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COMPOSITION AND WEIGHTED COMPOSITION OPERATORS ACTING ON SEQUENCE SPACES DEFINED BY MODULUS FUNCTIONS

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ABSTRACT

In this paper we study composition operators and weighted composition operators on sequence spaces defined by modulus functions.

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Mathematics subject classification*: Primary 47B20, Secondary 47B38.*

1. INTRODUCTION AND PRELIMINARIES:

A modulus function is a function $f:[0,\infty)\to[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$.

(ii) $f(x+y)\le f(x)+f(y)$ for all $x\ge 0, y\ge 0$.

(iii) f is increasing.

(iv) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example take $f(x)=\dfrac{x}{x+1}$, then $f(x)$ is bounded. If $f(x)=x^p, 0<p<1$ then the modulus function $f(x)$ is unbounded. Let f be a modulus function and $A=(a_{nk})$ be a non-negative matrix such that $\sup_n \sum_{k=1}^{\infty} a_{nk}$ is finite. If we denote by C, the space of all sequence $x=\{x_k\}$, then by $W_0(A,f)$, we mean the class of all sequence $x\in C$ such that $\lim_n \sum_{k=1}^{\infty} a_{n,k} f(|x_k|)=0.$ The class $W_0(A,f)$ is a linear space over the complex field C, For every $x\in W_0(A,f)$, we define $\|x\|_{A,f}=\sup_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|)$. Bhardwaj and Singh[1] proved that $\|.\|_{A,f}$ is a paranorm on $W_0(A,f)$ and $(W_0(A,f),\|.\|_{A.f})$ is a complete linear topological space. If $v:N\to N$ and $u:N\to C$ be two mappings. Then a bounded linear transformation $m_{u,v}:W_0(A,f)\to W_0(A,f)$ defined by $(m_{u,v}f)(x)=u(x)f(v(x))$ is called a weighted composition operator induced by (u,v). If we take $u(x)=1$, the constant one function, we write $m_{u,v}$ as T_v and call it a composition operator. In case $v(x)=x$, for every x we write $m_{u,v}$ as m_u and call it multiplication operator on $W_0(A,f)$ induced by u.

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H.Takagi and K.Yokouchi [9] initiated the study of multiplication and composition operators between L^p-spaces, whereas the study of weighted composition operators on some function spaces are considered by Carlson([2],[3]), Jamison and Rajagopalan [4], Kamowitz[5], Komal, B . S., Raj, Kuldip and Gupta Sunil[6], Takagi[8]. For more details see Singh and Manhas [7].

In this paper we plan to study composition and weighted composition operators acting on sequence spaces defined by modulus functions.

2. COMPOSITION OPERATORS ACTING ON SEQUENCE SPACES DEFINED BY MODULUS FUNCTION:

In this section we discuss composition operators acting on sequence spaces defined by modulus functions.

Theorem: 2.1 Let $T_v : W_0(A,f) \to W_0(A,f)$ be a linear transformation. Then T_v is bounded if there exists $M > o$ such that $\sum_{m\in v^{-1}(k)} a_{nm} \le M a_{nk}$ for all $k \in N$ and $n \in N$.

Proof: Suppose that the condition of the theorem is true. if $x \in W_0(A,f)$, then

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\sum_{m\in v^{-1}(k)} a_{n,m} f(|x_k|) \le M \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} f(|x_k|) = 0$$

Which shows that $T_v x \in W_0(A,f)$, Further

$$\|T_v x\|_{A,f} = \sup_n\left\{\sum_{k=1}^{\infty} a_{nk} f(|x_0 v(k)|)\right\}$$

$$\le M \sup_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|)$$

$$= M\|x\|_{A,f} \tag{1}$$

The continuity of T_v at origin follows from the inequality (i). Since T_v is linear, so it is continuous everywhere.

