq 1 \) and \( 0 \leq \theta < 2\pi \),

\[
|P(Re^{i\theta}) + \phi(R, r, \alpha, \beta)P(re^{i\theta})| \leq \left| Q(Re^{i\theta}) + \phi(R, r, \alpha, \beta)Q(re^{i\theta}) \right| - \left| R^n + \phi(R, r, \alpha, \beta) r^n \right|
\]

where \( m = \min_{|z|=1} |P(z)| \), \( Q(z) = z^n \overline{P(1/z)} \) and \( \phi(R, r, \alpha, \beta) \) is defined by (13). This implies,

\[
|P(Re^{i\theta}) + \phi(R, r, \alpha, \beta)P(re^{i\theta})| \leq R^n P\left( \frac{e^{i\theta}}{R} \right) + \phi(R, r, \alpha, \beta) P\left( \frac{e^{i\theta}}{r} \right)
\]

\[
- \left| R^n + \phi(R, r, \alpha, \beta) r^n \right|
\]

which gives,

\[
|P(Re^{i\theta}) + \phi(R, r, \alpha, \beta)P(re^{i\theta})| + \frac{m}{2} \left| R^n + \phi(R, r, \alpha, \beta) r^n \right| - \left| 1 + \phi(R, r, \alpha, \beta) \right|
\]

\[
\leq R^n P(e^{i\theta}/R) + \phi(R, r, \alpha, \beta) P\left( e^{i\theta}/r \right) - \frac{m}{2} \left| R^n + \phi(R, r, \alpha, \beta) r^n \right| - \left| 1 + \phi(R, r, \alpha, \beta) \right|
\]

Taking

\[
A = R^n P\left( \frac{e^{i\theta}}{R} \right) + \phi(R, r, \alpha, \beta) P\left( \frac{e^{i\theta}}{r} \right),
\]

\[
B = P(Re^{i\theta}) + \phi(R, r, \alpha, \beta) P\left( e^{i\theta}/r \right),
\]

and \( C = \left| R^n + \phi(R, r, \alpha, \beta) r^n \right| - \left| 1 + \phi(R, r, \alpha, \beta) \right| m \)

in Lemma 5 and noting by (26), that \( B+C \leq A \), we obtain for every real \( \gamma \),

\[
\begin{align*}
&\left\{ \left| R^n P\left( \frac{e^{i\theta}}{R} \right) + \phi(R, r, \alpha, \beta) P\left( \frac{e^{i\theta}}{r} \right) - \frac{R^n + \phi(R, r, \alpha, \beta) r^n}{2} \left| \left| 1 + \phi(R, r, \alpha, \beta) \right| \right| m \right| e^{i\gamma} \right. \\
&+ \left| P(Re^{i\theta}) + \phi(R, r, \alpha, \beta) P\left( e^{i\theta}/r \right) + \frac{R^n + \phi(R, r, \alpha, \beta) r^n}{2} \left| \left| 1 + \phi(R, r, \alpha, \beta) \right| \right| m \right| \right) e^{i\gamma} \\
\end{align*}
\]
This implies for each \( p > 0 \),

\[
\int_0^{2\pi} \left| F(\theta) + e^{i\gamma} G(\theta) \right|^p d\theta \leq \int_0^{2\pi} \left| R^n P(e^{i\theta} / R) + \phi(R, r, \alpha, \beta) r^n P(e^{i\theta} / r) \right| e^{i\gamma} d\theta + |P(R e^{i\theta}) + \phi(R, r, \alpha, \beta) P(r e^{i\theta})| \left| e^{i\gamma} \right| d\theta
\]

\[
F(\theta) = |P(R e^{i\theta}) + \phi(R, r, \alpha, \beta) P(r e^{i\theta})| + \left| R^n + \phi(R, r, \alpha, \beta) r^n - 1 + \phi(R, r, \alpha, \beta) \right| \left| e^{i\gamma} \right| d\theta
\]

\[
G(\theta) = \left| R^n \left( e^{i\theta} / R \right) + \phi(R, r, \alpha, \beta) r^n \left( e^{i\theta} / r \right) \right| + \left| R^n + \phi(R, r, \alpha, \beta) r^n - 1 + \phi(R, r, \alpha, \beta) \right| \left| e^{i\gamma} \right| d\theta
\]

