SOME DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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ABSTRACT

In the present paper we study difference sequence spaces defined by a sequence of modulus functions and examine some topological properties of these spaces.

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1. Introduction and Preliminaries:

A modulus function is a function \( f : [0, \infty) \to [0, \infty) \) such that

1. \( f(x) = 0 \) if and only if \( x = 0 \),
2. \( f(x + y) \leq f(x) + f(y) \) for all \( x \geq 0, y \geq 0 \),
3. \( f \) is increasing
4. \( f \) is continuous from right at 0.

It follows that \( f \) must be continuous everywhere on \([0, \infty)\). The modulus function may be bounded or unbounded. For example, if we take \( f(x) = \frac{x}{x+1} \) then \( f(x) \) is bounded. If \( f(x) = x^p, 0 < p < 1 \), then the modulus \( f(x) \) is unbounded. Subsequently, modulus function has been discussed in ([1], [7], [8]) and many others.

Let \( X \) be a linear metric space. A function \( p : X \to \mathbb{R} \) is called paranorm, if

1. \( p(x) \geq 0, \text{ for all } x \in X \),
2. \( p(-x) = p(x) \), for all \( x \in X \),
3. \( p(x + y) \leq p(x) + p(y) \), for all \( x, y \in X \),
4. if \( (\lambda_n) \) is a sequence of scalars with \( \lambda_n \to \lambda_n \) as \( n \to \infty \) and \( (x_n) \) is a sequence of vectors with \( p(x_n - x) \to 0 \) as \( n \to \infty \), then \( p(\lambda_n x_n - \lambda x) \to 0 \) as \( n \to \infty \).

A paranorm \( p \) for which \( p(x) = 0 \) implies \( x = 0 \) is called total paranorm and the pair \((X, p)\) is called a total paranorm space. It is well known that the metric of any linear metric space is given by some total paranorm (see [9], Theorem 10.4.2, p-183).

Let \( \omega \) be the set of all sequences, real or complex numbers and \( l_\omega, c \) and \( c_0 \) be respectively the Banach spaces of bounded, convergent and null sequences \( x = (x_k) \), normed by

\[ ||x|| = \sup_{k \in \mathbb{N}} |x_k|, \text{ where } k \in \mathbb{N}, \text{ the set of positive integers.} \]

Let \( \Lambda = (\lambda_n) \) be a non decreasing sequence of positive reals tending to infinity and \( \lambda_1 = 1 \) and \( \lambda_{n+1} \leq \lambda_n + 1 \). The generalized de la vallee-Poussin means is defined by

\[ t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k \]

where \( I_n = [n - \lambda_n + 1, n] \). A sequence \( x = (x_k) \) is said to be \((V, \lambda)\) summable to a number \( l \) if \( t_n(x) \to l \) as \( n \to \infty \) (see[4]). If \( \lambda_n = n \), \((V, \lambda)\) summability and strong \((V, \lambda)\) summability are reduced to \((C, 1)\) summability and \([C, 1]\) summability, respectively.

The idea of difference sequence spaces were introduced by Kizmaz. In [3], Kizmaz defined the sequence space

\[ X(D) = \{ x = (x_k) : (\Delta x_k) \in X \} \]

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for \( X = l_\sigma, c \) or \( c_0 \) where \( \Delta x = (\Delta x_k) = (x_k - x_{k+1}) \) for all \( k \in \mathbb{N} \).

Later, these difference sequence spaces were generalized by Et and Colak [2]. In [2] and Colak generalized the above sequence spaces to the sequence space as follows:

\[
X(\Delta^m) = \{ x = (x_k) : (\Delta^m x_k) \in X \}
\]

for \( X = l_\sigma, c \) or \( c_0 \) where \( m \in \mathbb{N} \), \( \Delta^0 x = (x_k) \), \( \Delta x = (x_k - x_{k+1}) \), \( \Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}) \) for all \( k \in \mathbb{N} \).

The generalized difference has the following binomial representation,

\[
\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}
\]

For all \( k \in \mathbb{N} \).

The following inequality will be used throughout the paper. If

\[
0 \leq p_k \leq \sup p_k = H, \quad D = \max(1, 2^{H-1})\text{ then } |a_k + b_k|^p_k \leq D(|a_k|^p_k + |b_k|^p_k)
\]

(1)

For all \( k \) and \( a_k, b_k \in \mathbb{C} \). Also \( |a|^p_k \leq \max(1, |a|^H) \) for all \( a \in \mathbb{C} \).

