ON LEFT DERIVATIONS OF $d$–ALGEBRAS

1N. Kandaraj * & 2M. Chandramouleeswaran**

1Department of Mathematics, Saiva Bhanu Kshatriya College, Aruppukottai-626101, India
2Department of Mathematics, Saiva Bhanu Kshatriya College, Aruppukottai-626101, India

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ABSTRACT

In this paper we investigate some properties of left derivations of $d$–algebras.

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1. INTRODUCTION

Y. Imai ([1], [2], [3]) and K.Isaki introduced two classes of abstract algebras: BCK algebras and BCI algebras. Q.P.Hu and X.Li introduced a broad class of abstract algebras: BCH algebras. ([4], [5]) J.Neggers and H.S.Kim introduced the notion of $d$–algebras. [6].

Y.B. Jun and X.L.X in [7] applied the notion of derivation in ring and near ring theory to BCI algebras and they also introduced a new concept called a regular derivation in BCI algebras. They investigated some of its properties, defined a $d$–invariant ideal and gave conditions for an ideal to be $d$–invariant. In non-commutative rings, the notion of derivations is extended to $d$–derivations, left derivations and central derivations.


In [9] H.A. Abujabal and Nora O.Alshehri introduced the notion of left derivations of BCI algebras and investigated regular left derivations in BCI algebras. Recently, we have [10] introduced the notion of derivations on a $d$–algebra. In this paper we introduced the notion of left derivations on $d$–algebras and they investigated regular left derivations.

2. PRELIMINARIES

Definition 2.1: A $d$–algebra is a non-empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:
1. $x * x = 0$
2. $0 * x = 0$
3. $x * y = 0$ and $y * x = 0$ \(\Rightarrow\) $x = y$.

Definition 2.2: Let $S$ be a non empty subset of a $d$–algebra $X$ then, $S$ is called $d$–sub algebra of $X$ if $x * y \in S$ for all $x, y \in S$.

Definition 2.3: Let $X$ be a $d$–algebra and $I$ be a subset of $X$ then $I$ is called $d$–ideal of $X$ if it satisfies the following conditions:
1. $0 \in I$
2. $x * y \in I$ and $y \in I$ \(\Rightarrow\) $x \in I$
3. $x \in I$ and $y \in X$ \(\Rightarrow\) $x * y \in I$.

Corresponding author: M. Chandramouleeswaran**

2Department of Mathematics, Saiva Bhanu Kshatriya College, Aruppukottai-626101, India

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Definition 2.4: Let $X$ be a $d$-algebra. A map $\theta : X \to X$ is a left-right derivation (briefly (l, r)-derivation) of $X$ if it satisfies the identity $\theta(x * y) = (\theta(x) * y) \land (x * \theta(y))$ for all $x, y \in X$. If $\theta$ satisfies the identity $\theta(x * y) = (x * \theta(y)) \land (\theta(x) * y)$ for all $x, y \in X$, then $\theta$ is a right-left derivation (briefly (r, l)-derivation) of $X$. Moreover, if $\theta$ is both a (l, r)- and (r, l)-derivation, then $\theta$ is a derivation of $X$.

Definition 2.5: Let $\theta$ be a derivation of a $d$-algebra $X$. An ideal $I$ of $X$ is said to be $\theta$-invariant if $\theta(I) \subseteq I$ where $\theta(I) = \{ \theta(x) \mid x \in I \}$.

Definition 2.6: A self map $\theta$ of a $d$-algebra $X$ is said to be regular if $\theta(0) = 0$.

Definition 2.7: Let $(X, *, 0)$ be a $d$-algebra and $x \in X$. Define $x * X = \{ x * a \mid a \in X \}$. $X$ is said to be edge $d$-algebra if for any $x \in X$, $x * X = \{ x, 0 \}$.

Lemma 2.8: Let $(X, *, 0)$ be an edge $d$-algebra, then $x * 0 = x$ for any $x \in X$.

Lemma 2.9: If $(X, *, 0)$ is an $d$-algebra, then the condition $(x * (x * y)) * y = 0$ for all $x, y \in X$ holds.

Lemma 2.10: If $(X, *, 0)$ is an $d$-algebra, then $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

Lemma 2.11: Let $(X, *, 0)$ be an $d$-algebra then $y * (y * x) = x \ \forall \ x, y \in X$.

3. LEFT DERIVATIONS

In this section we define the left derivations.