Integrating both sides of (27) with respect to \( \gamma \) from 0 to \( 2\pi \), we get with the help of lemma 4, for each \( p > 0, R > r \geq 1, |\alpha| \leq 1, |\beta| \leq 1 \) and \( \gamma \) real,

\[
\int_0^{2\pi} \int_0^{2\pi} \left| F(\theta) + e^{i\gamma} G(\theta) \right|^p d\theta d\gamma \leq \int_0^{2\pi} \int_0^{2\pi} \left| R^n P(e^{i\theta} / R) + \phi(R, r, \alpha, \beta) r^n P(e^{i\theta} / r) \right| e^{i\gamma} \left| e^{i\gamma} \right| d\theta d\gamma
\]

\[
\int_0^{2\pi} \int_0^{2\pi} \left| R^n P(e^{i\theta} / R) + \phi(R, r, \alpha, \beta) r^n P(e^{i\theta} / r) \right| e^{i\gamma} d\theta d\gamma
\]

Now for every real \( \gamma, t \geq 1 \) and \( p > 0 \), we have

\[
\int_0^{2\pi} \left| t + e^{i\gamma} \right|^p d\gamma \geq \int_0^{2\pi} \left| 1 + e^{i\gamma} \right|^p d\gamma.
\]

If \( F(\theta) \neq 0 \), we take \( t = \frac{G(\theta)}{F(\theta)} \), then by (26), \( t \geq 1 \) and we get
For $F(0) = 0$, this inequality is trivially true. Using this in (28), we conclude that for arbitrary real or complex numbers $\alpha, \beta$, with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$
\int_{0}^{2\pi} |1 + e^{i\gamma}|^p \gamma \int_{0}^{2\pi} \left\{ |P(e^{i\gamma}) + \phi(R, r, \alpha, \beta)P(Re^{i\gamma}) + \frac{R^n + \phi(R, r, \alpha, \beta) r^n - 1 + \phi(R, r, \alpha, \beta)}{2} m| \right\}^p d\theta
\leq \left\{ \int_{0}^{2\pi} R^n + \phi(R, r, \alpha, \beta) r^n + e^{i\gamma} \left(1 + \phi(R, r, \alpha, \beta)\right) \left|1 + \phi(R, r, \alpha, \beta)\right|^p d\gamma \right\} \left\{ \int_{0}^{2\pi} P(e^{i\gamma}) \gamma \right\}^p d\theta.
$$

(29)

Since

$$
\int_{0}^{2\pi} \left( R^n + \phi(R, r, \alpha, \beta) r^n + e^{i\gamma} \left(1 + \phi(R, r, \alpha, \beta)\right) \right) \left|1 + \phi(R, r, \alpha, \beta)\right|^p d\gamma = \int_{0}^{2\pi} \left( R^n + \phi(R, r, \alpha, \beta) r^n + e^{i\gamma} \left(1 + \phi(R, r, \alpha, \beta)\right) \right) \gamma \gamma^p d\gamma,
$$

From (29), we obtain

$$
\int_{0}^{2\pi} \left\{ \left| P(Re^{i\gamma}) + \phi(R, r, \alpha, \beta)P(Re^{i\gamma}) + \frac{R^n + \phi(R, r, \alpha, \beta) r^n - 1 + \phi(R, r, \alpha, \beta)}{2} m \right| \right\}^p d\theta
\leq \left\{ \int_{0}^{2\pi} |1 + e^{i\gamma}|^p \gamma \right\}^p d\gamma
$$

\[ \leq \int_{0}^{2\pi} \left\{ \int_{0}^{2\pi} P(e^{i\gamma}) \gamma \right\}^p d\theta \]

This gives for every real or complex number $\delta$ with $|\delta| \leq 1$,

$$
\left\| P(Rz) + \phi(R, r, \alpha, \beta)P(rz) + \delta \left\{ \frac{R^n + \phi(R, r, \alpha, \beta) r^n - 1 + \phi(R, r, \alpha, \beta)}{2} m \right\} \right\|^p \leq C_p \left\| 1 + e^{i\gamma} \right\|^p d\gamma.
$$

That proves the Theorem completely.

REFERENCES


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