Throughout \( E \) will represent a seminormed space, seminormed by \( q \). We define \( \omega(E) \) to be the vector space of all \( E \)-valued sequences. Let \( F = (f_k) \) be a sequence of strictly positive real numbers, \( A = a_{jk} \) be a non negative matrix such that

\[
\sup_j \sum_{k=1}^{\infty} a_{jk} < \infty
\]

and \( s, m \in \mathbb{N} \). Then we define the following sequence spaces:

\[
[V_A^E, A, \Delta_m^s, F, p, 0] = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [f_k(q(\Delta_m^s x_k))]^p_k = 0, \text{ uniformly in } j \right\}
\]

\[
[V_A^E, A, \Delta_m^s, F, p, 1] = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [f_k(q(\Delta_m^s x_k - L))]^p_k = 0, \text{ uniformly in } j \text{ for some } L \right\}
\]

and

\[
[V_A^E, A, \Delta_m^s, F, p, \infty] = \left\{ x \in \omega(E) : \sup_j \left( \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [f_k(q(\Delta_m^s x_k))]^p_k \right) < \infty \right\}
\]

For \( f_k(x) = x \), we have

\[
[V_A^E, A, \Delta_m^s, F, p, 0] = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [q(\Delta_m^s x_k)]^p_k = 0, \text{ uniformly in } j \right\}
\]

\[
[V_A^E, A, \Delta_m^s, F, p, 1] = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [q(\Delta_m^s x_k - L)]^p_k = 0, \text{ uniformly in } j \text{ for some } L \right\}
\]

and

\[
[V_A^E, A, \Delta_m^s, F, p, \infty] = \left\{ x \in \omega(E) : \sup_j \left( \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [q(\Delta_m^s x_k)]^p_k \right) < \infty \right\}
\]

For \( p_k = 1 \), we have

\[
[V_A^E, A, \Delta_m^s, F, 0] = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [f_k(q(\Delta_m^s x_k))] = 0, \text{ uniformly in } j \right\}
\]

\[
[V_A^E, A, \Delta_m^s, F, p, 1] = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [f_k(q(\Delta_m^s x_k - L))] = 0, \text{ uniformly in } j \text{ for some } L \right\}
\]

and

\[
[V_A^E, A, \Delta_m^s, F, p, \infty] = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in \mathbb{N}} a_{jk} [f_k(q(\Delta_m^s x_k))] < \infty \right\}
\]
For $f_k(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we have

$$[V^E, A, \Delta^s_m, p]_0 = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(q(\Delta^s_m x_k))] = 0, \text{ uniformly in } j \right\},$$

and

$$[V^E, A, \Delta^s_m, p]_1 = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(q(\Delta^s_m x_k - L))] = 0, \text{ uniformly in } j \text{ for some } L \right\},$$

For $m = 1$, we have

$$[V^E, A, \Delta^s, F, p]_0 = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(q(\Delta^s x_k))]^{p_k} = 0, \text{ uniformly in } j \right\},$$

and

$$[V^E, A, \Delta^s, F, p]_1 = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(q(\Delta^s x_k - L))]^{p_k} = 0, \text{ uniformly in } j \text{ for some } L \right\},$$

For $A = 1$, we have

$$[V^E, \Delta^s_m, F, p]_0 = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in I_n} [f_k(q(\Delta^s_m x_k))]^{p_k} = 0, \text{ uniformly in } j \right\},$$

and

$$[V^E, \Delta^s_m, F, p]_1 = \left\{ x \in \omega(E) : \lim_{n \to \infty} \frac{1}{A_n} \sum_{k \in I_n} [f_k(q(\Delta^s_m x_k - L))]^{p_k} = 0, \text{ uniformly in } j \text{ for some } L \right\},$$

For $E = \mathbb{C}, q(x) = |x|, f_k(x) = x, p_k = 1$, for all $k \in \mathbb{N}$, $s = 0$, $m = 0$ the spaces $[V^E, A, \Delta^s_m, F, p]_0, [V^E, A, \Delta^s_m, F, p]_1$ and $[V^E, f, \Delta^s_m, F, p]_\infty$ reduces to $[V, A]_0, [V, A]_1$ and $[V, A]_\infty$ respectively. These spaces are called as $\lambda$-strongly summable to zero, $\lambda$-strongly summable and $\lambda$ — strongly bounded by the de la Vallee-Poussin method. When $A_n = n$, for all $n = 1, 2, 3, \ldots$ the sets $[V, A]_0, [V, A]_1$ and $[V, A]_\infty$ reduce to the set $\omega_0, \omega_1$ and $\omega_\infty$ introduced and studied by Maddox [5]. Throughout this paper, we will denote any one of the notations $0, 1, \infty$ by $X$.