Definition 3.1: Let $X$ be a $d$-algebra. By a left derivation of $X$ we mean a self map $\theta$ of $X$ satisfying

$$\theta(x * y) = (\theta(x) * y) \land (\theta(y) * x) \ \forall \ x, y \in X.$$ 

Example 3.2: Let $X = \{ 0, 1, 2, 3 \}$ be a $d$-algebra with Cayley table defined by

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Define a map $\theta : X \to X$ by $\theta(x) = \begin{cases} 0 & \text{if } x = 0, 1, 3 \\ 3 & \text{if } x = 2 \end{cases}$

Then it is easily checked that $\theta$ is a left derivation of $X$.

Lemma 3.3: In any $d$-algebra $X$, the following properties hold for all $x, y, z \in X$.

1. $x * (x * (x * y)) = x * y$. 
2. $x * 0 = 0 \Rightarrow x = 0$. 
3. $((x * z) * (y * z)) * (x * y) = 0$. 
4. $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$. 
5. $((x * y) * (x * z)) * (z * y) = 0$. 
6. $(x * z) * (y * z) = x * y$. 
7. $(x * 0) * 0 = x$. 
8. $x * a = x * b \Rightarrow a = b$. 
9. $a * x = b * x \Rightarrow a = b$. 
10. $x * y = 0 \Rightarrow x = y$.

Definition 3.4: A left derivation $\theta$ of a $d$-algebra $X$ is said to be regular if $\theta(0) = 0$.
Lemma 3.5: Every left derivation of a $d$-algebra with $x \ast 0 = x$ is regular.

Proof: Now

\[
\begin{align*}
\theta(0) &= \theta(0 \ast x) \\
&= (\theta(0) \ast x) \land (\theta(x) \ast 0) \\
&= (\theta(0) \ast x) \land \theta(x) \quad (\therefore x \ast 0 = x) \\
&= \theta(x) \ast (\theta(x) \ast (\theta(0) \ast x)) \\
\theta(0) &= \theta(0) \ast x.
\end{align*}
\]

If $\theta(0) = 0$, then nothing to prove. If $\theta(0) \neq 0$, then $\theta(0) \ast \theta(0) \neq 0 \ast \theta(0) \neq 0$.

This is contradiction to the condition, $x \ast x = 0$.

Hence $\theta(0) = 0$. Therefore, every left derivation of a $d$-algebra with $x \ast 0 = x$ is regular.

Lemma 3.6: Let $\theta$ be a left derivation of a $d$-algebra $X$. Then for all $x, y \in X$ we have

1. $\theta(x) \ast x = \theta(y) \ast y$.
2. $\theta(x \ast y) = \theta(x) \ast y$.

Proof:

1. Let $x, y \in X$.

\[
\begin{align*}
\theta(0) &= \theta(x \ast x) \\
&= (\theta(x) \ast x) \land (\theta(x) \ast x) \\
&= (\theta(x) \ast x) \ast ((\theta(x) \ast x) \ast (\theta(x) \ast x)) \\
&= (\theta(x) \ast x) \ast 0 \\
&= \theta(x) \ast x \quad \cdots \cdots (1).
\end{align*}
\]

Similarly, $\theta(0) = \theta(y) \ast y \quad \cdots \cdots (2)$.

From (1) and (2), $\theta(x) \ast x = \theta(y) \ast y$.

2. Let $x, y \in X$. Since $\theta$ be a left derivation of $X$.

\[
\begin{align*}
\theta(x \ast y) &= (\theta(x) \ast y) \land (\theta(y) \ast x) \\
&= (\theta(y \ast x) \ast ((\theta(y) \ast x) \ast (\theta(x) \ast y)) \\
&= \theta(x) \ast y
\end{align*}
\]

Lemma 3.7: Let $\theta$ be a left derivation of a $d$-algebra $X$ such that $x \ast 0 = x$. Then $\theta(x) = x$ if and only if $\theta$ is regular.

Proof: Let $\theta$ be a regular.

That is $\theta(0) = 0$.

Now

\[
\begin{align*}
\theta(0) &= \theta(x \ast x) \\
&= (\theta(x) \ast x) \land (\theta(x) \ast x) \\
&= (\theta(x) \ast x) \ast ((\theta(x) \ast x) \ast (\theta(x) \ast x)) \\
&= (\theta(x) \ast x) \ast 0 \\
&= \theta(x) \ast x \\
&= 0
\end{align*}
\]

which implies $\theta(x) = x$.
Conversely, assume $\theta(x) = x$. Then it is clear that $\theta(0) = 0$, thus proving that $\theta$ is regular.

