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A DISCUSSION ON BOUNDS FOR 1-QUASI TOTAL COLOURINGS

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ABSTRACT

This manuscript commences with new perception of 1-quasi total colouring and diverse bounds of the 1-quasi total colourings. The upper bound for 1-quasi total chromatic number is observed as 2+ $\chi_l'(G)$ using list edge chromatics numbers. Also it has been identified that at most Δ +8log^8 Δ colours are required to properly coloured 1-quasi total graphs and to provide a polynomial time algorithm. Also it has been widened the Colin J.H, Mc.Diarmid and Abdon sanchez upper bound to1-quasi total graphs as $\left\lceil \frac{7}{5}\Delta + 3 \right\rceil$.

Key words: *1-quasi total colouring, list colouring, upper bounds, stable set.*

AMS Subject Classification: *05C15, 05C30, 05C99.*

0. INTRODUCTION

Behzad (1965) was pioneered the concept of Total Colouring for which Harry (1972) made an orifice on the concept of total graphs. In the year 1976 Bondy and Murthy studied the applications of total graphs. Later on S.Even (1975) made his contribution on algorithms in general graphs. Further Hind (1990) put his effort for improving the bounds for total chromatic graphs. In a while Chetwynd (1990) investigated on total colourings. In addition to that Beck (1994) gave an algorithmic approach to lovasz local lemma. Again Hind (1994) found some recent developments in total colourings. Then after Jesen (1995) conversed total colouring problems. Hind (1996) and others continued their contribution to total colouring yet again. After a while West DB (2002) developed the concept various applications of vertex edge colourings. Recently RVNSrinivas and JVRao (2012) made their efforts on 1- quasi total graphs.

Here all the graphs are finite and without loops .Let G=(V, E) be a graph ,when we discuss colouring of a subset of $V \cup E$, we always assume that it is proper ,means ,no two adjacent or incident elements assigned the same colour. A vertex colouring, edge colouring ,quasi total colouring is proper colouring of V,E, $V \cup E$ respectively .The chromatic number $\chi(G)$,edge chromatic number $\chi'(G)$ quasi total colouring $\chi_Q''(G)$ is the least no. of colours in a vertex ,edge ,quasi –total colouring of G, respectively. Here we restrict our proofs to simple graphs .Let $\Delta(G)$ be the maximum degree of a vertex in G. Here we follow the notations of West, D.B., 1996[11], any undefined notation follows that [11].

1. PRELIMINARIES

Definition 1.1: Let G be a graph with vertex set V(G) and edge set E(G). The 1–quasitotal graph, (denoted by $Q_1(G)$) of G is defined as follows: The vertex set of $Q_1(G)$, that is $V(Q_1(G)) = V(G) \cup E(G)$. Two vertices x, y in $V(Q_1(G))$ are adjacent if they satisfy one of the following conditions: (i). x, y are in V(G) and $\overline{xy} \in G$. (ii). x, y are in E(G) and x, y are incident in G.

Note 1.2: (i) G is a sub graph of $Q_1(G)$; and (ii)$Q_1(G)$ is a sub graph of T (G).

Example 1.3: Consider the graph G given in Fig. 1.3A.

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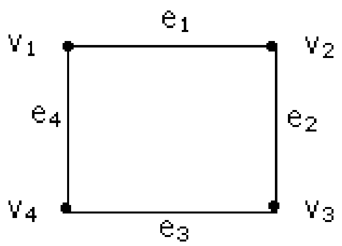
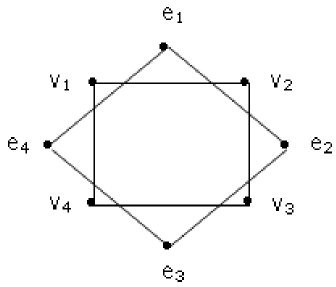


G

Fig 1.1



$Q_1(G)$

Fig 1.2

The 1-quasitotal graph Q_1 (G) is given by the Fig. 1.2

Now we define the of 1-quai total chromatic number.

