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SOME THEOREMS ON HOLOMORPHICALLY PROJECTIVE TRANSFORMATIONS IN TACHIBANA SPACES

U. S. Negi* & Aparna Rawat
Department of Mathematics, H.N.B. Garhwal University, Campus Badshahi Thaul,
Tehri Garhwal-249 199, Uttarakhand, India

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ABSTRACT

Otaba (1956) has studied affine transformation in an almost complex manifold with a natural connection. Ishihara (1957) has defined and studied the Holomorphically projective changes and there groups in an almost complex manifold. Further, Sumitomo (1959) has defined and studied on a holomorphically projective correspondence in an almost complex space. Singh and Samyal (2004) have defined and studied on a Tachibana space with parallel Bochner curvature tensor.

In the present paper, we have defined and studied some theorems on holomorphically projective transformations in Tachibana spaces and several theorems have been established.

Key words: *Kaehlerian, Tachibana, Projective, Recurrent, Symmetric, Transformation, Space.*

1. INTRODUCTION

An almost Tachibana space is first of all an almost complex space, that is, a 2n-dimensional space with an almost complex structure $F^h{}_i$:

$$F^i{}_j\, F^h{}_i = -\delta^h{}_j, \tag{1.1}$$

And always admits a positive definite Riemannian metric tensor g_{ji} satisfying:

$$F^a{}_j\, F^b{}_i\, g_{ab} = g_{ji}, \tag{1.2}$$

From which

$$F_{ji} = -F_{ij}, \tag{1.3}$$

where

$$F_{ji} \overset{\text{def}}{=} F^a{}_j\, g_{ai} \tag{1.4}$$

And finally has the property that the differential form

$F_{ji}\, d\xi^j \wedge d\xi^i$ is closed, that is,

$$F_{jih} \overset{\text{def}}{==} \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0$$

And finally has the property that the skew-symmetric F_{ih} is a Killing tensor

$$\nabla_j F_{ih} + \nabla_i F_{hj} = 0 \tag{1.5}$$

From which

$$\nabla j F^j{}_i + \nabla_i F^h{}_j = 0 \tag{1.6}$$

and

$$F_i = -\nabla_j F^j{}_i = 0 \tag{1.7}$$

Corresponding author: U. S. Negi*
Department of Mathematics, H.N.B. Garhwal University, Campus Badshahi Thaul,
Tehri Garhwal-249 199, Uttarakhand, India

Here ∇ denotes the operation of covariant differentiation with respect to the Riemannian connection $\{_j{}^h{}_i\}$.

The Nijenhuis tensor $N^h{}_{ji}$ is written in the form:

$$N^h{}_{ji} = -4(\nabla_j F^t{}_i) F^h{}_t + 2G^t{}_{ji} F^h{}_i + F^t{}_j G^h{}_{ti} - F^t{}_i G^h{}_{tj}. \tag{1.8}$$

A contravarient almost analytic vector field is defined as a vector field v^i, satisfying (Tachibana (1959):

$$£_v F^h{}_i \equiv v^j \partial_j F^h{}_i - F^j{}_i \partial_j v^h + F^h{}_j \partial_i v^j = 0,$$

Where $£_v$ stands for the Lie-derivative with respect to v^i.

Let $R^h{}_{kji}$ be the Riemannian curvature tensor and put

$$R_{ji} = R^r{}_{rji},\ R_{kjih} = R^r{}_{kji} g_{rh},\ R = R_{ji} g^{ji} \text{ and } S_{ji} = F^r{}_j R_{ri},$$

Then the following identities are satisfied (Yano 1957)

$$R^r{}_{kji} F^h{}_r = R^h{}_{kjr} F^r{}_i,\ R_{kjir} F^r{}_h = R_{kjhr} F^r{}_i \tag{1.9}$$

$$R_{kjih} = R_{kjtr} F^t{}_i F^r{}_h,\quad R_{ji} = R_{tr} F^t{}_j F^r{}_i \tag{1.10}$$

$$S_{ji} + S_{ij} = 0,\ S_{ji} = S_{tr} F^t{}_j F^r{}_i,\ S_{ji} = -\frac{1}{2} F^{tr} R_{trji}. \tag{1.11}$$

