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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF GENERALIZED SAKAGUCHI TYPE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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ABSTRACT

In this paper we introduce a class $K_s(\mathrm{A},\mathrm{B},s,t)$. *Further we obtain coefficient inequality for this class.*

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1. INTRODUCTION:

Let A denote the class of analytic and univalent functions f in $\Delta = \{z \in C : |z| < 1\}$ of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and S_s^* be the subclass of A consisting of univalent functions. For two functions $f, g \in A$, we say that the function f(z) is subordinate to g(z) in Δ and write $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w(z) with w(0) = 0 and $|w(z)| < 1$ $(z \in \Delta)$, such that $f(z) = g(w(z)), (z \in \Delta)$. In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Here we studied a Generalized Sakaguchi type class $S_s^*(\alpha, s, t)$.The function $f(z) \in A$ is said to be in the class $S_s^*(\alpha, s, t)$ if it satisfies,

$$\mathrm{Re}\left\{\frac{(s-t)zf'(z)}{f(sz)-f(zt)}\right\} > \alpha; \tag{2}$$

for some $\alpha \in [0,1), s, t \in C \; with \; t \neq s$, and for all $z \in \Delta$. The class $S_s^*(\alpha, 1, t)$ was introduced and studied by Owa et al. [6] and when we take $t = -1$ in above class, the class reduces in to $S_s^*(\alpha, 1, -1) = S_s^*(\alpha)$ which was introduced by Sakaguchi [4] and is called Sakaguchi Function of Order α, see [8,9], where as $S_s^*(0) = S_s^*$ of Starlike Functions with respect to symmetrical point in Δ.

Let $S_s^*(A, B, s, t)$, denote the class of functions of the form (1) and satisfying the condition,

$$\frac{(s-t)zf'(z)}{f(sz)-f(zt)} \prec (1+Az)/(1+Bz); \quad for \; s, t \in C, with \; t \neq s \text{ , where } (-1 \leq B < A \leq 1). \tag{3}$$

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In this paper, we consider be the class $K_s(\mathrm{A,B,s,t})$ of functions of the form (1) and satisfying the condition

$$\frac{(s-\mathrm{t})\mathrm{z}f'(\mathrm{z})}{g(\mathrm{sz})-g(\mathrm{zt})} \prec \frac{(1+\mathrm{Az})}{(1+\mathrm{Bz})}; \ for\ \mathrm{s,t} \in C\ with\ \mathrm{t} \neq s, g \in S_s^*(A,B,s,t), -1 \le \mathrm{B} < \mathrm{A} \le 1, z \in \Delta. \tag{4}$$

By the definition of subordination it follows that if $f \in K_s(A,B,s,t)$ if and only if

$$\frac{(s-\mathrm{t})\mathrm{z}f'(\mathrm{z})}{g(\mathrm{sz})-g(\mathrm{zt})} = \frac{1+\mathrm{Aw(z)}}{1+B\mathrm{w(z)}} = P(z); \ \left(s,t \in C, \mathrm{t} \neq s, g \in S_s^*(A,B,s,t), \left|w(z)\right| < 1, w \in \Delta\right) \tag{5}$$

where

$$P(\mathrm{z}) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{6}$$

In the present paper, we obtain the coefficient estimate of the class $K_s(A,B,s,t)$.

2. PRELIMINARY RESULT

To prove our main results, we need the following result:

Lemma 2.1: *If* $P(z)$ is given by (6), then

$$|p_n| \le (A-B) \quad \text{(See [3]).} \tag{7}$$

Lemma 2.2: Let $f \in S_s^*(A,B,s,t)$, then $n \ge 1$,

$$|a_n| \le \frac{(A-B)}{\left|(n-u_n)\right|}\left[1+(A-B)\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+(A-B)^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+.\ +(A-B)^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{8}$$

(See [10]), where $u_i = \dfrac{\left(s^i - t^i\right)}{(s-t)}$.

3. MAIN RESULT

We give the coefficients inequality for the class $K_s(A,B,s,t)$.

Theorem 3.1: *Let* $f \in K_s(A,B,s,t)$, *then for* $n \ge 1$,

$$|a_n| \le \frac{\alpha}{n}\left(1+\frac{|u_n|}{|(n-u_n)|}\right)\left[1+\alpha\sum_{i_1=2}^{n-1}\frac{|u_{i_1}|}{|(i_1-u_{i_1})|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{9}$$

where $\alpha = A-B$ and $u_i = \dfrac{\left(s^i - t^i\right)}{(s-t)}$.

Proof: Since $g \in S_s^*(A,B,s,t)$, this implies that $(s-t)zg'(z) = [g(sz)-g(tz)]K(z), z \in \Delta$, with $\mathrm{Re}\{K(z)\} > 0$, where $K(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ now equating coefficient of above equation, we get

$$b_2(2-u_2) = c_1 \tag{10}$$

$$b_3(3-u_3) = c_2 + \frac{c_1^2 u_2}{(2-u_2)} \tag{11}$$

$$b_4(4-u_4) = c_3 + \frac{c_1c_2u_2}{(2-u_2)} + \frac{c_1c_2u_3}{(3-u_3)} + \frac{c_1^3u_2u_3}{(2-u_2)(3-u_3)} \tag{12}$$

$$\begin{aligned} b_5(5-u_5) = c_4 &+ \frac{c_1c_3u_2}{(2-u_2)} + \frac{c_2^2u_3}{(3-u_3)} + \frac{c_1c_3u_4}{(4-u_4)} + \frac{c_1^2c_2u_2u_3}{(2-u_2)(3-u_3)} + \frac{c_1^2c_2u_2u_4}{(2-u_2)(4-u_4)} \\ &+ \frac{c_1^2c_2u_3u_4}{(3-u_3)(4-u_4)} + \frac{c_1^4u_2u_3u_4}{(2-u_2)(3-u_3)(4-u_4)}. \end{aligned} \tag{13}$$

