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ON EXPONENTIATED LOMAX DISTRIBUTION

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ABSTRACT

In this paper, we generalize the Lomax distribution by powering a positive real number α to the cumulative distribution function (CDF). This new family of distributions called exponentiated Lomax distribution (ELD). Some properties of this family will be discussed. The estimation of unknown parameters for ELD will be handled using maximum likelihood, Moments and L-moments methods.

Keywords: *Exponentiated distributions–Lomax distribution–Maximum Likelihood Estimation – L-Moments – Moments.*

1. INTRODUCTION

Exponentiated distributions can be obtained by three techniques:

- Powering a positive real number α to the cumulative distribution function (CDF), i.e, if we have CDF $F(x)$ of any random variable *X*, then the function

$G(z)=\left[F(z)\right]^{\alpha}, \alpha>0,$

is called an exponentiated distribution see [6], [7], [12] and [15].

- Using the formula $G(z)=1-\left[1-F(z)\right]^{\alpha}, \alpha>0$ see [14], or
- Using the transformation $z=\log(x)$, where *X* is non-negative random variables see [13].

For the same distribution, we can use more than one technique from previous techniques see [13] and [15].

A random variable X has Lomax distribution with two parameters θ and λ if PDF is:

$$f(x)=\theta\lambda\left[1+\lambda x\right]^{-(\theta+1)};\ x>0,\ \theta \text{ and } \lambda>0, \tag{1}$$

Then the CDF of $f(x)$ in (1) is given by:

$$F(x)=1-\left[1+\lambda x\right]^{-\theta};\ x>0,\ \theta \text{ and } \lambda>0, \tag{2}$$

In other hand, the method of L-moment estimators have been recently appeared in Hosking [8]. He used L-moment estimators for estimating the unknown parameters of log-normal, gamma, and generalized extreme value distribution.

Next, several estimations had been proposed to follow the work carried by Hosking [8]. L-moment estimators for generalized Rayleigh distribution was introduced by Kundu and Raqab [10]. Karvanen [9] applied the method of L-moment estimators to estimate the parameters of polynomial quintile mixture. He introduced two parametric families, the normal-polynomial quintile mixture and Cauchy-polynomial quintile mixture. Abdul-Moniem [1,2,3] applied the method of L-moment estimators to estimate the parameters of exponential distribution, Rayleigh distribution and Wiebull distribution, respectively also the estimate of unknown parameters for generalized Pareto distribution are discussed by Abdul-Moniem and Selim [4]. The standard method to compute the L-moment estimators is to equate the sample L-moments with the corresponding population L-moments.

In this paper, we depend on the first technique to obtain type I exponentiated Lomax distribution (ELD). Some properties of this family are presented. Estimation of unknown parameters ELD will be handled using Maximum Likelihood Estimators (MLE), method of moment estimators (MME) and L-Moments method (LMM).

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The paper is organized as follows. In Section 2, ELD is discussed. MLE is presented in Section 3. In Section 4, we handled MME. LMM will be discussed in Section 5. Numerical analyses obtained in Section 6. Section 7 presents our conclusion.

2. EXPONENTIATED LOMAX DISTRIBUTION

Using the first technique and equation (2), we can define the cumulative distribution function (CDF) of exponentiated Lomax distribution (ELD) as follows

$$F(x)=\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha};\ x>0,\ \alpha,\theta \text{ and } \lambda>0. \tag{3}$$

The PDF of ELD is

$$f(x)=\alpha\theta\lambda\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)};\ x>0,\ \alpha,\theta \text{ and } \lambda>0. \tag{4}$$

We can get the PDF for exponentiated Pareto, Pareto, and Lomax distributions by taking $\lambda=1$, $\lambda=\alpha=1$ and $\alpha=1$ respectively.