**Theorem 3.8:** Let $\theta$ be a left derivation of a $\sigma$-algebra $X$. Then $\theta$ is regular if and only if every ideal of $X$ is $\theta$-invariant.

**Proof:** Let $\theta$ be a regular left derivation of a $\sigma$-algebra $X$.

Then by lemma 3.7, $\theta(x) = x$ for all $x \in X$.

Let $y \in \theta(A)$, where $A$ is an ideal of $X$.

Then $y = \theta(x)$ for some $x \in A$.

Thus $y \ast x = \theta(x) \ast x = x \ast x = 0 \in A$.

Then $y \in A$ and $\theta(A) \subset A$.

Therefore $A$ is $\theta$-invariant.

Conversely, let every ideal of $X$ be $\theta$-invariant.

That is $\theta(A) \subset A$. Then $\theta(\{0\}) \subset \{0\}$. Hence $\theta(0) = 0$. Therefore $\theta$ is regular.

**Theorem 3.9:** Let $X$ be a $\sigma$-algebra. A self map $\theta$ of $X$ is left derivation if and only if it is derivation.

**Proof:** Assume that $\theta$ is a left derivation of a $\sigma$-algebra $X$.

$$\theta(x \ast y) = \theta(x) \ast y = (x \ast \theta(y)) \ast ((x \ast \theta(y)) \ast (\theta(x) \ast y)).$$

$$\theta(x \ast y) = (\theta(x) \ast y) \wedge (x \ast \theta(y)) \quad \cdots \quad (1).$$

$$\theta(x \ast y) = \theta(x) \ast y = (x \ast \theta(y))$$

$$= (\theta(x) \ast y) \ast ((\theta(x) \ast y) \ast (x \ast \theta(y)))$$

$$= (x \ast \theta(y)) \wedge (\theta(x) \ast y) \quad \cdots \quad (2).$$

From (1) and (2), $\theta$ is a derivation of $X$.

Conversely, let $\theta$ be a derivation of $X$. So it is a $(I, r)$-derivation of $X$.

Now: $\theta(x \ast y) = (\theta(x) \ast y) \wedge (x \ast \theta(y))$.

$$= (x \ast \theta(y)) \ast ((x \ast \theta(y)) \ast (\theta(x) \ast y))$$

$$= \theta(x) \ast y$$

$$= (\theta(y) \ast x) \ast ((\theta(y) \ast x) \ast (\theta(x) \ast y))$$

$$= (\theta(x) \ast y) \wedge (\theta(y) \ast x).$$

Hence $\theta$ is a left derivation of $X$.

**Definition 3.10:** Let $X$ be a $\sigma$-algebra and $\theta_1, \theta_2$ be two self maps of $X$. We have $\theta_1 \circ \theta_2 : X \to X$ as $(\theta_1 \circ \theta_2)(x) = \theta_1(\theta_2(x)) \forall x \in X$.

**Lemma 3.11:** Let $(X, \ast, 0)$ be a $\sigma$-algebra. Let $\theta_1$ and $\theta_2$ be two left derivations of $X$, then $\theta_1 \circ \theta_2$ is also a left derivation of $X$. 
Proof:

\[(\theta_1 \circ \theta_2)(x * y) = \theta_1(\theta_2(x * y)) = \theta_1((\theta_2(x) * y) \land (\theta_2(y) * x)) = \theta_1((\theta_2(y) * x) * [(\theta_2(y) * x) * \theta_2(x) * y]) = (\theta_1(\theta_2(x)) * y) \land \theta_1(\theta_2(y)) * x = ((\theta_1 \circ \theta_2)(x) * y) \land ((\theta_1 \circ \theta_2)(y) * x)\]

Hence \(\theta_1 \circ \theta_2\) is a left derivation of \(X\).

It can be easily proved that

**Theorem 3.12:** Let \(\{X, \ast, 0\}\) be a \(d\)-algebra and \(\theta_1, \theta_2\) are left derivations of \(X\). Then \(\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1\).