Theorem: 2.2 If T_v is bounded, then there exists $M > o$ such that

$$\sup_n \sum_{m\in v^{-1}(k)} a_{nm} \le M \sup_n \ a_{nk} \tag{2}$$

Proof: If the condition (ii) is not satisfied then for every positive integer k > 0,there exists positive integer p_k and n_k such that $\sup_n \sum_{m\in v^{-1}(k)} a_{nm} > k \sup_n a_{nk}$. Let $x^{pk} \in W_0(A,f)$ be such that

$$x^{pk}(s) = \begin{cases} f^{-1}\left(\dfrac{1}{\sup \sum_{m\in v^{-1}(k)} a_{nm}}\right) & if \quad s = p_k \\ 0, & elsewhere \end{cases}$$

Then

$$\|x^{pk}\|_{A,f} = \sup_n\left\{\sum_{m=1}^{\infty} a_{nm} f(|x^{pk}(m)|)\right\}$$

$$= \sup_n\left\{\sum_{m=1}^{\infty} a_{n,p_k} \frac{1}{\sup_n \sum_{m\in v^{-1}(pk)} a_{nm}}\right\}$$

$$= \frac{1}{k} \to 0 \text{ as } k \to \infty.$$

And

$$\|T_v x^{pk}\|_{A,f} = \sup_n\left\{\sum_{m=1}^{\infty} a_{nm} f(|x^{pk}(v(m))|)\right\}$$

$$=1.$$

This contradicts the continuity of T_v. Hence the condition (2) must be true.

Theorem: 2.3 Let $T_v \in B(W_{0(A,f)})$. Then T_v has closed range if there exists $\delta > 0$ such that

$$\sum_{m\in v^{-1}(k)} a_{nm} \ge \delta a_{nk} \tag{3}$$

for every $k \in N$ and $n \in N$.

Proof: We first assume that the condition (3) is true and then show that T_v has closed range. Let $x \in ranT_v$ and let $\{x^n\}$ be a sequence in $W_0(A,f)$ such that $T_v x^n \to x$. Then for every $\varepsilon > 0$ there

exists positive integer n_0 such that $\left\|T_v x^n - T_v x^m\right\|_{A,f} < \varepsilon$ for all $n,m \geq n_0$ or equivalently,

$$\varepsilon > \sup\left\{\sum_{k=1}^{\infty} \sum_{p\in v^{-1}(k)} a_{np} f\left(\left|x_n(p) - x_m(p)\right|\right)\right\}$$

$$\geq \delta \left|\sup_n \left\{\sum_{k=1}^{\infty} a_{nk} f\left(\left|x^n(p) - x^m(p)\right|\right)\right\}\right|$$

$$= \delta \left\|x^n - x^m\right\|_{A,f} \qquad (4)$$

for all $n,m \geq n_0$. From (4) it follows that $\{x_n\}$ is a cauchy sequence in $W_0(A,f)$. In view of completeness of $W_0(A,f)$ there exists $y \in W_0(A,f)$ such that $x^n \to y$. From the continuity of T_v, $T_v x^n \to T_v y$. Hence $x = T_v y$. So that $x \in ranT_v$. Hence ran T_v is closed.

Theorem: 4 Let $T_v x \in B(W_0(A,f))$. Then T_v is invertible if

(i) v is invertible

(ii) $\sup_n a_{nv(m)} \leq M \sup_n a_{nm}$ for every $n,m \in N$ where M is a constant such that $M > o$

Proof. Suppose T_v is an invertible operator. We show that v is invertible. If, $k \notin v(N)$ then $T_v e_k = 0$,

where e_k is defined as

$$e_k(s) = \begin{cases} f^{-1}\left(\dfrac{1}{\sup a_{nk}}\right) & if \quad s-k \\ 0, & elsewhere \end{cases}$$

So that T_v has non-trivial kernel. Hence v must be surjective. Next, if v is not injective,then v(n[1]) =v(n[2]) for two distinct positive integers n1 and n2 so that en1 does not belong to the range of T_v. Hence v must be injective. Let w be the inverse of v. Clearly T_w is the inverse of Tv. Since T_v is continuous, in view of theorem (2.2) we have

$$\sup_n \sum_{m\in v^{-1}(k)} a_{nm} \leq M \sup_n a_{nk} \text{ for every } k \in N.$$

Hence $a_{nv(k)} \leq M a_{nk}$ for every k.