The holomorphically projective curvature tensor $P^h{}_{kji}$, which will be briefly called HP-curvature tensor, is given by

$$P^h{}_{kji} = R^h{}_{kji} + \frac{1}{n+2}(R_{ki}\delta^h{}_j - R_{ji}\delta^h{}_k + S_{ki}F^h{}_j - S_{ji}F^h{}_k + 2S_{kj}F^h{}_i) \tag{1.12}$$

We can obtain the following identities

$$P^h{}_{(kj)I} = 0,\ P^h{}_{[kji]} = 0, \tag{1.13}$$

$$P^r{}_{rji} = 0, \tag{1.14}$$

$$P^r{}_{kji} F^h{}_r = P^h{}_{kjr} F^r{}_i,\ P^h{}_{rji} F^r{}_k == P^h{}_{rki} F^r{}_j \tag{115}$$

From which, we have

$$P^r{}_{kjr} = 0, \tag{1.16}$$

$$P^t{}_{rji} F^r{}_t = 0,\ P^t{}_{kjr} F^r{}_t = 0. \tag{1.17}$$

A necessary and sufficient condition for $P^h{}_{kji} = 0$, is that the space is a space of constant holomorphically curvature (Tashiro 1957), i.e., a space whose curvature tensor $R^h{}_{kji}$ takes the form

$$R^h{}_{kji} = -\frac{R}{n(n+2)}(g_{ki}\delta^h{}_i - g_{ji}\delta^h{}_k + F_{ki}F^h{}_j - F_{ji}F^h{}_k + 2F_{kj}F^h{}_j) \tag{1.18}$$

For a vector field V^i and a tensor field $\alpha^h{}_i$, the following identities are known (Yano 1957)

$$£_v \nabla_j \alpha^h{}_i - \nabla_j £_v \alpha^h{}_i = \alpha^r{}_i £_v \{_j{}^h{}_r\} - \alpha^h{}_r £_v \{_j{}^r{}_i\} \tag{1.19}$$

$$\nabla_k £_v \{_j{}^h{}_i\} - \nabla_j £_v \{_k{}^h{}_i\} = £_v R^h{}_{kji} \tag{1.20}$$

Where $£_v$ denotes the operator of Lie-differentiation with respect to $V^{i.}$

A Killing vector or an infinitesimal isometry V^i is defined by

$$£_v g_{ji} = \nabla_j V_i + \nabla_i V_j = 0.$$

Here we shall identify a contravariant vectors V^i with a covariant vector$V_i = g_{ir} V^r$. Hence we shall say V_i is a Killing vector, or that ρ^i is gradient, for example.

An infinitesimal affine transformation V^i is defined by

$$£_v \{ {}_j{}^h{}_i \} = \nabla_j \nabla_i V^h + R^h{}_{rji} V^r = 0.$$

We shall say a vector field V^i an infinitesimal holomorphically projective transformation or, for simplicity, an HP-transformation, if it satisfies

$$£_v \{ {}_j{}^h{}_i \} = \rho_j \delta^h{}_i + \rho_i \delta^h{}_j - \overline{\rho_j} F^h{}_i - \overline{\rho_i} F^h{}_j ,$$

Where ρ_i is a certain vector and $\overline{\rho_i} = F^r{}_i \rho_r$. In this case, we shall called ρ_i the associated vector of the transformation, If ρ_i vanishes, then the HP-transformation reduces to an affine one.

Contracting the last equation with respect to h and i, we get

$$\nabla_j \nabla_r V^r = (n+2) \rho_j ,$$

Which shows that the associated vector is gradient.

A vector field V^i is called contravariant analytic or, for simplicity, analytic, if it satisfies

$$£_v F^h{}_i \equiv - F^r{}_i \nabla_r V^h + F^h{}_r \nabla_i V^r = 0.$$

2. A GEOMETRICAL INTERPRETATION OF AN ANALYTIC HP-TRANSFORMATION

In a differentiable space M, we consider a tensor valued function V depending not only on a point P of M but also on k vectors $u_1, u_2, \ldots, u_k$ at the point and denote it by $V(P, u_1, u_2, \ldots, u_k)$. We assume that the value of this function V lies in the tensor space associated to the tengent space of M at P and that it depends differentiably on its arguments.

