International Journal of Mathematical Archive-3(5), 2012, 2157-2162
IJMA Available online through www.ijma.info ISSN 2229 – 5046

FIXED POINT THEOREMS FOR COMPATIBLE MAPPING SATISFYING A CONTRACTIVE CONDITION OF INTEGRAL TYPE

Ramakant Bhardwaj*
Department of Mathematics, Truba Instt. Of Engineering & Information Technology Bhopal, India

(Received on: 03-01-12; Accepted on: 26-01-12)

ABSTRACT

In this paper we prove some fixed point theorems for compatible mapping satisfying a contractive condition of integral type in complete metric spaces. Our results are version of some known results.

Key words: *Fixed point, Common Fixed point, complete metric space, Continuous Mapping, Compatible Mappings.*

AMS Subject Classification: *47H10, 54H25.*

INTRODUCTION

In 1976, Jungck [5] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states that, "let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point in X". This result was further generalized and extended in various ways by many authors. On the other hand Sessa [13] defined weak commutativity as follows: The mappings f and g are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all $x \in X$. further, Jungck [6] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. Let f and g be self mappings of a metric space(X, d). The mapping f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Clearly commuting, weakly commuting mappings are compatible but neither implication is reversible .Many authors have obtained a lot of fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of mappings.

In 2002, A. Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X, d) satisfying a general contractive condition of integral type.

Theorem 1.1: (Branciari[1])

Let (X, d) be a complete metric space, $c \in (0,1)$ and let $f: X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \xi(t)\, dt \leq c \int_0^{d(x,y)} \xi(t)\, dt \tag{1.2}$$

Where $\xi: [0, +\infty) \to [0, +\infty)$ is a Lesbesgue integrable mapping which is summable on each compact subset of $[0, +\infty)$, non negative, and such that for each $\epsilon > o$, $\int_0^{\epsilon} \xi(t)\, dt$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

After the paper of Branciari, a lot of a research works have been carried out on generalizing contractive conditions of integral type for a different contractive mapping satisfying various known properties. A fine work has been done by Rhoades [9] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(fx,fy)} \xi(t)\, dt \leq \int_0^{\max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}} \xi(t)\, dt \tag{1.3}$$

MAIN RESULTS

The purpose of this paper is to prove fixed point theorems by using rational contraction, Rhoades fixed point theorem [9], and Branciari result [1] to compatible maps.

***Corresponding author: Ramakant Bhardwaj*,**
Department of Mathematics, Truba Instt. Of Engineering & Information Technology Bhopal, India

Theorem 2.1: Let f and g be compatible self maps of a complete metric space (X,d) satisfying the following conditions:

$f(X) \subset g(X)$, g is continuous,

$$\int_0^{d(fx,fy)} \xi(t)\ dt \le a \int_0^{\max\left\{d(gx,gy),d(gx,fx),d(gy,fy),\frac{d(gx,fy)+d(gy,fx)}{2}\right\}} \xi(t)\ dt + b\int_0^{\frac{[d(gx,fx).d(gy,fy)]}{d(gx,gy)}} \xi(t)\ dt$$
$$+ c\int_0^{\frac{[d(gx,fy)d(gy,fx)]}{d(gx,gy)}} \xi(t)\ dt + d\int_0^{d(gx,gy)} \xi(t)\ dt \tag{2.1}$$

For each $x, y \in X$ with non negative reals $\alpha, \beta, \gamma, \delta$ such that $0 < a+b+c+d < 1$, where $\xi : \mathcal{R}^+ \to \mathcal{R}^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of $\mathcal{R}^+$, non negative, and such that

$$\text{for each } \epsilon > 0, \int_0^{\epsilon} \xi(t)\ dt \tag{2.2}$$

Then f and g have a unique common fixed point $\in X$.

Proof: Let $x_0 \in X$. since $f(X) \subset g(X)$, choose $x_1 \in X$ such that $gx_1 = fx_0$. In general, we construct a sequence x_{n+1} of element of X such that $y_n = gx_{n+1} = fx_n$ for $n = 0,1,2,3,\dots..$

For each integer $n \ge 1$ from 2.1

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\ dt = \int_0^{d(fx_n,fx_{n+1})} \xi(t)\ dt$$

$$\int_0^{d(fx_n,fx_{n+1})} \xi(t)\ dt \le a \int_0^{\max\left\{d(gx_n,gx_{n+1}),d(gx_n,fx_n),d(gx_{n+1},fx_{n+1}),\frac{d(gx_n,fx_{n+1})+d(gx_{n+1},fx_n)}{2}\right\}} \xi(t)\ dt$$
$$+ b\int_0^{\frac{[d(gx_n fx_n).d(gx_{n+1},fx_{n+1})]}{d(gx_n,gx_{n+1})}} \xi(t)\ dt + c\int_0^{\frac{[d(gx_n,fx_{n+1})d(gx_{n+1},fx_n)]}{d(gx_n,gx_{n+1})}} \xi(t)\ dt + d\int_0^{d(gx_n,gx_{n+1})} \xi(t)\ dt$$