Now from (4), we get,

$$(s-t)zf'(z) = [g(sz) - g(tz)]P(z) \tag{14}$$

using the value from (5) and (6), we get

$$\begin{aligned} &z + 2a_2z^2 + 3a_3z^3 + \ldots + 2na_{2n}z^{2n} + (2n+1)a_{2n+1}z^{2n+1} + \ldots \\ &= \left[z + b_2z^2u_2 + b_3z^3u_3 + \ldots + b_{2n}z^{2n}u_{2n} + b_{2n+1}z^{2n+1}u_{2n+1} + \ldots \right] \\ &\quad .(1 + p_1z + p_2z^2 + \ldots + p_{2n}z^{2n} + p_{2n+1}z^{2n+1} + \ldots). \end{aligned} \tag{15}$$

Equating the coefficients of various powers of z, we have,

$$2a_2 = p_1 + u_2b_2, \tag{16}$$

$$3a_3 = b_3u_3 + p_1b_2u_2 + p_2, \tag{17}$$

$$4a_4 = b_4u_4 + p_1b_3u_3 + p_2b_2u_2 + P_3, \tag{18}$$

$$5a_5 = b_5u_5 + p_1b_4u_4 + p_2b_3u_3 + p_3b_2u_2 + P_4, \tag{19}$$

on the similar manner, we can have

$$na_n = b_nu_n + b_{n-1}u_{n-1}p_1 + \ldots + b_2u_2p_{n-2} + P_{n-1}, \tag{20}$$

where, $u_i = \dfrac{(s^i - t^i)}{(s-t)}$.

Easily using Lemma 2.1 and Lemma 2.2 in (16) and (17) respective, we get

$$2|a_2| \le \alpha \left[1 + \frac{|u_2|}{|(2-u_2)|} \right], \tag{21}$$

$$3|a_3| \le \alpha \left[1 + \frac{|u_3|}{|(3-u_3)|} \right] \left\{ 1 + \alpha \frac{|u_2|}{|(2-u_2)|} \right\}, \tag{22}$$

Similarly using Lemma 2.1 and Lemma 2.2 in (18) and (19) respective, we get

$$4|a_4| \le \alpha \left[1 + \frac{|u_4|}{|(4-u_4)|} \right] \left\{ 1 + \alpha \left(\frac{|u_2|}{|(2-u_2)|} + \frac{|u_3|}{|(3-u_3)|} \right) + \alpha^2 \left(\frac{|u_2u_3|}{|(2-u_2)||(3-u_3)|} \right) \right\},$$

(23)

$$5|a_5|\le \alpha\left[1+\frac{|u_5|}{|(5-u_5)|}\right]\left\{\begin{array}{l}1+\alpha\left(\frac{|u_2|}{|(2-u_2)|}+\frac{|u_3|}{|(3-u_3)|}+\frac{|u_4|}{|(4-u_4)|}\right)\\ +\alpha^2\left(\frac{|u_2u_3|}{|(2-u_2)||(3-u_3)|}+\frac{|u_2u_4|}{|(2-u_2)||(4-u_4)|}+\frac{|u_3u_4|}{|(3-u_3)||(4-u_4)|}\right)\\ +\alpha^3\frac{|u_2u_3u_4|}{|(2-u_2)||(3-u_3)||(4-u_4)|}\end{array}\right\}. \tag{24}$$

It follows that from above equations Theorem 3.1 holds for n = 2, 3, 4 and 5. Now by Mathematical Induction, we can easily prove Theorem 3.1.

From (20) and Lemma 2.1 and Lemma 2.2, we get,

$$|a_n|\le\frac{\alpha}{n}\left(1+\frac{|u_n|}{|(n-u_n)|}\right)\left[1+\alpha\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{25}$$

where $\alpha = A-B$ and $u_i=\frac{(s^i-t^i)}{(s-t)}$.

Corollary 3.2: Let $f\in K_s(A,\ B,\ t)$, then for n ≥ 1,

$$|a_n|\le\frac{\alpha}{n}\left(1+\frac{|u_n|}{|(n-u_n)|}\right)\left[1+\alpha\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{26}$$

where $\alpha = A-B$ and $u_i=\frac{(1-t^i)}{(1-t)}$.

Proof: Let $S_s^*(A,B,t)$, the class of functions of the form (1) and satisfying the condition

$$\frac{(1-t)zf'(z)}{f(z)-f(zt)}\prec\phi(z);|t|\le 1,\ t\ne 1,\ \text{where}\ \varphi(z)=(1+Az)/(1+Bz),(-1\le B< A\le 1)$$

then after solving we get the required result,

$$|a_n|\le\frac{\alpha}{n}\left(1+\frac{|u_n|}{|(n-u_n)|}\right)\left[1+\alpha\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{27}$$

where $\alpha = A-B$ and $u_i=\frac{(1-t^i)}{(1-t)}$.

SPECIAL CASES

i. On putting s = 1 and t = –1 in Theorem 3.1, we get the known result [5].
ii. On the same manner if we put t = –1 in Corollary 3.2, we get the same result [5].

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