**Definition 3.13:** Let \(X\) be a \(d\)-algebra and \(\theta_1, \theta_2\) be two self maps of \(X\). We define \(\theta_1 \cdot \theta_2 : X \to X\) as \((\theta_1 \cdot \theta_2)(x) = \theta_1(x) \cdot \theta_2(x) \quad \forall \ x \in X\).

**Theorem 3.14:** Let \(\{X, \ast, 0\}\) be a \(d\)-algebra and \(\theta_1, \theta_2\) are left derivations of \(X\). Then \(\theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_1\).

Proof: Let \(X\) be a \(d\)-algebra and \(\theta_1, \theta_2\) are left derivations of \(X\).

\[
\text{Now } (\theta_1 \cdot \theta_2)(x \ast y) = \theta_1(x \ast y) \cdot \theta_2(x \ast y) = [(\theta_1(x) \ast y) \land (\theta_1(y) \ast x)] \cdot \theta_2(x \ast y) = 0 \quad \text{on simplification} \quad \cdots \cdots (1).
\]

Similarly \((\theta_2 \cdot \theta_1)(x \ast y) = \theta_2(x \ast y) \cdot \theta_1(x \ast y) = 0 \quad \cdots \cdots (2)\).

From (1) and (2), \((\theta_1 \cdot \theta_2)(x \ast y) = (\theta_2 \cdot \theta_1)(x \ast y)\).

Putting \(y = 0\) we get for all \(x \in X\),

\((\theta_1 \cdot \theta_2)(x) = (\theta_2 \cdot \theta_1)(x)\). Hence \(\theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_1\)

**Notation:** Der\(\{X\}\) denote the set of all left derivations on \(X\).

**Definition 3.15:** Let \(\theta_1, \theta_2 \in\) Der\(\{X\}\). Define the binary operation \(\land\) as

\[(\theta_1 \land \theta_2)(x) = \theta_1(x) \land \theta_2(x)\]

It is easy to prove that

**Lemma 3.16:** Let \(X\) be a \(d\)-algebra and \(\theta_1, \theta_2\) are left derivations of \(X\). Then \(\theta_1 \land \theta_2\) is also a left derivation of \(X\).

**Lemma 3.17:** Let \(X\) be a \(d\)-algebra. If \(\theta_1, \theta_2, \theta_3 \in\) Der\(\{X\}\). Then

\[\theta_1 \land (\theta_2 \land \theta_3) = (\theta_1 \land \theta_2) \land \theta_3\]

Proof: Let \(X\) be a \(d\)-algebra and \(\theta_1, \theta_2, \theta_3\) are left derivations of \(X\).

\[
\text{Now } ((\theta_1 \land \theta_2) \land \theta_3)(x \ast y) = (\theta_1 \land \theta_2)(x \ast y) \land \theta_3(x \ast y) = \theta_3(x \ast y) \ast (\theta_3(x \ast y) \ast (\theta_1 \land \theta_2)(x \ast y)) = (\theta_1 \land \theta_2)(x \ast y) = (\theta_2(x) \ast y) \ast ((\theta_2(x) \ast y) \ast (\theta_1(x) \ast y)) = \theta_1(x) \ast y \quad \cdots \cdots (1).\]
Also consider the following
\[
\theta_1 \land (\theta_2 \land \theta_3)(x \ast y) = \theta_1(x \ast y) \land (\theta_2 \land \theta_3)(x \ast y)
\]
\[
= \theta_1(x \ast y) \land [\theta_2(x \ast y) \land \theta_3(x \ast y)]
\]
\[
= \theta_1(x \ast y) \land [\theta_3(x \ast y) \ast ((\theta_3(x \ast y)) \ast (\theta_2(x \ast y)))]
\]
\[
= \theta_1(x \ast y) \quad \cdots \quad (2).
\]

This implies that \((\theta_1 \land (\theta_2 \land \theta_3))(x \ast y) = ((\theta_1 \land \theta_2) \land \theta_3)(x \ast y)\).

Put \(y = 0\), we have
\[
(\theta_1 \land (\theta_2 \land \theta_3))(x) = ((\theta_1 \land \theta_2) \land \theta_3)(x).
\]
\[
\Rightarrow \theta_1 \land (\theta_2 \land \theta_3) = (\theta_1 \land \theta_2) \land \theta_3.
\]

From the above two lemmas we obtain the following.

**Theorem 3.18** \((\text{Der} \,(X, \land)\) is a semi group.

**REFERENCES**


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