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\ dt \le a \int_0^{\max\left\{d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_n,y_n),\frac{d(y_{n-1},y_n)+d(y_n,y_n)}{2}\right\}} \xi(t)\ dt + b\int_0^{\frac{[d(y_{n-1},y_n).d(y_n,y_{n+1})]}{d(y_{n-1},y_n)}} \xi(t)\ dt$$
$$+ c\int_0^{\frac{[d(y_{n-1},y_{n+1})d(y_n,y_n)]}{d(y_{n-1},y_n)}} \xi(t)\ dt + d\int_0^{d(y_{n-1},y_n)} \xi(t)\ dt$$

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\ dt \le \left(\frac{a+d}{1-b}\right) \int_0^{d(y_{n-1},y_n)} \xi(t)\ dt \tag{2.3}$$

In this way we can write,

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\ dt \le \left(\frac{a+d}{1-b}\right)^n \int_0^{d(y_0,y_1)} \xi(t)\ dt \tag{2.4}$$

let $q = \left(\frac{a+d}{1-b}\right) < 1$, and as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^{d(y_n,y_{n+1})} \xi(t)\ dt = 0 \tag{2.5}$$

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d\big(y_{m(p)}, y_{n(p)}\big) \ge \epsilon, \quad d\big(y_{m(p)}, y_{n(p)-1}\big) < \epsilon \tag{2.6}$$

Now

$$d\big(y_{m(p)-1}, y_{n(p)-1}\big) < d\big(y_{m(p)-1}, y_{m(p)}\big) + d\big(y_{m(p)}, y_{n(p)-1}\big)$$

$$d\big(y_{m(p)-1}, y_{n(p)-1}\big) < d\big(y_{m(p)-1}, y_{m(p)}\big) + \epsilon \tag{2.7}$$

Hence

$$lim_{p\to\infty} \int_0^{d(y_{m(p)-1},y_{n(p)-1})} \xi(t)\ dt = \int_0^{\epsilon} \xi(t)\ dt \tag{2.8}$$

Using (2.3), (2.6), and (2.8) $we\ get$

$$\int_0^{\epsilon} \xi(t)\ dt \leq \int_0^{d(y_{m(p)},y_{n(p)})} \xi(t)\ dt \leq q\int_0^{d(y_{m(p)-1},y_{n(p)-1})} \xi(t)\ dt \leq q\int_0^{\epsilon} \xi(t)\ dt$$

Which is contradiction, since $q \in (0,1)$. therefore $\{y_n\}$ is a Cauchy, hence converges to $z \in X\ from$ 2.1, $we\ get$

$$\int_0^{d(fz,fx_n)} \xi(t)\ dt \leq a\int_0^{max\left\{d(gz,gx_n),d(gz,fz),d(gx_n,fx_n),\frac{d(gz,fx_n)+d(gx_n,fz)}{2}\right\}} \xi(t)\ dt + b\int_0^{\frac{[d(gz,fz).d(gx_n,fx_n)]}{d(gz,gx_n)}} \xi(t)\ dt$$
$$+ c\int_0^{\frac{[d(gz,fx_n)d(gx_n,fz)]}{d(gz,gx_n)}} \xi(t)\ dt + d\int_0^{d(gz,gx_n)} \xi(t)\ dt$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(fz,z)} \xi(t)\ dt \leq a\int_0^{max\left\{d(gz,z),d(gz,z),d(z,z),\frac{d(gz,z)+d(z,fz)}{2}\right\}} \xi(t)\ dt + b\int_0^{\frac{[d(gz,fz).d(z,z)]}{d(gz,z)}} \xi(t)\ dt$$
$$+ c\int_0^{\frac{[d(gz,z)d(z,fz)]}{d(gz,z)}} \xi(t)\ dt + d\int_0^{d(gz,z)} \xi(t)\ dt$$

Which implies $fz = z\ and\ gz = z$.

Now we show that z is a common fixed point of $f\ and\ g$. since $f\ and\ g$ are compatible, therefore,

$lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, which since $lim_{n\to\infty} gfx_n = gz$ implies that $lim_{n\to\infty} fgx_n = gz$.