1.4 Definition: A **1-Quasi total colouring** of a graph G is an assignment of colours to the vertices and edges of G such that distinct colours are assigned to (i) Every two adjacent vertices (ii) Every two adjacent edges.

A 1-quasi k-total coloring of a graph G is a quasi -total coloring of G from a set of k-colors of G. The 1-quasi total chromatic number of a graph G is the minimum positive integer k for which G is k-quasi total colorable denoted by $\chi''_{Q_1}(G)$ or $\chi(Q_1(G))$

1.5: Example: From the fig 1.2, the 1-quasi total chromatic number of G is 2

1.6 Definition *List colouring* [2] is a generalization of vertex colouring in which the set of colours available at each vertex is restricted. A graph is *k*-choosable if it has a proper list coloring no matter how one assigns a list of *k* colors to each vertex. The choosability (or list chromatic number) $\chi_l(G)$ of a graph *G* is the least number k such that *G* is *k*-choosable.

1.7 Note: Choosability $\chi_l(G)$ satisfies the following properties for a graph *G* with *n* vertices, chromatic number $\chi(G)$, and maximum degree $\Delta(G)$:

1. $\chi_l(G)) \geq \chi(G)$. A *k*-list-colorable graph must in particular have a list coloring when every vertex is assigned the same list of *k* colors, which corresponds to a usual *k*-coloring.

2. $\chi_l(G) \leq \Delta(G) + 1$.

First we have to discuss about lower bounds for 1-quasi total graphs

1.8 Definition: A graph is said to be contain *even hole* [3] if it contains an induced cycle with an even no of vertices.

1.9 Definition: A family F of graphs is called a color bounded family if for some function f(x) and any G from the family one has $\chi(G) \geq$ f(col(G)).

1.10. Theorem [Marksman, Gaspar an, and Reed (1996)]: Let G is a graph without any even-hole. Then

$$\chi''_{Q_1}(G) \geq \left\lfloor \frac{col(G)-1}{2} \right\rfloor.$$

1.11. Theorem [Manouchehr zakeer, 2008]: Let the maximum even-hole of a graph G be k. Then

$$\chi''_{Q_1}(G) \geq \frac{d(G)}{k}+1.$$

2. UPPER BOUNDS USING LIST EDGE CHROMATIC NUMBERS

If c is a total coloring of a graph G and v is a vertex of G with degv= $\Delta(G)$, then c must assign distinct colors to the $\Delta(G)$ edges incident with v as well as to v itself. This implies that $\chi''(G) \geq 1+\Delta(G)$ for every graph G. But this lower bound does not satisfy the 1-quasi total chromatic numbers .It can be justified with the following example.

2.1: Example: From the fig. 1.3A $\Delta(G)$ =2 and also satisfied $\chi''(G) \geq 1+\Delta(G)$, but fig 1.3B is the 1-quasi total graph of 1.3A and 1-quasi total chromatic number $\chi''_{Q_1}(G)$ =2, hence $\chi''_{Q_1}(G) \geq\neq 1+\Delta(G)$.

However ,in 1960s Mehdi Behzad and vizing independently conjectured ,similar to the upper bound for the chromatic index established by Vizing, that the total chromatic number cannot exceed this lower bound by more than 1.This conjecture has became known as the total coloring conjecture .

Here we may extend the bound to 1-quasi total graphs as , for any graph G, $\chi''_{Q_1}(G) \leq 2+\Delta(G)$ .Even though it is not known if 2+ $\Delta(G)$ is an upper bound for the 1-quasi total chromatic number of every graph, the number 2+ $\chi'_l(G)$.

Theorem.2.2: Every graph G, $\chi''_{Q_1}(G) \leq 2+\chi'_l(G)$.

Proof: Suppose that the list chromatic index $\chi'_l(G)$ =k.

We know that $\chi(G) \leq 1+\Delta(G) \leq 1+\chi'(G) \leq 1+\chi'_l(G) < 2+\chi'_l(G)$ =2+k.

