COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF GENERALIZED SAKAGUCHI TYPE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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(Received on: 04-04-12; Accepted on: 25-04-12)

ABSTRACT

In this paper we introduce a class \( K_s \) \((A,B,s,t)\). Further we obtain coefficient inequality for this class.

2000 Mathematics Subject Classification: 30C45.

Keywords: Analytic functions, Subordination, Starlike with respect to symmetric points, close-to-convex with respect to symmetric points, coefficient estimates

1. INTRODUCTION:

Let \( \Delta \) denote the class of analytic and univalent functions \( f \) in \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) of the form,

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

and \( S_s^* \) be the subclass of \( A \) consisting of univalent functions. For two functions \( f, g \in A \), we say that the function \( f(z) \) is subordinate to \( g(z) \) in \( \Delta \) and write \( f \prec g \) or \( f(z) \prec g(z) \), if there exists an analytic function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \((z \in \Delta)\), such that \( f(z) = g(w(z)), (z \in \Delta) \). In particular, if the function \( g \) is univalent in \( \Delta \), the above subordination is equivalent to \( f(0) = g(0) \) and \( f(\Delta) \subset g(\Delta) \).

Here we studied a Generalized Sakaguchi type class \( S_s^*(\alpha,s,t) \). The function \( f(z) \in A \) is said to be in the class \( S_s^*(\alpha,s,t) \) if it satisfies,

\[
\text{Re}\left\{ \left( \frac{(s-t)z f'(z)}{f(sz) - f(zt)} \right) \right\} > \alpha;
\]

for some \( \alpha \in [0,1), s, t \in \mathbb{C} with t \neq s \), and for all \( z \in \Delta \). The class \( S_s^*(\alpha,1,t) \) was introduced and studied by Owa et al. [6] and when we take \( t = -1 \) in above class, the class reduces in to \( S_s^*(\alpha,1,-1) = S_s^* \) which was introduced by Sakaguchi [4] and is called Sakaguchi Function of Order \( \alpha \), see [8,9], where as \( S_s^*(0) = S^*_s \) of Starlike Functions with respect to symmetrical point in \( \Delta \).

Let \( S_s^*(A,B,s,t) \), denote the class of functions of the form (1) and satisfying the condition,

\[
\frac{(s-t)z f'(z)}{f(sz) - f(zt)} < \left( 1 + A z \right) / \left( 1 + B z \right); \text{ for } s, t \in \mathbb{C}, \text{ with } t \neq s , \text{ where } (-1 \leq B < A \leq 1).
\]

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In this paper, we consider the class $K_s(A, B, s, t)$ of functions of the form (1) and satisfying the condition

$$\frac{(s-t)zf'(z)}{g(sz) - g(zt)} < \frac{(1+Az)}{(1+Bz)}; \text{ for } s, t \in C \text{ with } t \neq s, g \in S^*_s(A, B, s, t), -1 \leq B < A \leq 1, z \in \Delta.$$  

(4)

By the definition of subordination it follows that if $f \in K_s(A, B, s, t)$ if and only if

$$\frac{(s-t)zf'(z)}{g(sz) - g(zt)} = \frac{1 + Aw(z)}{1 + Bw(z)} = P(z); \text{ } (s, t \in C, t \neq s, g \in S^*_s(A, B, s, t), \left|w(z)\right| < 1, w \in \Delta)$$

(5)

where

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$  

(6)

In the present paper, we obtain the coefficient estimate of the class $K_s(A, B, s, t)$.

2. PRELIMINARY RESULT

To prove our main results, we need the following result:

**Lemma 2.1:** If $P(z)$ is given by (6), then

$$|p_n| \leq (A - B) \text{ (See [3])}. \tag{7}$$

**Lemma 2.2:** Let $f \in S^*_s(A, B, s, t)$, then $n \geq 1,

$$|a_n| \leq \left(\frac{(A-B)}{n-u_n}\right) \left[1 + (A-B)\sum_{i=2}^{n-1}\frac{|u_i|}{(i-u_i)} + (A-B)^2\sum_{i=2}^{n-2} \sum_{i=2}^{n-1}\frac{|u_iu_{i-1}|}{(i-i-1)|i-i-1|} + \ldots + (A-B)^n\prod_{i=2}^{n-1}\frac{|u_i|}{(i-i-1)}\right],$$

(8)

(See [10]), where $u_i = \frac{(s'-t')}{(s-t)}$.

