**g**∗α-CLOSED SETS IN BITOPOLOGICAL SPACES

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**ABSTRACT**

In this paper we introduce a new class of sets (i, j)-**g**∗α-closed sets in bitopological spaces. Properties of these sets are investigated and we introduce new bitopological spaces (i, j)-**T**1/2 and (i, j)-**αT**1/2 as applications.

Key words: (i, j)-g-closed sets, (i, j)-g*-closed sets, (i, j)-**g**∗α-closed sets; (i, j)-**T**1/2 spaces, and (i, j)-**αT**1/2 space.

1. INTRODUCTION

A triple (X, τ₁, τ₂) where X is non-empty set and τ₁ and τ₂ are topologies on X is called a bitopological space and Kelly [10] initiated the study of such spaces. In 1985, Fukutake introduced the concepts of g-closed sets [8] in bitopological spaces. Recently Veera Kumar introduced and studied the concepts of g* -closed set [19] and g* -continuity in topological spaces.

The purpose of this paper is to introduce the new class of sets, namely **g**∗α-closed sets. Applying these sets, the author introduced the new class of spaces, namely (i, j)-**T**1/2 spaces, (i, j)-**αT**1/2 spaces for bitopological spaces and investigate some of their properties. In this paper we study the relationship of **g**∗α-closed sets with the class of closed sets namely (i, j)-g*-closed[17], (i, j)-gs-closed[5], (i, j)-αg-closed [14], (i, j)-gsp-closed[5], (i, j)-rg-closed set[16], (i, j)-sg-closed[6], and (i, j)-gpr-closed sets. Also we study the relationship of (i, j)-**T**1/2, (i, j)-**αT**1/2 spaces with (i, j)-**T**1/2 [7], (i, j)-**T**1/2 [17], (i, j)-**T**b spaces.

2. PRELIMINARIES

If A is a subset of X with a topology τ, then the closure of A is denoted by τ-cl (A) or cl(A), the interior of A is denoted by τ-int(A), or int(A) and the complement of A in X is denoted by A^c.

**Definitions 2.1:** A subset A of a bitopological space (X, τ₁, τ₂) is called
(i) a generalized open (g-open) set[17] if F ⊆ int(A) whenever F ⊆ A and F is closed in (X, τ).
(ii) a α-open set[15] if A ⊆ int(cl(int(A))).
(iii) a regular closed set[16] if A = cl(int(A)).
(iv) a semi-open set[12] if A ⊆ cl(int(A)).
(v) a semi-pre-open set[1] if cl(int(cl(A))) ⊆ A.
(vi) a prereclosed set[17] if A = int(cl(A)).

**Definition 2.2:** A subset A of a bitopological space (X, τ₁, τ₂) is called
(i) (i, j)-g*-closed[17] if τ₁-cl(A) ⊆ U whenever A ⊆ U and U ∈ GO(X, τ)
(ii) (i, j)-gs-closed[5] if τ₁-scl(A) ⊆ U whenever A ⊆ U and U is open in τ₁.
(iii) (i, j)-αg-closed[14] if τ₁-αcl(A) ⊆ U whenever A ⊆ U and U ∈ τ₁.
(iv) (i, j)-gsp-closed[5] if τ₁-spcl(A) ⊆ U whenever A ⊆ U and U is open in τ₁.
(v) (i, j)-rg-closed[16] if τ₁-cl(A) ⊆ U whenever A ⊆ U is and U is regular open in τ₁.
(vi) (i, j)-sg-closed[7] if τ₁-scl(A) ⊆ U whenever A ⊆ U and U is semi open in τ₁.
(vii) (i, j)-gpr-closed[9] if τ₁-pcl(A) ⊆ U whenever A ⊆ U and U is regular open in τ₁.
The family of all \((i, j)\)-g-closed (resp \((i, j)\)-gsp-closed, \((i, j)\)-gp-closed, \((i, j)\)-αgr-closed, \((i, j)\)-α-g-closed, \((i, j)\)-gs-closed) subsets of a bitopological space \((X, \tau_1, \tau_2)\) is denoted by \(D(i, j)\).