Thus G is (k+2)-colorable .let a (k+2)-coloring c of G be given. For each edge e=uv of G, let L (e) be a list of k+2 colors and let $L'(e)$ =L (e)-{c (u), c (v)}.

Since $|L'(e)| \geq k$ for each edge e of G and $\chi'_l(G)$ =k, it follows that there is a proper edge coloring c of G such that $c'(e) \in L'(e)$ and so $c'(e) \notin \{c(u), c(v)\}$.Hence the 1-quasi total coloring c''_{Q_1} of G defined by

c''_{Q_1} (x)= $\begin{Bmatrix} c(x), if x \in V(G) \\ c'(G), if x \in E(G) \end{Bmatrix}$ is a (k+2)1-quasi total coloring of G and so $\chi''_{Q_1}(G) \leq 2+k \leq 2+\chi'_l(G)$, as desired.

The list colouring conjecture states that $\chi'(G) = \chi'_l(G)$ for every nonempty graph G if this conjecture is true then $\chi'_l(G) \leq 1+\Delta(G)$ by Vizing's theorem and by the theorem 2.2 , so $\chi''_{Q_1}(G) \leq 3+\Delta(G)$.

3. COLOURING WITH Δ+POLY (LOG Δ) COLOURS

In this section providing a polynomial time algorithm, finds a quasi total colouring of any graph with Δ sufficiently large, using at most $\Delta+8\log^8\Delta$ colours

As the concept of total coloring was introduced independently by Behzad [1965]and Vizing was each conjectured that any graph with maximum degree Δ has a Δ+2 total coloring .Note that if true, this conjecture is tight as every such graph requires at least Δ+1 colour. The first Δ+o(Δ) bound on the quasi total chromatic number of such a graph was $\Delta+2\sqrt{\Delta}$,due to Hind[1990]. In this chapter we tightened the bound $\Delta+18\Delta^{1/3}\log(3\Delta)$ of Haggkvit and Chetwynd [1995], for of 1-quasi total graphs as $\Delta+8\log^8\Delta$.

3.1Theorem: If G has maximum degree Δ then $\chi''_{Q_1}(G) \leq \Delta + 8\log^8\Delta$.

Remarks: Our proof is probabilistic and make use of the Lovasz Local lemma .the proof can be made constructive ,providing an O($n^3\log^{o(1)}n$) randomized algorithm ,and a polytime deterministic algorithm to find such a total colouring.

The total chromatic number conjecture is reminder of Vizings's theorem which states that if G has maximum degree Δ then the edge chromatic number of G, $\chi'(G)$ is either Δ or Δ+1.It is also reminiscent of list colouring conjecture .in Fact ,a slightly weaker form of the total colouring conjecture follows from the list colouring conjecture.

The list edge chromatic number of a graph G, $\chi_l'(G)$, is the minimum number r with the following property: For any mapping f: E(G)→S where S is a collection of sets of colours each size of r, G has a proper edge –colouring where for each edge e, the colour of e lies in f(e).The list colouring conjecture is that $\chi_l'(G) = \chi'(G)$.

Recall that for any graph G, $\chi(G) \leq \Delta$ +1.Now consider any Δ +1 colouring c:V(G) →{1,2,…., Δ +1}.For each edge e=(u,v) define f(e) to be the set {1,…., Δ +3}-{c(u),c(v)}.Now the size of f(e) is Δ +1 for each e ,and so if the list colouring conjecture holds we can use such coloring to provide a Δ +3 quasi total colouring of G. Therefore, the list colouring conjecture implies $\chi''_{Q_1}(G) \leq \Delta$ +3.

Inspired by the this implication, we say that a proper vertex coloring is extendable to a *t*- qusi total coloring if there is a quasi total colouring of size t whose restriction to V(G) is that vertex coloring. Thus ,we have seen that the list coloring conjecture implies every Δ +1 vertex coloring of G is extendible to a Δ +3 quasi total coloring of G .According to Hind [1994] has shown that there exist graphs having Δ +1 vertex coloring which is not extendible to a Δ +2 qusi total coloring.