Theorem: 2.5 Let $T_v \in B(W_0(A,f))$. Then T_v is an isometry if $\sum_{m\in v^{-1}(k)} a_{nm} = a_{nk}$ for every $k,n \in N$..

Proof: If the condition of the theorem is satisfied, then for every $x \in W_0(A,f)$, we have

$$\left\|T_v x\right\|_{A,f} = \sup_n \left\{\sum_{k=1}^{\infty} \sum_{m\in v^{-1}(k)} a_{nm} f\left|x_k\right|\right\}$$

$$= \sup_n \left\{\sum_{m=1}^{\infty} a_{nk} f\left(\left|x^{pk}(m)\right|\right)\right\}$$

$$= \left\|x\right\|_{A,f}$$

This proves the theorem.

Theorem: 2.6 Let $T_v : W_0(A,f) \to W_0(A,g)$ be linear transformation. Then T_v is bounded if there exists $M > o$ such that

$$\sum_{m\in v^{-1}(k)} a_{nm} g(y) \leq M a_{nk} f(y)$$

for all $k,n \in N.$ and $y \in R^+.$ (5)

Proof: Suppose the condition (5) is true. Then for $x \in W_0(A,f)$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \sum_{m\in v^{-1}(k)} a_{nm} g\left(\left|x_k\right|\right) \leq \lim_{n\to\infty} M \sum_{k=1}^{\infty} a_{nk} f\left(\left|x_k\right|\right)$$

$$=0$$

Which proves that $T_v x \in B(W_0(A,g))$ Further,

$$\left\|T_v x\right\|_{A,g} = \sup_n \left\{\sum_{k=1}^{\infty} a_{nk}\, g\left(\left|x(v(x))\right|\right)\right\}$$

$$= M \sup_n \left\{ \sum_{k=1}^{\infty} a_{nk} f\left(|x_k|\right) \right\}$$

$$= \|x\|_{A,f}.$$

This shows that T_v is continuous at the origin. From linearity of T_v it is continuous everywhere.

Theorem: 2.7 Let $T_v \in B(W_0(A,f), W_0(A,g))$ Then T_v has closed range if there exists $\delta > 0$ such that

$$\sum_{m \in v^{-1}(k)} a_{nm} g(y) \geq \delta a_{nk} f(y) \text{ for all } k, n \in N. \text{ and } y \in R^+.$$

Proof: Assume that the condition of the theorem is satisfied. We show that T_v has closed range.

Let $x \in ranT_v$. Then there exists a sequence $\{x_n\} \subset W_0(A,f)$ such that $T_v x_n \to x$. So for every $\varepsilon > o$ there exists a positive integer n_0 such that $\|T_v x_n - T_v x_m\|_{A,g} < \varepsilon$ for all $n, m \geq n_0$ or equivalently,

$$\varepsilon > \sup \left\{ \sum_{k=1}^{\infty} \sum_{p \in v^{-1}(k)} a_{np} g\left(|x_n(p) - x_m(p)|\right) \right\}$$

$$= \delta \|x_n - x_m\|_{A,f}$$

It follows that $\{x_n\}$ is a cauchy sequence in $W_0(A,f)$. In view of completeness of $W_0(A,f)$, there exists $y \in W_0(A,f)$ such that $x_n \to y$. From the continuity of T_v we have $T_v x_n \to T_v y$ Thus $x = T_v y$ Hence $x \in ramT_v$. So that $ranT_v$ is closed.

3. WEIGHTED COMPOSITION OPERATORS ACTING BETWEEN SEQUENCE SPACES DEFINED BY MODULUS FUNCTION

In this section we discuss weighted composition operators acting between sequence spaces defined by modulus functions.