**Definition 2.3:** (i) A bitopological space \((X, \tau_1, \tau_2)\) is said to be an \((i, j)\)-\(T_{1/2}\) space if every \((i, j)\)-g*-closed set is \(\tau_i\)-closed.
(ii) A bitopological space \((X, \tau_1, \tau_2)\) is said to be an \((i, j)\)-\(T_b\) space if every \((i, j)\)-gs-closed set is \(\tau_j\)-closed.

3. \((i, j)\) - \(g^{*\alpha}\) - CLOSED SETS

In this section we introduce the concept of \((i, j)\) - \(g^{*\alpha}\) - closed sets in bitopological spaces.

**Definition 3.1:** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be an \((i, j)\) - \(g^{*\alpha}\) - closed set if \(\tau_i - \alpha cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U \in GO(X, \tau_i)\).

We denote the family of all \((i, j)\) - \(g^{*\alpha}\) - closed sets in \((X, \tau_1, \tau_2)\) by \(D^{*\alpha}(i, j)\).

**Remark 3.2:** By setting \(\tau_1 = \tau_2\), in Definition 3.1, a \((i, j)\) - \(g^{*\alpha}\) - closed set is a \(g^{*}\) - closed set.

If \(A\) is \((i)\) \(\tau_j\)-closed, \((ii)\) \(\tau_j\)-\(\alpha\)-closed, \((iii)\) \((i, j)\) - \(g^{*}\) - closed subset of \((X, \tau_1, \tau_2)\), then \(A\) is \((i,j)\) - \(g^{*\alpha}\) - closed.

The following examples show that the reverse implications of the above proposition are not true.

**Example 3.4:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}, \tau_2 = \{\phi, \{a\}, \{a, b\}, X\}\). Then the subset \(A = \{b\}\) is \((1, 2)\) - \(g^{*\alpha}\) - closed but not \(\tau_2\)-closed in \((X, \tau_1, \tau_2)\).

**Example 3.5:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}, \tau_2 = \{\phi, \{a\}, X\}\). Then the subset \(\{a, c\}\) is \(g^{*\alpha}\) - closed but not \(\tau_2\)-\(\alpha\)-closed.

**Example 3.6:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}; \tau_2 = \{\phi, \{a\}, \{a, b\}, X\}\). The subset \(\{b\}\) is \((1, 2)\) - \(g^{*\alpha}\) - closed but not \((1, 2)\)-g*-closed.

**Proposition 3.7:** In a bitopological space \((X, \tau_1, \tau_2)\) every \((i, j)\) - \(g^{*\alpha}\) - closed set is \((i)\) \((i, j)\)-gs-closed, \((ii)\) \((i, j)\)-\(\alpha\)-closed, \((iii)\) \((i, j)\)-\(\alpha\)gr-closed, \((iv)\) \((i, j)\)-gsp-closed.

The following examples support that the converse of the above theorem is not true.

**Example 3.8:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}, \tau_2 = \{\phi, \{a\}, \{a, b\}, X\}\). In this example, the subsets \(\{a, b\}, \{a, c\}\) are \((1, 2)\) - gs-closed but not \((1, 2)\) - \(g^{*\alpha}\) - closed.

**Example 3.9:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}, \tau_2 = \{\phi, \{a\}, \{c\}, X\}\). The subset \(\{a, c\}\) is \((1, 2)\) - \(g^{*\alpha}\) - closed but not \((1, 2)\) - \(g^{*\alpha}\) - closed.

**Example 3.10:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}\). The subsets \(\{b\}, \{c\}, \{a, b\}, \{a, c\}\) are \((1, 2)\) - \(g^{*}\) - closed sets but not \((1, 2)\) - \(g^{*\alpha}\) - closed.

**Example 3.11:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{c\}, \{a, b\}, X\}\). The subsets \(\{a, b\}, \{a, c\}\) are \((1, 2)\) - gsp-closed but not \((1, 2)\) - \(g^{*\alpha}\) - closed.

**Proposition 3.12:** In a bitopological space \((X, \tau_1, \tau_2)\) every \((i, j)\) - \(g^{*\alpha}\) - closed set is \((i)\) \((i, j)\)-gpr-closed, \((ii)\) \((i, j)\)-gp-closed, \((iii)\) \((i, j)\)-pre-semi-closed.