In this paper we study some topological properties and inclusion relations between above defined sequence spaces.

### 2. MAIN RESULTS:

**Theorem 2.1** Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. Then the sequence spaces $[V^E, A, \Delta^s_m, F, p]_0, [V^E, A, \Delta^s_m, A, F, p]_1$ and $[V^E, \Delta^s_m, A, F, p]_\infty$ are linear spaces.

**Proof:** Let $x, y \in [V^E, \Delta^s_m, A, F, p]_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive number $M_\alpha$ and $N_\beta$ such that $|\alpha| \leq M_\alpha$ and $|\beta| \leq N_\beta$. Since $f_k$ is subadditive and $\Delta^s$ is linear, we have

$$\frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(q(\Delta^s_m (\alpha u_k y_k + \beta u_k y_k)))]^{p_k} \leq \frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(|\alpha| q(\Delta^s_m u_k y_k))]^{p_k} + f_k(|\beta| q(\Delta^s_m (u_k y_k)))^{p_k} \leq \Delta(M_\alpha)^H \frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(q(\Delta^s_m u_k y_k))]^{p_k} + \Delta(N_\beta)^H \frac{1}{A_n} \sum_{k \in I_n} a_{jk} [f_k(q(\Delta^s_m y_k))]^{p_k} \to 0 \text{ as } n \to \infty.$$

This proves that $[V^E, \Delta^s_m, A, F, p]_0$ is linear space. Similarly we can prove that $[V^E, \Delta^s_m, A, F, p]_1$ and $[V^E, \Delta^s_m, A, F, p]_\infty$ are linear spaces in view of the above proof.
Theorem 2.2 Let $F = (f_k)$ be a sequence of modulus functions. Then

$$[V^E_{\lambda}, A, F, p]_0 \subset [V^E_{\lambda}, A, F, p]_1 \subset [V^E_{\lambda}, A, F, p]_2.$$  

Proof: The first inclusion is obvious. For the second inclusion, let $x \in [V^E_{\lambda}, A, F, p]_1$. Then by definition, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k))]^{p_k} = \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k) - L + L)]^{p_k} \leq D \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k - L))]^{p_k} + D^2 \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(L)]^{p_k}.$$  

Now, there exists a positive number $A$ such that $L \leq A$. Hence we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k))]^{p_k} \leq D \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k) - L)]^{p_k} + D \frac{1}{\lambda_n} \sum_{k \in I_n} [A f_k(1)]^H \lambda_n \sum_{k \in I_n} a_k.$$  

Since $x \in [V^E_{\lambda}, A, F, p]_1$, we have $x \in [V^E_{\lambda}, A, F, p]_v$. Therefore, $[V^E_{\lambda}, A, F, p]_1 \subset [V^E_{\lambda}, A, F, p]_v$. This completes the proof.

Theorem 2.3 Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. Then $[V^E_{\lambda}, A, F, p]_0$ is a paranormed space with

$$g(x) = \sup_n \left( \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k))]^{p_k} \right)^\frac{1}{p_k}$$  

where $K = \max(1, \sup p_k)$.

Proof: Clear $g(x) = g(-x)$, it is trivial that $\Delta^s_{\lambda}x_k = 0$ for $x = 0$. Since $f(0) = 0$, we get $g(x) = 0$ for $x = 0$. Since $\frac{p_k}{K} \leq 1$, using the Minkowski’s inequality, for each $n$, we have

$$\left( \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k + \Delta^s_{\lambda}y_k))]^{p_k} \right)^\frac{1}{p_k} \leq \left( \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}x_k))]^{p_k} \right)^\frac{1}{p_k} \frac{1}{p_k} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}y_k))]^{p_k} \right)^\frac{1}{p_k}.$$

Hence $g(x)$ is subadditive. For, the continuity of multiplication, let us take any complex number $a$. By definition, we have

$$g(x) = \sup_n \left( \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}ax_k))]^{p_k} \right)^\frac{1}{p_k} \leq C_a^{\frac{H}{K}} g(x),$$  

where $C_a$ is a positive integer such that $|a| \leq C_a$. Now, let $a \to 0$ for any fixed $x$ with $g \neq 0$. By definition for $|a| < 1$, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(a \Delta^s_{\lambda}x_k))]^{p_k} < \epsilon \text{ for } n > n_0(\epsilon) \quad (2)$$

Also, for $1 \leq n \leq n_0$, taking $a$ small enough, since $f$ is continuous, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(a \Delta^s_{\lambda}x_k))]^{p_k} < \epsilon. \quad (3)$$

Now, eqn. (2) and (3) together imply that $g(ax) \to 0$ as $a \to 0$.