The survival (reliability) function $S(x)$, the hazard rate function (HRF) $h_\alpha(x)$ and the reversed hazard rate function (RHRF) $h_\alpha^*(x)$ for ELD are in the following forms:

$$S(x)=1-\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha};\ x>0,\ \alpha,\theta \text{ and } \lambda>0, \tag{5}$$

$$h_\alpha(x)=\frac{\alpha\theta\lambda\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)}}{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha}};\ x>0,\ \alpha,\theta \text{ and } \lambda>0, \tag{6}$$

and

$$h_\alpha^*(x)=\frac{\alpha\theta\lambda(1+\lambda x)^{-(\theta+1)}}{\left[1-(1+\lambda x)^{-\theta}\right]}=\alpha h^*(x);\ x>0,\ \alpha,\theta \text{ and } \lambda>0, \tag{7}$$

Where $h^*(x)$ is the RHRF for the Lomax distribution.

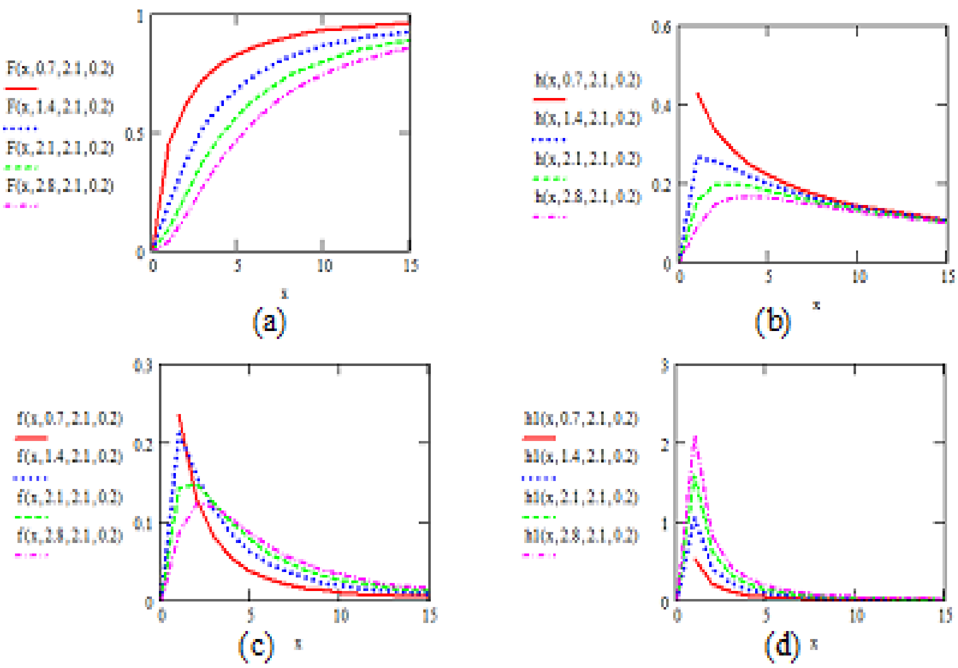

Figure 1: (a) the ELD CDF. (b) the ELD HRF. (c) the ELD PDF. (d) the ELD RHRF.
For $\theta=2.1,\ \lambda=0.2$ when $\alpha=0.7,\ 1.4,\ 2.1,\ 2.8.$.

3. MAXIMUM LIKELIHOOD ESTIMATORS (MLE)

In this section, we consider maximum likelihood estimators (MLE) of ELD. Let $x_1, x_2, ..., x_n$ be a random sample of size n from ELD, then the log-likelihood function $L(\lambda, \theta, \alpha)$ can be written as

$$L(\lambda,\theta,\alpha) \propto n\left[\ln(\lambda)+\ln(\theta)+\ln(\alpha)\right]+(\alpha-1)\sum_{i=1}^{n}\ln\left[1-(1+\lambda x_i)^{-\theta}\right]-(\theta+1)\sum_{i=1}^{n}\ln(1+\lambda x_i) \tag{8}$$

The normal equations become

$$\frac{\partial L}{\partial \lambda}=\frac{n}{\lambda}+\theta(\alpha-1)\sum_{i=1}^{n}\frac{x_i(1+\lambda x_i)^{-(\theta+1)}}{\left[1-(1+\lambda x_i)^{-\theta}\right]}-(\theta+1)\sum_{i=1}^{n}\frac{x_i}{(1+\lambda x_i)} \tag{9}$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}-(\alpha-1)\sum_{i=1}^{n}\frac{(1+\lambda x_i)^{-\theta}\ln(1+\lambda x_i)}{\left[1-(1+\lambda x_i)^{-\theta}\right]}-\sum_{i=1}^{n}(1+\lambda x_i) \tag{10}$$