The following examples support that the reverse implications of the above propositions are not true.

**Example 3.13:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b\}, X\}, \tau_2 = \{\phi, \{c\}, X\}\). Then the subsets \(\{a, c\}, \{b, c\}\) are \((1, 2)\) - gpr-closed but not \((1, 2)\) - \(g^{*\alpha}\) - closed.

**Example 3.14:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}, \tau_2 = \{\phi, \{b\}, \{a, c\}, X\}\). Then the subsets \(\{a\}, \{c\}, \{a, b\}, \{b, c\}\) are \((1, 2)\) - gp-closed but not \((1, 2)\) - \(g^{*\alpha}\) - closed.

**Example 3.15:** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b\}, X\}, \tau_2 = \{\phi, \{b\}, \{a, c\}, X\}\). Then the subsets \(\{a\}, \{c\}, \{a, b\}, \{b, c\}\) are \((1, 2)\) - pre semi-closed but not \((1, 2)\) - \(g^{*\alpha}\) - closed.

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Remark 3.16: The following examples show that (i, j) - g-closed sets and (i, j)- $g^{\alpha}$- closed sets are independent.

Example 3.17: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then the subsets $A = \{a, b\}$, $\{a, c\}$ are (1, 2)-g-closed but not (1, 2)- $g^{\alpha}$-closed.

Example 3.18: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$. Then the subset $B = \{b\}$ is (1, 2)- $g^{\alpha}$-closed but not (1, 2)-g-closed.

Remark 3.19: The following examples show that (i, j) - $g^{\alpha}$- closed sets and (i, j) - rg- closed sets are independent.

Example 3.20: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, c\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$. Then the subsets $A = \{b\}$, $\{c\}$ are (1, 2)- $g^{\alpha}$-closed sets but not (1, 2) - rg-closed sets.

In the same example 3.20, the subset $B = \{a, c\}$ is a (1, 2)-rg-closed set but not a (1, 2)- $g^{\alpha}$-closed set.

Remark 3.21: The following examples show that (i, j) - sg-closed sets and (i, j) - $g^{\alpha}$- closed sets are independent.

Example 3.22: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{a, c\}, X\}$. Then the subsets $A = \{c\}$, $\{b, c\}$ are (1, 2)-sg-closed but not (1, 2)- $g^{\alpha}$-closed.

Example 3.23: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then the subset $B = \{a\}$ is (1, 2)- $g^{\alpha}$-closed set but not a (1, 2)-sg-closed set.

All the above results can be represented by the following diagram.

Where $A \to B$ (resp. $A \leftrightarrow B$) represents A implies B but not conversely (resp. A and B are independent).

Proposition 3.24: If $A, B \in D^*(i, j)$, then $A \cup B \in D^*(i, j)$.

Remark 3.25: $D^*(1, 2)$ is generally not equal to $D^*(2, 1)$. This is proved by an example below.

Example 3.26: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$. Then the subset $\{b\}$ is in $D^*(1, 2)$ but not in $D^*(2, 1)$.

Proposition 3.27: If $\tau_1 \subseteq \tau_2$, then $D^*(2, 1) \subseteq D^*(1, 2)$.

The converse of the above proposition is not true as seen from the following example.

Example 3.28: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$. Then $D^*(2, 1) \subseteq D^*(1, 2)$ but the subsets $\{b\}$, $\{a, b\}$ of $\tau_1$ not contained in $\tau_2$. 
Proposition 3.29: If A is (i, j) - $g^{α, γ}$- closed set, then $τ_j$ - acI(A) – A contains no non-empty $τ_j$ - g-closed set.

Proof: Let A be (i, j)-g$^{α, γ}$-closed set and F be a $τ_j$-g-closed set such that $F ⊆ τ_j$-acI(A)-A then $F^c ⊄ A$. And $F^c$ is g-open. We have $τ_j$-acI(A) $⊆ F^c$. Thus $F ⊆ τ_j$-acI $(A) \cap (τ_j$-acI$(A)^c = $ $φ$

Corollary 3.30: If A is (i, j) - $g^{α, γ}$-closed in $(X, τ_1, τ_2)$, then A is $τ_j$-α-closed if and only if $τ_j$-acI(A)-A is $τ_j$-g-closed.