Assuming the space M to be affinely connected, we take an arbitrary curve C: $x^i = x^i$

(t) and denote its successive derivatives by

$$\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \frac{d^3x^i}{dt^3} \ldots\ldots \tag{2.1}$$

Then if we substitute (2.1) into the function V instead of $u_1, u_2 \ldots u_k$. we have a family of tensors

$$V(C) = V\left(\dot{x} , \frac{dx}{dt} , \ldots\ldots, \frac{d^k x}{dt^k} \right) \text{ along the curve C.}$$

Let V^i be an infinitesimal transformation, i.e., a vector field, and $'x^i = x^i + \varepsilon v_i$ be the infinitesimal point transformation determined by v^i, ε being an infinitesimal constant. Given a curve C: $x^i = x^i$ (t), the image $'C$ of F is expressed by

$$x^i = x^i (t) + \varepsilon V^i (x(t)).$$

We shall call the limiting value

$$£_v V(C) \equiv \lim_{\varepsilon \to 0} \frac{V('C) - 'V(C)}{\varepsilon}$$

The Lie-derivative of V(C) with respect to V^i, where we have denoted by $'V(C)$ the family of tensors induced from V(C) by the transformation

$$'x^k = x^i + \varepsilon V^i$$

In a Tachibana space, a curve $x^i = x^i (t)$ defined by

$$\frac{d^2x^h}{dt^2} + \left\{ {h \atop j \quad i} \right\} \frac{dx^j}{dt} \frac{dx^i}{dt} = \alpha \frac{dx^h}{dt} + \beta F_j^h \frac{dx^j}{dt} \tag{2.2}$$

is, by definition, a holomorphically planar curve, or an H-plane curve, where α and β are certain functions of t.

Let V^i be an infinitesimal transformation and assume that any ε the infinitesimal point transformation $'x^i = x^i\ \varepsilon\ V^i$ maps any H-plane curves.

Now we ask for the condition that V^i preserve that H-plane curves. For such a vector V^i taking account of (2.2), we have

$$£_v\left[\frac{d^2x^h}{dt^2} + \begin{Bmatrix} h \\ j\ \ i \end{Bmatrix}\frac{dx^j}{dt}\frac{dx^i}{dt} - \alpha\frac{dx^h}{dt} - \beta F_j^h\frac{dx^j}{dt}\right] = \gamma\frac{dx^h}{dt} + \delta F_j^h\frac{dx^j}{dt} \tag{2.3}$$

along any H-plane curve, where γ and δ are certain functions of t.

Denoting the Lie-derivative of the Christoffel's symbols and the complex structure $F^h{}_i$, respectively, by

$$t^h{}_{ji} = £_v\begin{Bmatrix} h \\ j\ \ i \end{Bmatrix}, \quad \alpha^h{}_i = £_v\ F^h{}_i,$$

we have from (2.3)

$$t^h{}_{ji}\ \dot{x}^j\ \dot{x}^i + \alpha\,\dot{x}^h + b\,F^h{}_j\,\dot{x}^j - \beta\,\alpha^h{}_j\,\dot{x}^j = 0 \tag{2.4}$$

where we have put

$$a = -(\gamma + £_v\,\alpha),\ b = -(\delta + £_v\,\beta), \quad \dot{x} = \frac{dx^i}{dt}$$

Since the relation (2.4) holds for any H-plane curve C, it must hold identically for any values of x^i and $\dot{x}^i$.

By means of the definition of the H-plane curve, we see further that the identity (2.4) holds for any value of the coefficient β.

Taking account of these arguments, we can easily see that relation

$$a^h{}_j\,\dot{x}^j = f\ \dot{x}^h + g\,F^h{}_j\,\dot{x}^j, \tag{2.5}$$

$$t^h{}_{ji}\,\dot{x}^j\,\dot{x}^i = p\,\dot{x}^h + q\,F^h{}_j\,\dot{x}^j, \tag{2.6}$$

hold for any values x^i and $\dot{x}^i$, where f,g,p and q are certain functions of x^i and $\dot{x}^i$.