3. MAIN RESULT

We give the coefficients inequality for the class $K_s(A, B, s, t)$.

**Theorem 3.1:** Let $f \in K_s(A, B, s, t)$, then for $n \geq 1,

$$|a_n| \leq \frac{\alpha}{n} \left[1 + \frac{|u_n|}{(n-u_n)}\right] \left[1 + \alpha \sum_{i=2}^{n-1}\frac{|u_i|}{(i-i-1)} + \alpha^2 \sum_{i=2}^{n-2} \sum_{i=2}^{n-1}\frac{|u_iu_{i-1}|}{(i-i-1)|i-i-1|} + \ldots + \alpha^{n-2} \prod_{i=2}^{n-1}\frac{|u_i|}{(i-i-1)}\right],$$

(9)

where $\alpha = A - B$ and $u_i = \frac{(s'-t')}{(s-t)}$.

**Proof:** Since $g \in S^*_s(A, B, s, t)$, this implies that $(s-t)zg'(z) = [g(sz) - g(tz)]K(z), z \in \Delta$, with $\text{Re}\{K(z)\} > 0$, where $K(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots$ now equating coefficient of above equation, we get

$$b_2(2-u_2) = c_1$$

(10)
\[ b_3(3 - u_3) = c_2 + \frac{c_1^2 u_2}{(2 - u_2)} \] (11)

\[ b_4(4 - u_4) = c_3 + \frac{c_1 c_2 u_3}{(2 - u_2)} + \frac{c_1 c_2 u_3}{(3 - u_3)} + \frac{c_1^3 u_2 u_3}{(2 - u_2)(3 - u_3)} \] (12)

\[ b_3(5 - u_3) = c_4 + \frac{c_1 c_2 u_3}{(2 - u_2)} + \frac{c_1^2 u_3}{(3 - u_3)} + \frac{c_1 c_2 u_4}{(4 - u_4)} + \frac{c_1^2 c_2 u_4}{(2 - u_2)(3 - u_3)} + \frac{c_1^3 c_2 u_4}{(2 - u_2)(4 - u_4)} + \frac{c_1^2 c_2 u_4}{(3 - u_3)(4 - u_4)} \] (13)

Now from (4), we get,

\[(s - t)z'f'(z) = \left[g(sz) - g(tz)\right]P(z) \] (14)

using the value from (5) and (6), we get

\[ z + 2a_2z^2 + 3a_3z^3 + \ldots + 2na_nz^{2n} + (2n + 1)a_{2n+1}z^{2n+1} + \ldots \]
\[ = \left[z + b_2z^2u_2 + b_3z^3u_3 + \ldots + b_{2n}z^{2n}u_{2n} + b_{2n+1}z^{2n+1}u_{2n+1} + \ldots \right] \]
\[ .(1 + p_1z + p_2z^2 + \ldots + p_{2n}z^{2n} + p_{2n+1}z^{2n+1} + \ldots). \] (15)

Equating the coefficients of various powers of \( z \), we have,

\[ 2a_2 = p_1 + u_2b_2, \] (16)

\[ 3a_3 = b_2u_3 + p_1b_2u_2 + p_2, \] (17)

\[ 4a_4 = b_3u_4 + p_1b_3u_3 + p_2b_2u_2 + p_3, \] (18)

\[ 5a_5 = b_4u_5 + p_1b_4u_4 + p_2b_3u_3 + p_2b_2u_2 + p_4, \] (19)

on the similar manner, we can have

\[ na_n = b_nu_n + b_{n-1}u_{n-1}p_1 + \ldots + b_{2n}u_{2n}p_{n-2} + P_{n-1}, \] (20)

where, \( u_j = \frac{(s' - t')}{(s - t)} \).