Now from 2.1,

$$\int_0^{d(fgx_n,gx_n)} \xi(t)\ dt \leq a\int_0^{max\left\{d(ggx_n,gx_n),d(ggx_n,fgx_n),d(gx_n,gx_n),\frac{d(ggx_n,gx_n)+d(gx_nfgx_n)}{2}\right\}} \xi(t)\ dt$$
$$+ b\int_0^{\frac{[d(ggx_n,fgx_n).d(gy,gx_n)]}{d(ggx_n,gx_n)}} \xi(t)\ dt + c\int_0^{\frac{[d(ggx_n,fy)d(gy,fgx_n)]}{d(ggx_n,gx_n)}} \xi(t)\ dt + d\int_0^{d(ggx_n,gx_n)} \xi(t)\ dt$$

Taking $lim\ as\ n \to \infty$ we obtain $z = gz$.

Again from 2.1, we can show that, $z = fz$ and hence $z\ is$ common fixed point of $f\ and\ g\ in\ X$.

UNIQUENESS

Let us w is another fixed point of $f\ and\ g$ in X different from z $i.e.\ z \neq w$, then from 2.1 we have,

$$\int_0^{d(fw,fz)} \xi(t)\ dt \leq a\int_0^{max\left\{d(gw,gz),d(gw,fw),d(gz,fz),\frac{d(gw,fz)+d(gz,fw)}{2}\right\}} \xi(t)\ dt + b\int_0^{\frac{[d(gw,fw).d(gz,fz)]}{d(gw,gz)}} \xi(t)\ dt$$
$$+ c\int_0^{\frac{[d(gw,fz)d(gz,fw)]}{d(gwgz)}} \xi(t)\ dt + d\int_0^{d(gw,gz)} \xi(t)\ dt$$

$$\int_0^{d(w,z)} \xi(t)\ dt \leq (a+d)\int_0^{d(w,z)} \xi(t)\ dt$$

Which contradiction,

So that, z is unique common fixed point of $f\ and\ g$.

REMARKS

i. Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting $\xi(t) = 1\ over\ \mathcal{R}^+$.

ii. On setting $\xi(t) = 1\ and\ a = b = c = 0$, in 2.1, then the result can be significantly improved Jungck's fixed point theorem [5] by employing compatible maps instead on commutativity of maps.

iii. On setting $\xi(t) = 1\ and\ b = c = d = 0,\ f = g$ in 2.1, then the result can be significantly improved Rhoades fixed point theorem [9].

iv. On setting $\xi(t) = 1$ and $a = c = d = 0,\ f = g$ in 2.1, then the result can be significantly improved Jaggi fixed point theorem.

We also extend and generalize the theorem of Branciari for a pair of compatible mappings. In a similar we can generalize order results related to contractive conditions of same kind.

We prove our next theorem by using rational contraction in integral type mapping. In fact our next result is as follows,

Theorem 2.2: Let f and g be compatible self maps of a complete metric space (X, d) satisfying the following conditions:

$f(X) \subset g(X),\ g\ is\ continuous,$

$$\int_0^{d(fx,fy)} \xi(t)\ dt \ \le a \int_0^{max\left\{\begin{matrix}\frac{[d(gx,fx).d(gy,fy)]}{d(gx,gy)}, \frac{[d(gx,fy)d(gy,fx)]}{d(gx,gy)}, \\ \frac{[d(gx,fx).d(gx,fy)]}{d(gx,gy)}, \frac{[d(gy,fx).d(gy,fy)]}{d(gx,gy)}, \\ d(gx,gy)\end{matrix}\right\}} \xi(t)\ dt \tag{2.9}$$

For each $x, y \in X$ with non negative reals $\alpha, \beta, \gamma, \delta$ such that $0 < a < 1$, where $\xi : \mathcal{R}^+ \to \mathcal{R}^+$ is a lesbesgue-integrable mapping which is summable on each compact subset of $\mathcal{R}^+$, non negative, and such that

$$for\ each\ \epsilon > 0, \int_0^{\epsilon} \xi(t)\ dt \tag{2.10}$$

Then f and g have a unique common fixed point $\in X$.