In[6] Hind and others define a proper vertex coloring to be a β -frugal if no vertex has more than β members of any color class in its neighborhood To prove our main theorem we use the following Theorem

3.2 Theorem: Every graph G with maximum degree $\Delta \geq \Delta_0 = e^{10^7}$ has a $\log^5 \Delta - frugal(\Delta+1)$ vertex coloring.

Here we show that every $\log^5 \Delta - frugal(\Delta+1)$ vertex coloring is extendable to an $\Delta + 8\log^8 \Delta$ quasi total coloring thus proving our the theorem3.1.

Here we assume that $\Delta \geq e^{10^7}$ for each vertex v, N (v) denotes neighborhood of v.

To prove our theorem 3.1 we use the following lemma

3.3 Lemma: suppose G is a graph with maximum degree at most D $\geq 8\log^8 \Delta$. Suppose further that we are given $S_1, S_2, S_3, \ldots\ldots S_{\frac{D}{2}} \subseteq V(G)$ such that for all $v \in V(G)$,1 ≤ i ≤ D/2, $|N(v) \cap S_i| \leq \log^5 \Delta$. Then there exists a sequence of edge-disjoint matchings in G, $M_1, M_2, M_3, \ldots\ldots M_{\frac{D}{2}}$ such that

1. M_i misses S_i;

2. $G' = G - \cup_{i=1}^{\frac{D}{2}} M_i$ has maximum degree at most $\frac{D}{2} + 2\log^7 \Delta$.

Repeated iterations of lemma 3.3 will prove our theorem3.1.

Proof of theorem3.1: Take $S_1, S_2, S_3, \ldots\ldots S_{\Delta+1}$ to be the colour classes of any $\log^5 \Delta - frugal(\Delta+1)$ colouring of G, as guaranteed by theorem **A.** Set G_0=G, $\Delta = \Delta_0$, and repeatedly apply lemma 1 until $\Delta_j < 8\log^8 \Delta$,setting $G_{j+1} = G'$, $\Delta_{j+1} = \Delta_{j/2} + 2\log^7 \Delta \leq \frac{\Delta}{2^j} + 4\log^7 \Delta$,and choosing $S^{(j)}{}_1, \ldots\ldots\ldots S_{\Delta_j/2}$ from previously unused members of { $S_1, S_2, \ldots\ldots\ldots S_{\Delta+1}$ },all the while forming colour classes from the pairs $S_i \cup M_i$. As there are at most $\log \Delta$ Iterations, $\sum \Delta_i / 2 \leq \Delta - 4\log^8 \Delta + \log \Delta (4\log^7 \Delta) < \Delta + 1$, and so we will have produced fewer than Δ +1 colour classes. Therefore, an $8\log^8 \Delta$ edge-colouring of the final G' will provide our $\Delta + 8\log^8 \Delta$ 1-quasi total colouring.

3.4. Algorithmic considerations:

We desaibe that our proof can make algorithmic using the techniques by Beck [7], at the price of increasing our lower bound on Δ.Set n=$|V(G)|$.In [6] Provides an O ($n^3 \log^{o(1)} n$) randomized algorithm and a polytime deterministic algorithm to find a $\log^5 \Delta$-frugal (Δ+1) –c0louring of G. After doing this we must find the set X_i and the matching M_i ,we need find the 8$\log^8 \Delta$ edge colouring used in the proof of the thorem3.1.The latter step can be done in O(n^4) steps ,or we can find a 16 $\log^8 \Delta$ edge –colouring in O(n^2) steps. Each M_i can be found in O($n^{2.5}$) steps ,as in [9] O($n^2 \log^{o(1)} n$) steps using an algorithm particularly the same as that discussed in [7]. the only amendment needed s to allow for sampling with probability p_i here rather than with probability $\frac{1}{2}$ as in [7] .Also we have a O($n^{2.5} \log^{o(1)} n$) time randomized algorithm and a polytime deterministic algorithmic for finding a Δ+16 $\log^8 \Delta$ total colouring of G.