Easily using Lemma 2.1 and Lemma 2.2 in (16) and (17) respective, we get

\[ 2|a_2| \leq \alpha \left[1 + \frac{|u_2|}{|2 - u_2|}\right], \] (21)

\[ 3|a_3| \leq \alpha \left[1 + \frac{|u_3|}{|3 - u_3|}\right] \left[1 + \alpha - \frac{|u_2|}{|2 - u_2|}\right], \] (22)

Similarly using Lemma 2.1 and Lemma 2.2 in (18) and (19) respective, we get

\[ 4|a_4| \leq \alpha \left[1 + \frac{|u_4|}{|4 - u_4|}\right] \left[1 + \alpha \left(\frac{|u_2|}{|2 - u_2|} + \frac{|u_3|}{|3 - u_3|}\right) + \alpha^2 \left(\frac{|u_2u_3|}{|2 - u_2||3 - u_3|}\right)\right], \]
It follows that from above equations Theorem 3.1 holds for \( n = 2, 3, 4 \) and \( 5 \). Now by Mathematical Induction, we can easily prove Theorem 3.1.

From (20) and Lemma 2.1 and Lemma 2.2, we get,

\[
\left| a_n \right| \leq \frac{\alpha}{n} \left( 1 + \frac{|u_n|}{|(n-u_n)|} \right) \left[ 1 + \alpha \sum_{i=2}^{n-1} \left| \frac{u_i}{(i-u_i)} \right| + \alpha^2 \sum_{i=2}^{n-1} \sum_{j=i+1}^{n-1} \left| \frac{u_i u_j}{(i-u_i)(j-u_j)} \right| + \ldots + \alpha^{n-2} \prod_{i=2}^{n-1} \left| \frac{u_i}{(i-u_i)} \right| \right],
\]

where \( \alpha = A - B \) and \( u_i = \frac{(s-i')}{(s-t)} \).

**Corollary 3.2:** Let \( f \in K_1(A, B, t) \), then for \( n \geq 1 \),

\[
\left| a_n \right| \leq \frac{\alpha}{n} \left( 1 + \frac{|u_n|}{|(n-u_n)|} \right) \left[ 1 + \alpha \sum_{i=2}^{n-1} \left| \frac{u_i}{(i-u_i)} \right| + \alpha^2 \sum_{i=2}^{n-1} \sum_{j=i+1}^{n-1} \left| \frac{u_i u_j}{(i-u_i)(j-u_j)} \right| + \ldots + \alpha^{n-2} \prod_{i=2}^{n-1} \left| \frac{u_i}{(i-u_i)} \right| \right],
\]

where \( \alpha = A - B \) and \( u_i = \frac{(1-t')}{(1-t)} \).

**Proof:** Let \( S^*_1(A, B, t) \), the class of functions of the form (1) and satisfying the condition

\[
\frac{(1-t)zf'(z)}{f(z)-f(zt)} < \phi(z); |t| \leq 1, t \neq 1, \text{ where } \phi(z) = (1+Az)/(1+Bz), (-1 \leq B < A \leq 1)
\]

then after solving we get the required result,

\[
\left| a_n \right| \leq \frac{\alpha}{n} \left( 1 + \frac{|u_n|}{|(n-u_n)|} \right) \left[ 1 + \alpha \sum_{i=2}^{n-1} \left| \frac{u_i}{(i-u_i)} \right| + \alpha^2 \sum_{i=2}^{n-1} \sum_{j=i+1}^{n-1} \left| \frac{u_i u_j}{(i-u_i)(j-u_j)} \right| + \ldots + \alpha^{n-2} \prod_{i=2}^{n-1} \left| \frac{u_i}{(i-u_i)} \right| \right],
\]

where \( \alpha = A - B \) and \( u_i = \frac{(1-t')}{(1-t)} \).

**SPECIAL CASES**

i. On putting \( s = 1 \) and \( t = -1 \) in Theorem 3.1, we get the known result [5].

ii. On the same manner if we put \( t = -1 \) in Corollary 3.2, we get the same result [5].
REFERENCES


[10] Chaurasia, V. B. L., Dubey, Ravi Shanker, Subclass of Sakaguchi type functions with respect to symmetric conjugate points, accepted and in press.

Source of support: Nil, Conflict of interest: None Declared