Proof: Let $x_0 \in X$. since $f(X) \subset g(X), choose\ x_1 \in X$ such that $gx_1 = fx_0$. In general, we construct a sequencex_{n+1} of element of X such that $y_n = gx_{n+1} = fx_n$ for $n = 0,1,2,3, \ldots ..$

For each integer $n \ge 1$ from 2.9

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\ dt \ = \ \int_0^{d(fx_n,fx_{n+1})} \xi(t)\ dt$$

$$\int_0^{d(fx_n,fx_{n+1})} \xi(t)\ dt \ \le \ a \int_0^{max\left\{\begin{matrix}\frac{[d(gx_n,fx_n).d(gx_{n+1},fx_{n+1})]}{d(gx_n,gx_{n+1})}, \frac{[d(gx_n,fx_{n+1})d(gx_{n+1},fx_n)]}{d(gx_n,gx_{n+1})}, \\ \frac{[d(gx_n,fx_n).d(gx_n,fx_{n+1})]}{d(gx_n,gx_{n+1})}, \frac{[d(gx_{n+1},fx_n).d(gx_{n+1},fx_{n+1})]}{d(gx_n,gx_{n+1})}, \\ d(gx_n,gx_{n+1})\end{matrix}\right\}} \xi(t)\ dt$$

$$\int_0^{d(y_n,fx_{n+1})} \xi(t)\ dt \ \le \ a \int_0^{max\left\{\begin{matrix}\frac{[d(y_{n-1},y_n).d(y_n,y_{n+1})]}{d(y_{n-1},y_n)}, \frac{[d(y_{n-1},y_{n+1})d(y_n,y_n)]}{d(y_{n-1},y_n)}, \\ \frac{[d(y_{n-1},y_n).d(y_{n-1},y_{n+1})]}{d(y_{n-1},y_n)}, \frac{[d(y_n,y_n).d(y_n,y_{n+1})]}{d(y_{n-1},y_n)}, \\ d(y_{n-1},y_n)\end{matrix}\right\}} \xi(t)\ dt$$

$$\int_0^{d(y_n,fx_{n+1})} \xi(t)\ dt \ \le \ a \int_0^{max\{d(y_n,y_{n+1}),0,d(y_{n-1},y_{n+1}),0,d(y_{n-1},y_n)\}} \xi(t)\ dt$$

$$\int_0^{d(y_n,fx_{n+1})} \xi(t)\ dt \ \le \ a \int_0^{d(y_{n-1},y_n)} \xi(t)\ dt \tag{2.11}$$

In this way we can write,

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\ dt \ \ \le a^n \int_0^{d(y_0,y_1)} \xi(t)\ dt \tag{2.12}$$

as $n \to \infty$, we have

$$lim_{n\to\infty} \int_0^{d(y_n,y_{n+1})} \xi(t)\ dt \ = \ 0 \tag{2.13}$$

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < \ n(p) < \ m(p+1)$ with

$$d(y_{m(p)}, y_{n(p)}) \ \ge \epsilon, \qquad d(y_{m(p)}, y_{n(p)-1}) \ < \ \epsilon \tag{2.14}$$

Now

$$d(y_{m(p)-1}, y_{n(p)-1}) < d(y_{m(p)-1}, y_{m(p)}) + d(y_{m(p)}, y_{n(p)-1})$$

$$d(y_{m(p)-1}, y_{n(p)-1}) < d(y_{m(p)-1}, y_{m(p)}) + \epsilon \tag{2.15}$$

Hence

$$lim_{p\to\infty} \int_0^{d(y_{m(p)-1}, y_{n(p)-1})} \xi(t)\ dt = \int_0^{\epsilon} \xi(t)\ dt \tag{2.16}$$

Using (2.9), (2.13), and (2.15) $we\ get$

$$\int_0^{\epsilon} \xi(t)\ dt \le \int_0^{d(y_{m(p)}, y_{n(p)})} \xi(t)\ dt \le a \int_0^{d(y_{m(p)-1}, y_{n(p)-1})} \xi(t)\ dt \le a \int_0^{\epsilon} \xi(t)\ dt$$

Which is contradiction, since $a \in (0,1)$. therefore $\{y_n\}$ is a Cauchy, hence converges to $z \in X\ from$ 2.9, $we\ get$

$$\int_0^{d(fz,fx_n)} \xi(t)\ dt \le a \int_0^{max\left\{\begin{array}{c}\frac{[d(gz,fz).d(gx_n,fx_n)]}{d(gz,gx_n)}, \frac{[d(gz,fx_n)d(gx_n,fz)]}{d(gz,gx_n)}, \\ \frac{[d(gx,fx).d(gx,fx_n)]}{d(gz,gx_n)}, \frac{[d(gx_n,fz).d(gx_n,fx_n)]}{d(gz,gx_n)}, \\ d(gz,gx_n)\end{array}\right\}} \xi(t)\ dt$$

Taking limit as $n \to \infty$, we get, $fz = z\ and\ gz = z$.