Let $\alpha^i{}_j$ be a tensor on V such that

$$F^r{}_j\,\alpha^i{}_r + \alpha^r{}_j\,F^i{}_r = 0,$$

We obtain by means of (2.5)

$$\alpha^h{}_i \equiv £_v\ F^h{}_i = 0. \tag{2.7}$$

On the other hand, If we substitute (2.7) and $\nabla_j F_j^i = 0$ into the identify

$$\nabla_j £_V\ F_i^h - £_V \nabla_j F_i^h = F_r^h £_V\{{}_j{}^r{}_i\} - F_i^r £_V\{{}_j{}^h{}_r\},$$

Then we get

$$t_{ji}^r F_r^h = t_{jr}^h F_i^r. \tag{2.8}$$

From (2.6) and (2.8), taking account of the fact that

$$t_{ji}^h = \alpha_j\delta_i^h + \alpha_i\delta_j^h - \overline{\alpha}_j F_i^j - \overline{\alpha}_i F_j^h,$$

Where α_i is certain vector and $\overline{\alpha}_I = F_i^r\alpha_r$, we get

$$t_{ji}^h = £_V\{{}_j{}^h{}_i\} = \rho_j\delta_i^h + \rho_i\delta_j^h - \overline{\rho}_j\,F_i^h - \overline{\rho}_i F_j^h, \tag{2.9}$$

Where ρ_i is a certain vector field. Therefore, the infinitesimal transformation Vi is an analytic HP-transformation.

Conversely, it is obvious that an analytic HP-transformation preserves the H-plane curves. Thus we have the following:

Theorem 2.1: In a Technibana space, an infinitesimal transformation preserves the H-plane curves, if and only if it is an analytic HP-transformation.

3. SOME PROPERTIES OF HP-TRANSFORMATIONS

Let V^i be an HP-transformation, then it holds

$$£_V\{_j{}^h{}_i\} \equiv \nabla_j\nabla_i V^h + R^h_{rji}V^r = \rho_j\delta^h_i - \rho_i\delta^h_j - \bar{\rho}_j F^h_i - \bar{\rho}_i F^h_j \,. \tag{3.1}$$

Transvecting (3.1) with g^{ji}, we have

$$\nabla^r\nabla_r V'^i + R^h_r V^r = 0\,. \tag{3.2}$$

Hence, by virtue of the well known theorem on analytic vectors, Yano (1957), Lichnerowiez (1957), we have the following:

Theorem 3.1: In a compact Tachibana space an HP-transformation is analytic. In a compact Tachibana space, M, it holds that

$$\int_M (R_{ji}V^jV^i)d\sigma \geqq 0$$

For an analytic vector V^i, where $\boldsymbol{d\sigma}$ denote the volume element of M amd the equality holds when and when only V^i is parallel. Therefore, if the Ricci's from $R_{ji}\,\xi^j\xi^i$ is negative definite, then there exists no non-trival HP-transformation provided that the space is compact.

Taking account of the identity (1.19), we have for a vector field V^i

$$£_v\nabla_j F^h_i - \nabla_j £_v F^h_i = F^r_i £_v\{_j{}^h{}_r\} - F^h_r £_v\{_j{}^r{}_i\}\,,$$

Which implies

$$\nabla_j £_v F^h_i = F^h_r £_v\{_j{}^r{}_i\} - F^r_i £_v\{_j{}^h{}_r\}\,,$$

because of $\nabla_j F^h_i = 0$. If the vector field V^i is an HP-transformation, it is easily verified that the right hand-side of the last equation vanishes. Thus we have the following theorem by the virtue of Obata's theorem Obata (1956):

Theorem 3.2: In an irreducible Tachibana space admitting no quaternion structure, any HP-transformation is analytic.

It is known that a Tachibana space having non-vanishing Ricci tensor vanishes identically Obata (1956). Thus we have

Theorem 3.3: In an irreducible Tachibana space having non-vanishing Ricci tensor any HP-transformation is analytic.

Corollary 3.1: In an irreducible Tachibana Einstein space if its scalar curvature is non-vanishing, any HP-transformation is analytic.

In the following part of this section, we shall give some formulae on analytic HP-transformation which will be useful in the further study.