Theorem: 3.1 Let $m_{u,v} : W_0(A,f) \to W_0(A,g)$ be a linear transformation. Then $m_{u,v}$ is bounded if there exists $M > o$ such that

$$\sum_{m \in v^{-1}(p)} a_{nm} g\left(|u(p)|\right) \leq M a_{np} f(y) \tag{6}$$

for all $p, n \in N$ and $y \in R^+$.

Proof: Suppose that the condition (6) is true. If $x \in W_0(A,f)$ then

$$\lim_{n\to\infty} \sum_{p=1}^{\infty} \sum_{m \in v^{-1}(p)} a_{nm} g\left(|x_p|\right) \leq M \lim_{n\to\infty} \sum_{p=1}^{\infty} a_{np} f\left(|x_p|\right) := 0,$$

Which shows that $m_{u,v} x \in W_0(A,g)$. Further,

$$\|m_{u,v} x\|_{A,f} = \sup_n \left\{ \sum_{p=1}^{\infty} a_{np} g\left(|u(p) x o v(p)|\right) \right\}$$

$$\leq M \sup_n \sum_{p=1}^{\infty} a_{np} f\left(|x_p|\right)$$

$$= M \geq \|x\|_{A,f} \tag{7}$$

The continuity of $m_{u,v}$ at the origin follows from the inequality (7). Since $m_{u,v}$ is linear so it is continuous everywhere.

Corollary: 3.2. If $m_{u,v}$ is bounded, then there exists $M > o$ such that

$$\sup_n \sum_{m \in v^{-1}(p)} a_{nm} g\left(|u(p) y|\right) \leq M \sup_n a_{np} f(y)$$

for all $p, n \in N$ and $y \in R^+$.

Theorem: 3.3 Let $m_{u,v} \in B(W_0(A,f), W_0(A,g))$. Then $m_{u,v}$ has closed range if there exists $\delta > 0$ such that

$$\sum a_{nm} g\left(|u(k) y|\right) \geq \delta a_{nk} f(y) \quad \text{for all } k, n \in N \text{ and } y \in R^+.$$

Proof: Assume that the condition of the theorem is satisfied. We show that $m_{u,v}$ has closed range.

Let $x \in ranm_{u,v}$. Then there exists a sequence $\{x_n\} \subset W_0(A,f)$ such that $m_{u,v} x^n \to x$ So for every

$\varepsilon > 0$ there exists n_0 such that $\|m_{u,v}x_n - m_{u,v}x_m\|_{A,g} < \varepsilon$ for all $n, m \geq n_0$ or equivalently,

$$\varepsilon > \sup\left\{\sum_{k=1}^{\infty} \sum_{p\in v^{-1}(k)} a_{np} g\left(\left|u(p)(x_n(p) - x_m(p))\right|\right)\right\}$$

$$\geq \delta\left|\sup_n\left\{\sum_{k=1}^{\infty} a_{nk} f\left(\left|x_n(p) - x_m(p)\right|\right)\right\}\right|$$

$$= \delta\|x_n - x_m\|_{A,f}.$$

It follows that $\{x_n\}$ is a cauchy sequence in $W_0(A, f)$ In view of completeness of $W_0(A, f)$ there exists $y \in W_0(A, f)$ such that $x_n \to y$. From the continuity $m_{u,v}$ we have $m_{u,v}x_n \to m_{u,v}y.$ Hence $x = (m_{u,v}y) \in ranm_{u,v}$. So that ran $m_{u,v}$ is closed.

Corollary: 3.4 Let $m_{u,v} \in B(W_0(A, f), W_0(A, g))$ be such that $m_{u,v}$ has closed range if there exists $\delta > 0$ such that

$$\sup_n \sum_{m\in v^{-1}(p)} a_{nm} g\left(\left|u(k)y\right|\right) \geq \delta \sup_n a_{nk} f(y) \quad \text{for all } k, n \in N$$

and $y \in R^+$.

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