Theorem 2.4 Let $F = (f_k)$ be a sequence of modulus functions and $m \geq 1$, then the inclusion $[V^E_{\lambda}, \Delta^s_{\lambda}^{-1}, A, F]_X \subset [V^E_{\lambda}, \Delta^s_{\lambda}, A, F]_X$ is strict. In general $[V^E_{\lambda}, \Delta^s_{\lambda}^i, A, F]_X \subset [V^E_{\lambda}, \Delta^s_{\lambda}, A, F]_X$ for all $i = 1, 2, \ldots, s - 1$ and the inclusion is strict.

Proof: Let $x \in [V^E_{\lambda}, \Delta^s_{\lambda}^{-1}, A, F]_v$. Then we have

$$\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}^{-1}x_k))] < \infty.$$  

By definition, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}^{-1}x_k))] = \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}^{-1}x_k))] + \frac{1}{\lambda_n} \sum_{k \in I_n} a_k[f_k(q(\Delta^s_{\lambda}^{-1}x_{k+1}))] \leq \infty.$$  

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Thus \([V^E_{\lambda}, \Delta_{m}^{-1}, A, F]_\infty \subset [V^E_{\lambda}, \Delta_{m}^{r}, A, F]_\infty\). Proceeding in this way, we have
\([V^E_{\lambda}, \Delta_{m}^{i}, A, F]_\infty \subset [V^E_{\lambda}, \Delta_{m}^{i}, A, F]_\infty\) for all \(i = 1, 2, \ldots, m - 1\). Let \(E = \mathbb{C}\) and \(\lambda_n = n\) for each \(n \in \mathbb{N}\). Then the sequence \(x = (x^n) \in [V^E_{\lambda}, \Delta_{m}^{r}, A, F]_\infty\) but does not belong to \([V^E_{\lambda}, \Delta_{m}^{r-1}, A, F]_\infty\) for \(f_k(x) = x\).
Similarly, we can prove for the case \([V^E_{\lambda}, \Delta_{m}^{r}, A, F]_0\) and \([V^E_{\lambda}, \Delta_{m}^{r}, A, F]_1\) in view of the above proof.

**Corollary 2.5** Let \(F = (f_k)\) be a sequence of modulus functions. Then
\([V^E_{\lambda}, \Delta_{m}^{r-1}, A, F, p]_1 \subset [V^E_{\lambda}, \Delta_{m}^{r}, A, F, p]_0\).

**Theorem 2.6** Let \(F = f_k\) be a sequence of modulus functions and \(s\) be a positive integer. Then we have
\([V^E_{\lambda}, \Delta_{m}^{r}, A, F, q]_\infty \subset [V^E_{\lambda}, \Delta_{m}^{r}, A, F, p]_\infty\).

**Proof:** (i) Let \(\varepsilon > 0\) and choose \(\delta\) with \(0 < \delta < 1\) such that \(f(t) < \varepsilon\) for \(0 \leq t \leq \delta\). Write \(y_k = f_k^{-1}(\varepsilon)\) and consider
\[
\sum_{k \in I_n} a_{jk}[f_k(y_k)]^{pk} \leq \sum_{k \in I_n, y_k \leq \delta} a_{jk} + \sum_{k \in I_n, y_k > \delta} a_{jk}. 
\]

Since \(f_k\) is continuous, we have
\[
\sum_{k \in I_n, y_k \leq \delta} a_{jk}[f_k(y_k)]^{pk} \leq \varepsilon^{1+2} \sum_{k \in I_n, y_k > \delta} a_{jk}
\]
and for \(y_k > \delta\), we use the fact that
\(y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}\).

By the definition, we have for \(y_k > \delta\),
\(f_k(y_k) < 2f_k(1)\frac{y_k}{\delta}\).

Hence
\[
\frac{1}{\lambda_n} \sum_{k \in I_n, y_k \leq \delta} a_{jk}[f_k(y_k)]^{pk} \leq \max(1, (2f_k(1)\delta^{-1})^{1/2}) \frac{1}{\lambda_n} \sum_{k \in I_n, y_k > \delta} a_{jk}[y_k]^{pk}. 
\]

From eqn. (4) and (5), we have
\([V^E_{\lambda}, \Delta_{m}^{r}, A, F, q]_\infty \subset [V^E_{\lambda}, \Delta_{m}^{r}, A, F, p]_\infty\).

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