Let V^i be an HP-transformation. Substituting (3.1) into the identity

$$\nabla_k £_V g_{ji} - £_V\nabla_k g_{ji} = g_{ri} £_V\{_k{}^r{}_j\} + g_{jr}\, £_V\{_k{}^r{}_i\}\,,$$

We find

$$\nabla_k £_V\, g_{ji} = \rho_j\, g_{ki} + \rho_i g_{kj} - \bar{\rho}_j\, F_{ki} - \bar{\rho}_i\, F_{kj} + 2\rho_k\, g_{ji}\,. \tag{3.3}$$

Which will be used in **§5**.

If we substitute (3.1) into (1.20), then we have

$$£_V R^h_{kji} = \delta^h_j \nabla_k \rho_i - \delta^h_k \nabla_j \rho_i - F^h_j \nabla_k \bar{\rho}_i + F^h_k \nabla_j \bar{\rho}_i - (\nabla_k \bar{\rho}_j - \nabla_j \bar{\rho}_k) F^h_i , \tag{3.4}$$

Contracting the last equation with respect to h and k we find

$$£_V R_{ji} = -n \nabla_j \rho_i - 2F^r_j F^t_i \nabla_r \rho_t \tag{3.5}$$

Now we shall assume that V^i is an analytic HP-transformation. Then we have $£_V R_{ji} = £_V (R_{rt} F^r_j F^t_i)$

By virtue of (2.1). Hence from (3.5) it follows

$$\nabla_j \rho_i = F^r_j F^t_i \nabla_r \rho_t. \tag{3.6}$$

Since n>2. The last equation also is written in the form:

$$£_V F^h_i \equiv -F^r_i \nabla_r \rho^h + F^h_r \nabla_i \rho^r = 0 ,$$

Which shows that ρ^i is analytic. Moreover, according to (3.6) we have

$$\nabla_j \bar{\rho}_i + \nabla_i \bar{\rho}_j = F^r_i (\nabla_j \rho_r - F^t_j F^s_r \nabla_t \rho_s) = 0 , \tag{3.7}$$

Which means that $\bar{\rho}^i$ is a Killing vector. Thus we get the following:

Theorem 3.5: If a vector ρ_i is the associated vector of an analytic HP-transformation, then ρ^i is analytic and $\bar{\rho}^i$ is a Killing vector.

From (3.5) and (3.6) it follows

$$£_V R_{ji} = -(n+2) \nabla_j \rho_i , \tag{3.8}$$

From which we have

$$£_V S_{ji} = (n+2) \nabla_j \bar{\rho}_i . \tag{3.9}$$

On the other hand, from (3.4) and (3.7) we get

$$£_V R^h_{kji} = \delta^h_j \nabla_k \rho_i - \delta^h_k \nabla_j \rho_i - F^h_j \nabla_k \bar{\rho}_i + F^h_k \nabla_j \rho_i - 2F^h_i \nabla_k \bar{\rho}_j . \tag{3.10}$$

If we substitute (3.8) and (3.9) into (3.10). Then we can verify Ishihara (1957)

$$£_V P^h_{kji} = 0. \tag{3.11}$$

In the next place, substitute (3.1) and (3.8) into the identify

$$£_V \nabla_k R_{ji} - \nabla_k £_V R_{ji} = -R_{rt} £_V \{ {}_k{}^r{}_j \} - R_{jr} £_V \{ {}_k{}^r{}_i \} ,$$

We have

$$£_V \nabla_k R_{ji} = -(n+2) \nabla_k \nabla_j \rho_i - R_{ki} \rho_j - R_{kj} \rho_i + S_{ki} \bar{\rho}_j + S_{kj} \bar{\rho}_i - 2R_{ji} \rho_k . \tag{3.12}$$

Hence we put

$$P_{kji} = \frac{1}{n+2} (\nabla_k R_{ji} - \nabla_j R_{ki}) . \tag{3.13}$$

It holds

$$£_V P_{kji} = P^r_{kji} \rho_r . \tag{3.14}$$

4. AN ANALYTIC HP-TRANSFORMATION WHICH LEAVES INVARIANT THE COVARIANT DERIVATIVE OF THE HP-CURVATURE TENSOR

In this section, we shall show an analogous theorem to the one obtained by T. Sumitomo for an infinitesimal projective transformation in a Riemannian space Yano and Nagano (1957), Sumitomo (1959).