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left[1-(1+\lambda x_i)^{-\theta}\right] \tag{11}$$

The MLE of λ, θ and α can be obtain by solving the equations (9), (10), and (11) using $\frac{\partial L}{\partial \lambda}=0$, $\frac{\partial L}{\partial \theta}=0$ and $\frac{\partial L}{\partial \alpha}=0$.

4. TRADITIONAL MOMENTS FOR ELD

The r^{th} traditional moments for ELD is

$$\begin{aligned}\mu_r' = E(X^r) &= \alpha\theta\lambda\int_0^\infty x^r(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}dx\\ &=\frac{\alpha}{\lambda^r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i B\left(1-\frac{1}{\theta}(r-i),\alpha\right),\end{aligned} \tag{12}$$

The first three moments can be obtain by taking r=1, 2 and 3 in (12) as follows:

$$\begin{aligned}\mu_1' &= \frac{\alpha}{\lambda}\left[B\left(1-\frac{1}{\theta},\alpha\right)-B(1,\alpha)\right]\\ &=\frac{\alpha}{\lambda}\left[B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}\right]\end{aligned} \tag{13}$$

$$\begin{aligned}\mu_2' &= \frac{\alpha}{\lambda^2}\left[B\left(1-\frac{2}{\theta},\alpha\right)-2B\left(1-\frac{1}{\theta},\alpha\right)+B(1,\alpha)\right]\\ &=\frac{\alpha}{\lambda^2}\left[B\left(1-\frac{2}{\theta},\alpha\right)-2B\left(1-\frac{1}{\theta},\alpha\right)+\frac{1}{\alpha}\right],\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mu_3' &= \frac{\alpha}{\lambda^3}\left[B\left(1-\frac{3}{\theta},\alpha\right)-3B\left(1-\frac{2}{\theta},\alpha\right)+3B\left(1-\frac{1}{\theta},\alpha\right)-B(1,\alpha)\right]\\ &=\frac{\alpha}{\lambda^3}\left[B\left(1-\frac{3}{\theta},\alpha\right)-3B\left(1-\frac{2}{\theta},\alpha\right)+3B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}\right].\end{aligned} \tag{15}$$

The variance and coefficient of variation (CV) for ELD are

$$\text{Var (X)} = \left(\frac{\alpha}{\lambda}\right)^2 \left[\frac{1}{\alpha} B\left(1-\frac{2}{\theta},\alpha\right) - B^2\left(1-\frac{1}{\theta},\alpha\right)\right], \tag{16}$$

and

$$\text{CV} = \frac{\sqrt{\frac{1}{\alpha} B\left(1-\frac{2}{\theta},\alpha\right) - B^2\left(1-\frac{1}{\theta},\alpha\right)}}{B\left(1-\frac{1}{\theta},\alpha\right) - \frac{1}{\alpha}}. \tag{17}$$

We can obtain the estimator of $\lambda,\ \theta$ and α by using the following equations:

$$\mu'_r = \frac{\sum_{i=0}^{n} x_i^r}{n}, \quad r = 1,2,3$$

i.e

$$\frac{\hat{\alpha}}{\hat{\lambda}}\left[B\left(1-\frac{1}{\hat{\theta}},\hat{\alpha}\right) - \frac{1}{\hat{\alpha}}\right] = \bar{x}, \tag{18}$$

$$\frac{\hat{\alpha}}{\hat{\lambda}^2}\left[B\left(1-\frac{2}{\hat{\theta}},\hat{\alpha}\right) - 2B\left(1-\frac{1}{\hat{\theta}},\hat{\alpha}\right) + \frac{1}{\hat{\alpha}}\right] = \frac{\sum_{i=1}^{n} x_i^2}{n}, \tag{19}$$

and

$$\frac{\hat{\alpha}}{\hat{\lambda}^3}\left[B\left(1-\frac{3}{\hat{\theta}},\hat{\alpha}\right) - 3B\left(1-\frac{2}{\hat{\theta}},\hat{\alpha}\right) + 3B\left(1-\frac{1}{\hat{\theta}},\hat{\alpha}\right) - \frac{1}{\hat{\alpha}}\right] = \frac{\sum_{i=1}^{n} x_i^3}{n}. \tag{20}$$