4. EXTENSION OF VERTEX COLOUIRNG TO QUASI TOTAL COLOURING

In this section we give an upper bound on the no. of colours required to extend a given vertex colouring of a graph to a quasi total colouring. Colin J.H, Mc.Diarmid and abdon Sanchez proved the total colouring for any simple graph is at most $\frac{7}{5}\Delta+3$.Here we extend the concept to 1-quasi total graphs and proved that for any simple graph there is 1-quasi total colouring using at most $\left\lceil \frac{7}{5}\Delta+3 \right\rceil$colours, where Δ is maximum degree.

Trivially $\chi'(G) \geq \Delta$ and $\chi''(G) \geq \Delta$ +1.It is also well known that $\chi(G) \leq \Delta+1$ and $\chi'(G) \leq \Delta+1$. Also from the total colouring conjecture $\chi''(G) \leq \Delta+2$ [8].

We know that total coloring conjecture has been verified for several graphs. We may find for Complete graphs in [1].We aim here in upper bounds on the quasi total chromatic number $\chi''_Q(G)$.Of course clearly $\chi''_Q(G) \leq \chi(G)+\chi'(G)$.It is also proved that $\chi''(G) \leq \frac{3}{2}\Delta$ for multiple graph with $\Delta \geq 6$[1]. Hind[4] has reviled that $\chi''(G) \leq \chi'(G)+2\sqrt{\chi}$,it has also been expressed that if k is an integer ,with $\angle k$ at least number of vertices then $\chi''(G) \leq \chi'(G)+k+1$ and that most graphs satisfy $\chi''(G)=\Delta+1$(see1,8).Now we describe our main result.

4.1. Theorem: For any graph G, any vertex colouring with p colours extends to a 1-quasi total colouring $\chi''_Q(G) \leq \{\max(p,\chi')+\frac{2}{5}\min(p,\chi')+\frac{8}{5}\}$

To prove the theorem first we have to use the following lemmas.

Lemma4.2: [Colin J.H, Mc.Diarmid, 1991] Let G= (V, E) be a graph with $\Delta(G) \leq 2$,let $W \subseteq V$, and let $\phi : W \rightarrow \{1,2,3,4\}$ be a partial vertex colouring of G. Then there is an edge colouring $\psi : E \rightarrow \{1,2,3,4\}$ such that $\phi \cup \psi$ is a proper colouring of $W \cup E$, and if $\psi(e)=1$ then $e \subseteq W$.

Lemma 4.3: Let G=(V,E) be a graph ,let $W \subseteq V$ and let $\phi : W \rightarrow \{1,2,3,4\}$ be a partial vertex colouring .Suppose that there are k matchings in G such that the sub graph H is obtained by deleting them has $\Delta(H) \leq 2$ (certainly ,this is true if $k=\chi'(G)-2$).Then there is an edge colouring $\psi : E \rightarrow \{1,......,k+4\}$ such that $\phi \cup \psi$ is a proper colouring of $W \cup E$,and if $\psi(e)=1$ then $e \subseteq W$.

Proof: If $\Delta(H) \leq 2$, then the result follows immediately from lemma 4.1 .so, suppose that $\Delta(H) \geq 3$.Let E_5 ,….., E_{k+4} be k matchings such that the graph H obtained by deleting them has $\Delta(H) \leq 2$.Now apply lemma 4.2 to H.

Now we define t(0),t(1),…,t(4) to be 1, $\frac{8}{5},\frac{6}{5},\frac{4}{5},\frac{2}{5}$, respectively, for any integer n let t(n)=t(i), where $n \equiv i(\bmod 5)$, $0 \le i \le 4$.

Lemma 4.4:Let G=(V,E) be a graph , let $V_1,\ldots\ldots V_p$ be disjoint stable sets of vertices, and let $W = V_1 \cup \ldots.. \cup V_p$.Then there is a colouring ϕ of $W \cup E$ using at most

[$\max(p,\chi') + \frac{2}{5}\min(p,\chi') + t(\min(p,\chi'))$] colours such that $\phi(v) = i$ for $v \in V_i$, for each i=1,…p.

