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**ABSTRACT**

In this paper we prove that the set of all minimal dominating sets which induced from the neighbourhoods $N(v)$ of Clepsch graph with parameters $(16,5,0,2)$ are blocks of partial balanced incomplete block designs, and we generalize this result for minimal dominating sets $N(v)$ for all strongly regular graphs without triangles.

**Keywords:** Clebsch graph, Strongly regular graph without triangles, Minimal dominating set, Partial balanced incomplete block design.

Mathematics Subject Classification (2000):05C15.

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1. **INTRODUCTION:**

The strongly regular graphs and its relation with partial balanced incomplete block designs (PBIBD's) were studied in [1], [2] and [3] it was shown that the strongly regular graphs are emerged from PBIBD with two association schemes. In this paper, we consider different way to establish a link between PBIBD and strongly regular graph through the number of some minimal dominating sets, in the folded 5-cube $(16,5,0,2)$ graph and we generalize this result to all SRNT graphs.

We refer to [4], [6] and [8] for the necessary background about strongly regular graphs, dominating set, and PBIBD's.

2. **DEFINITIONS AND NOTATIONS:**

**Definition: 2.1**

A strongly regular graph with no triangles (SRNT graph) $G$ with the parameters $(n,k,0,\mu)$ is $k$ - regular graph with $n$ vertices such that for any two adjacent vertices have no common neighbours, and any two non-adjacent vertices have $\mu$ common neighbours.

**Definition: 2.2**

Given $v$ objects a relation satisfying the following conditions is said to be an association scheme with $m$ classes:

(i) any two objects are either first associates, second associates, ..., $m^{th}$ associates, the relation of association being symmetric.

(ii) each object has $n_i$ $i^{th}$ associates ($i=1,2$).

(iii) if two objects $\alpha$ and $\beta$ are $i^{th}$ associates, then the number of objects common to the $j^{th}$ associates of the first and $k^{th}$ associates of the second is $p_{jk}^i$ and is independent of the pair of $i^{th}$ associates $\alpha$ and $\beta$. Also $p_{jk}^i = p_{kj}^i$.

If we have association scheme for the $v$ objects we can define a PBIBD as the following definition.

**Definition: 2.3**

The PBIB design is arrangement of $v$ objects into $b$ sets of size $k$ where $k < v$ such that:

(i) every object is contained in exactly $r$ blocks.

(ii) each block contains $k$ distinct objects.

(iii) any two objects which are $i^{th}$ associates occur together in exactly $\lambda_i$ blocks.

The numbers $v, b, r, k, \lambda_i (i = 1, 2, ..., m)$ are called the parameters of PBIBD with $m$ association.

**Definition: 2.4**

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A subset $D$ of $V(G)$ is said to be dominating set if every vertex not in $D$ is adjacent to at least one vertex in $D$. The domination number $\gamma(G)$ of a graph $G$ is defined to be the minimum cardinalities taken over all dominating sets of $G$, and the set $D$ in $G$ is said to be minimum dominating set if $|D| = \gamma(G)$. 

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Definition: 2.5
A dominating set $D$ is called minimal dominating set of a graph $G$ if no proper subset of $D$ is dominating set.

3. RESULTS:
Proposition: 3.1
The minimal dominating sets which induced from the neighbourhood of vertices of Clebsch graph in Figure 1 are blocks of PBIB design.

Proof: Each vertex of the Clebsch graph belongs to exactly 5 minimal dominating sets $N(v)$, and the list of minimal dominating sets given below:

$$
\begin{align*}
\{2, 5, 8, 10, 12\}, & \{1, 3, 6, 9, 13\} \\
\{2, 4, 7, 9, 11\}, & \{3, 5, 6, 8, 15\} \\
\{2, 4, 7, 9, 11\}, & \{2, 4, 11, 12, 16\} \\
\{3, 5, 12, 13, 16\}, & \{1, 4, 13, 14, 16\} \\
\{2, 5, 14, 15, 16\}, & \{1, 3, 11, 15, 16\} \\
\{5, 6, 10, 13, 14\}, & \{1, 6, 7, 14, 15\} \\
\{2, 7, 8, 11, 15\}, & \{3, 8, 9, 11, 12\} \\
\{4, 9, 10, 12, 13\}, & \{6, 7, 8, 9, 10\}
\end{align*}
$$

By consider the minimal dominating sets above as blocks, and the two association scheme can be defined as the two elements $\alpha$ and $\beta$ are 1st associates if they are adjacent vertices in $G$ and they are second associates otherwise. Thus we have

<table>
<thead>
<tr>
<th>Second Associates</th>
<th>First Associates</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>3,4,6,7,9,11,13,14,15,16</td>
<td>2,8,12,10,5</td>
<td>1</td>
</tr>
<tr>
<td>4,5,7,8,10,11,12,14,15,16</td>
<td>1,3,6,9,13</td>
<td>2</td>
</tr>
<tr>
<td>1,5,6,8,9,11,12,13,15,16</td>
<td>2,4,7,10,14</td>
<td>3</td>
</tr>
<tr>
<td>1,2,7,9,10,11,12,13,14,16</td>
<td>3,5,6,8,15</td>
<td>4</td>
</tr>
</tbody>
</table>

From the above association it is easy to verify that the minimal dominating sets of Clebsch graph which induced from the neighbourhood of the vertices forms PBIBWD with the parameters $v = 16, b = 5, r = 5, k = 5, \lambda_1 = 0, \lambda_2 = 2$.

$$
\begin{align*}
P_1 & = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}, \\
P_2 & = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}.
\end{align*}
$$

Thus the sets of minimal dominating sets $N(v)$ of Clebsch graph are the blocks of PBIB design with the above parameters.

Theorem: 3.2
Let $G$ be SRNT graph with the parameters $(n,k,0,\mu)$ and let $v$ be any vertex of $G$ then $N(v)$ is minimal dominating set of $G$.

Proof: Let $u$ be any vertex in $V(G) - N(v)$ Then we have two cases, either $u = v$ or $u$ not adjacent to $v$.

Case 1. If $u = v$ then $u$ has $k$ adjacent vertices in $N(v)$.

Case 2. If $u$ not adjacent to $v$, then from the definition of SRNT graph $u$ adjacent to $\mu$ vertices in $N(v)$ Thus $N(v)$ is dominating set. Now we want to prove that $N(v)$ is minimal dominating. Suppose $N(v)$ is not dominating set of $G$ then there is vertex $w \in N(v)$ such that
$N(v) - w$ is dominating set, hence $w$ adjacent to at least one vertex in $N(v)$, and this is contradiction with the definition of SRNT graph. Therefore $N(v)$ is minimal dominating set of $G$.

**Theorem: 3.3**
The set of minimal dominating set $N(v)$ in SRNT graph $G$ with the parameters $(n,k,0,\mu)$, where $v$ any vertex of $G$ are blocks of PBIB design.

**Proof:** We can defined the PBIB design as following:

**Point set** is the vertices of $G$, and the **block set** is the set of minimal dominating sets $N(v)$, where $v$ any vertex in $G$, and for any points $\alpha$ and $\beta$ are $1^{st}$ associates if they are adjacent in $G$ and $2^{nd}$ associates otherwise and it is clear that each vertex of the SRNT graph belong to exactly $k$ minimal dominating sets, so the parameters of PBIB design are $v, r, k, \lambda_1, \lambda_2$ where $v = b = n$, $r = k' = k$, $\lambda_1 = 0$, $\lambda_2 = \mu$.

**REFERENCES:**