We can obtain $\hat{\lambda},\ \hat{\theta}$ and $\hat{\alpha}$ by solving nonlinear equations (18), (19) and (20) numerically.

5. L-moments for LED

A population L-moment (L_r) is defined as a certain linear function of the expectations of the order statistics $Y_{1:r}$, $Y_{2:r}$, ..., $Y_{r:r}$ in a conceptual random sample of size *r* from the underlying population. For example, $L_1 = E\left(Y_{1:1}\right)$, which is the same as the population mean. L_1 is defined in terms of a conceptual sample of size $r = 1$, while $L_2 = \frac{1}{2} E\left(Y_{2:2} - Y_{1:2}\right)$, an alternative to the population standard deviation. Also L_2 is defined in terms of a conceptual sample of size $r = 2$. Similarly, the L-moments L_3 and L_4 are alternatives to the un-scaled measures of skewness and kurtosis μ_3 and μ_4 respectively. Compared to the conventional moments, L-moments have lower sample variances and are more robust against outliers [16].

In this section, we will discuss the population L-moment of order *r* for the *ELD*. Also we study the sample L-moments and L-moments estimators.

5.1 Population L-moments

The formula of population L-moments of order *r* is given by Hosking [8] as follows:

$$L_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(X_{r-k:r}\right), \qquad r = 1,2,... \tag{21}$$

where $E\left(X_{r-k:r}\right)$ is the expectation of $\left(r-k\right)^{th}$ order statistics from sample of size r and we obtain it as follows:

$$E\left(X_{r-k:r}\right)=\frac{r!}{(r-k-1)!k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{i}\,\alpha\lambda\theta\int_{0}^{\infty}x\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha(r-k+i)-1}(1+\lambda x)^{-(\theta+1)}dx$$

$$=\frac{r!}{(r-k-1)!k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{i}\frac{\alpha}{\lambda}\left\{B\left(1-\frac{1}{\theta},\alpha(r-k+i)\right)-\frac{1}{\alpha(r-k+i)}\right\} \tag{22}$$

Substituting from (22) in (21), the population L-moments of order *r* for the *ELD* is given by:

$$L_r=\sum_{k=0}^{r-1}\frac{(-1)^k\binom{r-1}{k}(r-1)!}{(r-k-1)!(k)!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{i}$$
$$\times\frac{\alpha}{\lambda}\left\{B\left(1-\frac{1}{\theta},\alpha(r-k+i)\right)-\frac{1}{\alpha(r-k+i)}\right\}. \tag{23}$$

The first two L-moments can be obtained by taking $r=1,\ 2,\ 3$ in (23) as follows

$$L_1=\frac{\alpha}{\lambda}\left\{B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}\right\}, \tag{24}$$

$$L_2=\frac{\alpha}{\lambda}\left\{2B\left(1-\frac{1}{\theta},2\alpha\right)-B\left(1-\frac{1}{\theta},\alpha\right)\right\}, \tag{25}$$

and

$$L_3=\frac{\alpha}{\lambda}\left\{6B\left(1-\frac{1}{\theta},3\alpha\right)-6B\left(1-\frac{1}{\theta},2\alpha\right)+B\left(1-\frac{1}{\theta},\alpha\right)\right\}. \tag{26}$$

The L-CV for ELD is

$$L-CV=\frac{2B\left(1-\frac{1}{\theta},2\alpha\right)-B\left(1-\frac{1}{\theta},\alpha\right)}{B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}}., \tag{27}$$