Now we show that z is a common fixed point of $f\ and\ g$. since $f\ and\ g$ are compatible, therefore,

$lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, which since $lim_{n\to\infty} gfx_n = gz$ implies that $lim_{n\to\infty} fgx_n = gz$.

Now from 2.9,

$$\int_0^{d(fgx_n,gx_n)} \xi(t)\ dt \le a \int_0^{max\left\{\begin{array}{c}\frac{[d(ggx_n,fgx_n).d(gx_n,gx_n)]}{d(ggx_n,gx_n)}, \frac{[d(gx,fy)d(gy,fx)]}{d(ggx_n,gx_n)}, \\ \frac{[d(ggx_n,fgx_n).d(ggx_n,gx_n)]}{d(ggx_n,gx_n)}, \frac{[d(gx_n,fgx_n).d(gx_n,gx_n)]}{d(ggx_n,gx_n)}, \\ d(ggx_n,gx_n)\end{array}\right\}} \xi(t)\ dt$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(fz,z)} \xi(t)\ dt \le a \int_0^{max\left\{\begin{array}{c}\frac{[d(gz,fz).d(z,z)]}{d(gz,z)}, \frac{[d(gz,z)d(z,fz)]}{d(gz,z)}, \\ \frac{[d(gz,fz).d(gz,z)]}{d(gz,z)}, \frac{[d(z,fz).d(z,z)]}{d(gz,z)}, \\ d(gz,z)\end{array}\right\}} \xi(t)\ dt$$

Taking $lim\ as\ n \to \infty$ we obtain $z = gz$.

Again from 2.1, we can show that, $z = fz$ and hence $z\ is$ common fixed point of $f\ and\ g\ in\ X$.

UNIQUENESS

Let us w is another fixed point of $f\ and\ g$ in X different from z $i.e.\ z \neq w$, then from 2.1 we have,

$$\int_0^{d(fw,fz)} \xi(t)\ dt \le a \int_0^{max\left\{\begin{array}{c}\frac{[d(gw,fw).d(gy,fz)]}{d(gw,gz)}, \frac{[d(gw,fz)d(gz,fw)]}{d(gw,gz)}, \\ \frac{[d(gw,fw).d(gw,fz)]}{d(gw,gz)}, \frac{[d(gz,fw).d(gz,fz)]}{d(gw,gz)}, \\ d(gw,gz)\end{array}\right\}} \xi(t)\ dt$$

$$\int_0^{d(w,z)} \xi(t)\ dt \le a \int_0^{d(w,z)} \xi(t)\ dt$$

Which contradiction,

So that, z is unique common fixed point of f and g.

ACKNOWLEDGEMENT

This work is done under the project titled "Application of fixed point theory in pure and Applied Mathematics" funded by MPCOST Bhopal (M.P.)

REFERENCES

[1] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math Sci., 29:9(2002), 531-536.

[2] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, Fund. Math.3, (1922)133181 (French).

[3] S. K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci. 25(1972), 727-730.

[4] K. Goebel and W. A. Kirk, Topiqs in Metric Fixed Point Theory, Combridge University Press, New York, 1990.

[5] G. Jungck, Commuting mappings and fixed point, Amer. Math. Monthly, 83(1976), 261-263.

[6] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci., 9 (1986), 771-779.

[7] G. Jungck, Compatible mappings and common fixed points (2), Internat. J. Math. and Mat. Sci., 11(1988), 285-288.

[8] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., 60(1968), 71-76.

[9] B. E . Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type, International Journal of Mathematics and Mathematical Sciences, 63, (2003), 4007 - 4013.

[10] B. E. Rhoades, A Comparison of Various Definitions of Contractive Mappings, Trans. Amer. Math. Soc. 226 (1977), 257-290.

[11] B. E. Rhoades, Contractive definitions revisited, Topological Methods in Nonlinear Functional Analysis (Toronto, Ont., 1982), Contemp. Math., 21, American Mathematical Society, Rhode Island, (1983), 189-203.

[12] B. E. Rhoades, Contractive Definitions, Nonlinear Analysis, World Science Publishing, Singapore, 1987, 513-526.

[13] S. Sessa, On a weak commutativity conditions of mappings in fixed point consideration, Publ. Inst. Math. Beograd, 32:46(1982), 146-153.

[14] O. R. Smart, Fixed Point Theorems, Cambridge University Press, London, 1974.

[15] T. Zamfrescu, Fixed point theorems in metric spaces, Arch. Math. (Basel) 23(1972), 292-298.

Source of support: MPCOST Bhopal (M.P.), Conflict of interest: None Declared