Let V^i be an analytic HP-transformation. If we substitute (3.1) and (3.11) into the identify

$$£_V \nabla_l P_{kji}^h - \nabla_l £_V P_{kji}^h = P_{kji}^r\, £_V \{ {}_l{}^h{}_r \} - P_{kri}^h\, £_V \{ {}_l{}^r{}_j \} - P_{kjr}^h\, £_V \{ {}_l{}^r{}_i \} \ ,$$

Then we obtain

$$£_V \nabla_l P_{kji}^h = T_{lkji}^h \quad ,$$

Where we have put

$$T_{lkji}^h = \delta_l^h P_{kji}^r \rho_r - 2\rho_l P_{kji}^h - \rho_k P_{lji}^h - \rho_j P_{kli}^h - \rho_i P_{kjl}^h - F_l^h\ P_{kji}^r \bar{\rho}_r + F_l^r (\bar{\rho}_k P_{rji}^h + \bar{\rho}_j\ P_{kri}^h + \bar{\rho}_i P_{kjr}^h)\ .$$

Now we shall assume that $£_V \nabla_l P_{kji}^h = 0$. Then we have

$$T_{lkji}^h = 0. \tag{4.1}$$

Contracting this equation with respect to h and l, we can verify

$$P_{kji}^r\ \rho_r = 0\ ,$$

By virtue of (1.13) ~ (1.17).

Substituting the last equation into (4.1) and taking account of $P_{kji}^r \bar{\rho}_r = 0$, we obtain the equation

$$2\rho_t P_{kji}^h + \rho_k P_{lji}^h + \rho_j P_{kli}^h + \rho_i P_{kjl} = F_l^r (\bar{\rho}_k P_{rji}^h + \bar{\rho}_l P_{kri}^h + \bar{\rho}_i\ P_{kjr}^h)\ .$$

Transvecting this equation with

$$\rho^l P^{kji} = \rho^l P_{rsth}\, g^{rk} g^{sj} g^{ti}$$

And taking account of (1.13) ~(1.17), we obtain

$$(\rho_l P_{kji}^h)(\rho^l P_h^{kji}) + 2(\rho^l P_{ljih})(\rho_r P^{rjih}) + (\rho^l P_{kjrl})(\rho_t P^{kjtr}) = 0\ ,$$

After some complicated calculation.

Since the each term in the left hand side of the last equation is non-negative, it must hold $\rho_l P_{kji}^h = 0$, from which we get the following:

Theorem 4.1: If a Tachibana space admits and analytic non-affine HP-transformation which leaves invariant the covariant derivative of the curvature tensor, then the space is a space of constant holomorphic curvature.

In a symmetric Tachibana space, i.e., in a Tachibana space satisfying $\nabla_l P_{kji}^h = 0$, the equation $£_V \nabla_l P_{kji}^h = 0$ trivially holds, so we have

Corollary 4.1: If a symmetric Tachibana space admits an analytic non-affine HP-transformation, then the space is a space of constant holomorphic curvature.

5. AN ANALYTIC HP-TRANSFORMATION IN A TACHIBANA SPACE SATISFYING $\nabla_k R_{ji} = 0$.

In this section, we shall obtain a theorem on an analytic HP-transformation in a Tachibana space satisfying $\boldsymbol{\nabla_k R_{ji} = 0}$.

The method used here is analogous to the one use by Sumitomo (1959) for an infinitesimal projective transformation in a Riemannian space.

At the first place, we have a well known (Sumitomo 1959).

Lemma 5.1: A necessary and sufficient condition for a Riemannian space to be an Einstein one is that the following equation holds:

$$R_{ji}R^{ji} = \frac{R^2}{n}.$$

This follows from the identify

$$Z_{ji}Z^{ji} = R_{ji}R^{ji} - \frac{R^2}{n}$$

Where

$$Z_{ji} = R_{ji} - \left({}^{R}/_{n}\right)g_{ji}.$$

Now consider a Tachibana space such that $\nabla_k R_{ji} = 0$ and let V^i be an analytic HP-transformation. Then, from (3.12) we have

$$(n+2)\nabla_k\nabla_j\,\rho_i = -R_{ki}\,\rho_j - R_{kj}\,\rho_i + S_{ki}\bar{\rho}_j + S_{kj}\bar{\rho}_i - 2R_{ji}\,\rho_k\,. \tag{5.1}$$