5.2 Sample L-moments and L-moments estimators

L-moments can be estimated from the following formula Elamir and Seheult [5]:

$$l_r=\frac{1}{r}\sum_{i=1}^{n}\frac{\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}}{\binom{n}{r}}x_{i:n},\qquad r=1,2,3,\ldots \tag{28}$$

L-moment estimator for $\lambda,\ \theta$ and α can be found as follows:

$$L_r=l_r,\quad r=1,2,3.$$

i.e

$$\frac{\alpha}{\lambda}\left\{B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}\right\}=\bar{x}\ , \tag{29}$$

$$\frac{\alpha}{\lambda}\left\{2B\left(1-\frac{1}{\theta},2\alpha\right)-B\left(1-\frac{1}{\theta},\alpha\right)\right\}=\frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{i:n}-\bar{x}\ , \tag{30}$$

and

$$\frac{\alpha}{\lambda}\left\{6B\left(1-\frac{1}{\theta},3\alpha\right)-6B\left(1-\frac{1}{\theta},2\alpha\right)+B\left(1-\frac{1}{\theta},\alpha\right)\right\}=$$
$$\sum_{i=1}^{n}\frac{(i-1)(i-2)-4(i-1)(n-i)+(n-i)(n-i-1)}{n(n-1)(n-2)}x_{i:n}\ . \tag{31}$$

6. NUMERICAL ILLUSTRATION

Using generating samples of sizes 10 (10)50 with 1000 replications, we estimate the mean square error (MSE) of the unknown parameters θ and λ at $\alpha = 0.5,\ 1.5,$ and 2. We have used the methods: MLE (solving equations (9) and (10) by taking $\frac{\partial L}{\partial \lambda}=0$ and $\frac{\partial L}{\partial \theta}=0$), MME (solving equations (18) and (19)), LME (solving equations (29) and (30)).

The results of MSE are summarized in the following table:

Table (1): The values of MSE for different method of estimations.

Methods N		MLE		MME		LME	
		θ	λ	θ	λ	θ	λ
10	$\alpha=0.5$	2.92109	0.92192	59.7517	0.00288	1.14558	0.03921
	$\alpha=1.5$	2.12180	0.01668	47.6173	0.00246	1.70763	0.00494
	$\alpha=2$	2.22728	0.01022	42.5264	0.00235	2.05637	0.00363
20	$\alpha=0.5$	2.15944	0.04745	38.6115	0.00218	2.9296	0.00795
	$\alpha=1.5$	1.76953	0.00314	29.1128	0.00217	3.31781	0.00155
	$\alpha=2$	1.44735	0.00361	24.154	0.00206	3.32662	0.00182
30	$\alpha=0.5$	2.35535	0.00968	33.5892	0.00288	3.71653	0.00302
	$\alpha=1.5$	1.74968	0.00217	22.7259	0.00196	2.97624	0.00137
	$\alpha=2$	1.62217	0.00173	19.1598	0.00187	3.63701	0.00122
40	$\alpha=0.5$	2.50962	0.00707	27.8639	0.00195	3.74199	0.00229
	$\alpha=1.5$	1.40814	0.00147	15.7202	0.00177	2.75139	0.00112
	$\alpha=2$	1.16955	0.00149	10.7886	0.00172	2.19911	0.00124
50	$\alpha=0.5$	1.99372	0.00314	23.528	0.00191	4.22286	0.00171
	$\alpha=1.5$	1.20498	0.00133	9.38163	0.00164	2.23101	0.00112
	$\alpha=2$	0.91023	0.00113	7.05986	0.00156	1.8053	0.00102

From the table, we can see:

1. The values of MSE decrease as the sample size n increases.
2. For the scale parameter λ, we can note that the LME is the best method.
3. For the large samples, MLE is better than MME.

7. CONCLUSIONS

In this paper we have proposed a new family of distributions called exponentiated Lomax distribution (ELD). We get the probability density functions for exponentiated Pareto, Pareto, and Lomax distributions as a special cases from ELD. We have derived some interesting properties of this family. Estimation of unknown parameters ELD will be handled using Maximum Likelihood Estimators (MLE), method of moment estimators (MME) and L-Moments method (LMM). More work is needed in these direction.

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