Proof: Necessity: If A is $τ_j$-α-closed then, $τ_j$-acI(A) = A. i.e., $τ_j$-acI(A)-A = $φ$ and hence $τ_j$-acI(A)-A is $τ_j$-g-closed.

Sufficiency: Suppose $τ_j$-acI(A)-A = $F \neq φ$. Then by the proposition 3.29, F contains a g- closed set B. $τ_j$-acI(A)-A = $φ$, since A is $g^{α, γ}$-closed. Therefore A is $τ_j$-α-closed.

Proposition 3.31: If A is an (i, j) - $g^{α, γ}$- closed set in $(X, τ_1, τ_2)$, then $τ_j$ - clI(x)∩A $≠ φ$ holds for each x$∈$τ$_j$-acI(A).

Proof: If A is (i, j) - $g^{α, γ}$ -closed set. And let x$∈$τ$_j$-acI(A). Suppose $τ_j$ - $clI(x)∩A = φ$. Then, A $⊆ (τ_j$ - $clI(x))^c = U$ where U is open in $τ_j$. Thus $A ⊆ U$ where U is g-open in $τ_j$. Then $τ_j$-acI(A) $⊆ U$, since A is (i, j)-$g^{α, γ}$-closed. Hence $τ_j$- $clI(x)∩A$ $≠ φ$.

Proposition 3.32: If A is (i, j) - $g^{α, γ}$-closed set of $(X, τ_1, τ_2)$ such that A $⊆ B$ ⊆ $τ_j$-acI(A), then B is also a (i, j)-$g^{α, γ}$-closed set of $(X, τ_1, τ_2)$.

Proof: If A is (i, j) - $g^{α, γ}$-closed of $(X, τ_1, τ_2)$ such that A $⊆ B$ ⊆ $τ_j$-acI(A). Let B$⊆ U$ where U is g-open in $τ_j$. Since A$⊆ B$, $A ⊆ U$ where U is g-open in $τ_j$. Then $τ_j$-acI(A) $⊆ U$, since A is (i, j)-$g^{α, γ}$-closed. i.e., B $⊆ τ_j$-acI(A) $⊆ U$. Hence $τ_j$-acI(B) $⊆ U$. Therefore B is (i, j) - $g^{α, γ}$-closed set of $(X, τ_1, τ_2)$.

4. (i, j)$T^α_1/2$ and (i, j)-$g^{α*}T^α_1/2$ spaces

In this section, we introduce (i, j)$T^α_1/2$ and (i, j)-$g^{α*}T^α_1/2$ bitopological spaces.

Definition 4.1: A bitopological space $(X, τ_1, τ_2)$ is said to be an (i, j)$T^α_1/2$ space if every (i, j) - $g^{α, γ}$-closed set is $τ_j$-closed.

Proposition 4.2: If $(X, τ_1, τ_2)$ is said to be an (i, j) - $T^α_1/2$ space. Then it is an (i, j) - $T^α_1/2$ space but not conversely.

Example 4.3: Let $X= \{a, b, c\}, τ_1 = \{φ, \{a\}, X\}, τ_2 = \{φ, \{a\}, \{a, b\}, X\}$. Then $(X, τ_1, τ_2)$ is $(1, 2)$ - $T^α_1/2$ space, since the subsets $\{b\}, \{c\}, \{b, c\}$ are $(1, 2)$-g*-closed and $τ_2$-closed, but $(X, τ_1, τ_2)$ is not $(1, 2)$- $T^α_1/2$, since the subset $\{b\}$ is $(1, 2)$ - $g^{α, γ}$-closed but not $τ_2$-closed.

Proposition 4.4: Every (i, j)-$T^α_b$- space is (i, j) - $T^α_1/2$ space but not conversely.