Transvecting this equation with g^{kj}, we get

$$\nabla^r\nabla_r\,\rho_i = -\frac{1}{n+2}\left(2R_i^r\rho_r + R\,\rho_i\right) \tag{5.2}$$

On the other hand, since ρ^i is analytic, we have

$$\nabla^r\nabla_r\,\rho_i + R_i^r\,\rho_r = 0.$$

Comparing the last two equations, we find

$$R_i^r\rho_r = \frac{R}{n}\rho_i \tag{5.3}$$

Which shows that ρ^i is a Ricci's direction. Thus it follows:

$$R^{tr}\nabla_t\rho_r = \frac{R}{n}\nabla_t\,\rho^t. \tag{5.4}$$

Lemma 5.2: If a Tachibana space satisfying $\nabla_k R_{ji}$ =0 is not an Einstein space , then the associated vector ρ^t of an analytic HP-transformation satisfies

$$\nabla_i\rho^i = 0.$$

Proof: By applying the Ricci's identity to R_{ji} , we find

$$R^r_{lkj}R_{ri} + R^r_{lki}R_{jr} = 0. \tag{5.5}$$

Transvecting this with g^{ki} , we have

$$R_{ltrj}R^{tr} = R^r_l R_{jr}\,. \tag{5.6}$$

From (5.5) it follows

$$\left(\pounds_V R^r_{lkj}\right)R_{ri} + R^r_{lkj}\pounds_V\,R_{ri} + (\pounds_V R^r_{lki})R_{jr} + R^r_{lki}\pounds_V\,R_{jr} = 0\,.$$

If we transvect this equation with $R^{jk}g^{il}$, then we get

$$\left(R^k_h R^{ij} + R_{th}R^{th}g^{ik}\right)\pounds_V\,R^h_{kji} = 0$$

By virtue of (5.6).

Now let V^i be an analytic HP-transformation. If we substitute (3.10) into the last equation, then we can verify

$$(\nabla_r \rho^r) R_{ji} R^{ji} - R R_{ji} \nabla^j \rho^i = 0, \tag{5.7}$$

after some calculation.

From (5.4) and (5.7) it follows

$$\left(R_{ji} R^{ji} - \frac{R^2}{n}\right) \nabla_r \rho^r = 0,$$

Which implies together with Lemma (5.1) the lemma.

Theorem 5.3: If a Tachibana space satisfying $\nabla_k R_{ji} = 0$ admits an analytic non-affine HP-transformation, it is a Tachibana Einstein one.

Proof: Since $\nabla_k R_{ji} = 0$, $R_{ji}\ R^{ji}$ is constant, and also we have

$$\begin{aligned} 0 = £_V\left(R_{ji} R^{ji}\right) &= \left(£_V R_{ji}\right) R^{ji} + R_{ji} £_V (R_{rt}\ g^{rj} g^{ti}) \\ &= 2\left[\left(£_V R_{ji}\right) R^{ji} + R^r_j R_{rt}\ £_V\ g^{jt}\right], \end{aligned}$$

Where V^i is a vector field.

Now let V^i be an analytic non-affine HP-transformation. If we substitute (3.8) and (5.4) into the last equation, then we find

$$-\frac{n+2}{n} R\, \nabla_r\ \rho^r + R^r_j\ R_{rt}\ £_V\ g^{jt} = 0.$$

If we assume that our space is not an Einstein one, then we have

$$R^r_j\ R_{rt}\ £_V\ g^{jt} = 0, \tag{5.8}$$

By virtue of Lemma (5.2). By means of $£_V\left(g^{jt}\ g_{jt}\right) = 0$, (5.8) can be written in the Form:

$$R^{jr}\ R^t_r\ £_V\ g_{jt} = 0,$$

Operating ∇_i to the both sides and then substituting (3.3), we get

$$\tfrac{1}{2}(\rho_i \rho^i),\ \left(R_{jr} R^{jr}\right) + \left(R_{jr}\ \rho^j\right)(R^r_i\ \ \rho^i) = 0$$

Since the each term of the left hand side is non-negative, we have $\rho_i R_{jr} = 0$, which contradicts to our assumption.

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