Proof: we have to know the result for the cases $p \le \chi'$ and $p > \chi'$.

Let $p > \chi'$. Apply the known result to $V_1,\ldots. V_{\chi'}$ and then use $p - \chi'$ new colours for $V_{\chi'+1},\ldots.,V_p$.This uses at most $\left[p + \frac{2}{5}\chi' + t(\chi')\right]$ colours i.e. $\left[\chi' + \frac{2}{5}\chi' + t(\chi')\right] + (p - \chi')$ Colours, as required. Hence it suffices to consider the case $p \le \chi'$.

Assume first that p=5k-1, where k≥2.Let $\phi_0 : E \to \{1,\ldots\ldots.\chi'\}$ be an edge colouring of G. For 1,…, k let $W_j=V_{4j-3}$.For j=1,….,k apply lemma 4.2 to the vertex sets V_{4j-3},….. V_{4j} in the graph on V with edge set E_j. Thus ,we find a colouring $\phi_1 : W_1 \cup \ldots \cup E \to \{1,\ldots\ldots.\chi' + 2k\}$ such that $\phi_1(v) = i$ for $v \in V_i$,i=1,……..,4k and each edge e coloured 4j satisfies $e \subseteq W_j$,for i=1,…..,k. We must next remove colour i from each edge e incident with V_i, for each i=4k+1,….,p.

Suppose then that $\phi_1(e) = 1$, where i=4k+j, 1≤j≤k-1, and e is incident with V_i,

let $\phi_2(e) = \left\{\begin{matrix} 4k if e \in (V_i, W_j) \\ 4j if e \in (V_i, V - W_j) \end{matrix}\right\}$, apart from these edges let ϕ_2 agree with ϕ_1.Now we may elaborate ϕ_2 to a colouring ϕ_3 of $W \cup E$ by setting $\phi_3(v) = i$ for $v \in V_i$,i=4k+1,…..,p.

To see that the edges coloured 4k by ϕ_3 from a matching note that the sets of vertices W_k, ($V_{4k+j} \cup W_j$) j=1,…k-1 are pair wise disjoint. Also for each j=1,…k-1 the edges coloured 4j form a matching, since the sets of vertices W_j and ($V_{4k+j} \cup$ V-W_j) are disjoint. Observe that ϕ_3 uses at most $\chi' + 2k$ colours.

Finally, we write p=5k-1+r, where k $\ge$ 0 and 0≤r≤4. From th e abo v ecase and lemma 1.2 it follo ws that there is a colouring, as desired, with number of colours at most $\chi' + 2k + \min(r,2)$,this is equals to $\chi' + \frac{2}{5}p + t(p)$ for all p>1.

4.5 Note: In this lemma we put W=V we obtain our theorem4.1.

By Brooks theorem, $\chi(G) \le \Delta$ for any connected graph G which is not complete or an odd cycle hence, using also Vizing's theorem we get the following corollary

4.6. Corollary: For any simple graph, $\chi''_Q(G) \le \left\lceil \frac{7}{5}\Delta + 3 \right\rceil$.

Proof: In a simple graph there is at most one edge between every pair of vertices .Hence from theorem4.1, we get $\chi''_Q(G) \le \{\max(p,\chi') + \frac{2}{5}\min(p,\chi') + \frac{8}{5}\}$

$$\leq \chi' + \frac{2}{5}\chi + \frac{8}{5}$$

$$\leq (\Delta+1) + \frac{2}{5}\Delta + \frac{8}{5}$$

$$= \frac{7}{5}\Delta + \frac{13}{5} \leq \left\lceil \frac{7}{5}\Delta + 3 \right\rceil.$$

5. CONCLUSION

This paper, presents a new concept of 1-quasi total colouring and this chromatic number accomplishes new upper bounds as 2+ $\chi_l'(G)$ using list edge chromatic numbers Δ +8log^8 Δ ,and $\left\lceil \frac{7}{5}\Delta + 3 \right\rceil$ to 1-quasi total graphs. Also this manuscript contributes a polynomial time algorithm for the upper bound Δ +8log^8 Δ .

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