Example 4.5: Let $X= \{a, b, c\}, τ_1 = \{φ, \{c\}, \{a, c\}, X\}, τ_2 = \{φ, \{a\}, X\}$. Then the space $(X, τ_1, τ_2)$ is $(1, 2)$ - $T^α_1/2$ space since the subsets $\{a\}, \{b, c\}$ are $(1, 2)$- $g^{α, γ}$-closed and $τ_2$-closed but $(X, τ_1, τ_2)$ is not $(1, 2)$-$T^α_b$, since the subsets $\{b\}, \{c\}, \{a, b\}, \{a, c\}$ are $(1, 2)$-gs-closed but not $τ_2$-closed.

Definition 4.12: A bitopological space $(X, τ_1, τ_2)$ is said to be an (i, j)-$g^{α*}T^α_1/2$ space if every (i, j) - $g^{α, γ}$-closed set is (i, j) - $g^{α*}$-closed.

Proposition 4.13: Every (i, j) - $T^α_1/2$ space is (i, j)-$g^{α*}T^α_1/2$ but not conversely.

Example 4.14: Let $X= \{a, b, c\}, τ_1 = \{φ, \{c\}, \{a, c\}, X\}, τ_2 = \{φ, \{a\}, X\}$. Then $(X, τ_1, τ_2)$ is an $(1, 2)$-$g^{α*}T^α_1/2$ space, since the subsets $\{b\}, \{c\}, \{a, b\}, \{b, c\}$ are $(1, 2)$ - $g^{α, γ}$-closed and $g^{α*}$-closed but not an $(1, 2)$-$T^α_1/2$-space, since the subset $\{b\}$ is $(1, 2)$-$g^{α*}$-closed but not $τ_2$-closed.

Proposition 4.15: Every (i, j)-$T^α_b$- space is (i, j)-$g^{α*}T^α_1/2$ space but not conversely.

Example 4.16: Let $X= \{a, b, c\}, τ_1 = \{φ, \{c\}, \{a, c\}, X\}, τ_2 = \{φ, \{a\}, X\}$. Then $(X, τ_1, τ_2)$ is an $(1, 2)$-$g^{α*}T^α_1/2$ space since the subsets $\{b\}, \{c\}, \{a, b\}, \{b, c\}$ are $(1, 2)$ - $g^{α, γ}$-closed and $g^{α*}$-closed but not an $(1, 2)$-$T^α_b$-space, since the subsets $\{b\}, \{c\}, \{a, b\}$ are $(1, 2)$-gs-closed and not $τ_2$-closed.
Proposition 4.17: \((i, j) - T^{*}_{1/2}\) and \((i, j)^{-*}_{T^{*}_{1/2}}\) are independent as seen from the following two examples.

**Example 4.18:** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{c\}, \{a, c\}, X\}\), \(\tau_2 = \{\emptyset, \{a\}, X\}\). Then \((X, \tau_1, \tau_2)\) is an \((1, 2)^{-*}_{T^{*}_{1/2}}\) space since the subsets \(\{b\}, \{c\}, \{a, b\}, \{b, c\}\) are \((1, 2)^{-*}_{g^{*}*}\)-closed and \(g^*-\)closed but not an \((1, 2)^{-*}_{T^{*}}\)-space, since the subsets \(\{b\}, \{c\}\) are \((1, 2)^{-*}_{g^{*}*}\)-closed and not \(\tau_2\)-closed.

**Example 4.19:** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{a\}, X\}\), \(\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}\). Then \((X, \tau_1, \tau_2)\) is \((1, 2)^{-*}_{T^{*}_{1/2}}\) space since the subsets \(\{c\}, \{b, c\}\) are \((1, 2)^{-*}_{g^{*}*}\)-closed and \(\tau_2\)-closed but not an \((1, 2)^{-*}_{T^{*}_{1/2}}\) space, since the subset \(\{b\}\) is \((1, 2)^{-*}_{g^{*}*}\)-closed and not \(\tau_2\)-closed.

All the above results can be represented by the following diagram.

![Diagram](image)

Where \(A \rightarrow B\) (resp. \(A \leftrightarrow B\)) represents A implies B but not conversely (resp. A and B are independent)

**REFERENCES**


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