

## SOME SPECIAL TYPES OF SQUARE DIFFERENCE GRAPHS

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### ABSTRACT

I defined a new labeling and a new graph called square difference labeling and the square difference graph. Let  $G$  be a  $(p, q)$  graph.  $G$  is said to be a square difference graph if there exists a bijection  $f: V(G) \rightarrow \{0, 1, \dots, p-1\}$  such that the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  given by  $f^*(uv) = |f(u)^2 - f(v)^2|$  for every  $uv \in E(G)$  are all distinct. A graph which admits square difference labeling is called square difference graph. I discussed the square difference labeling for some graphs like cycles, complete graphs, cycle cactus, ladder, lattice grids and wheels. In this paper my discussion is some graphs like  $mK_3$ ,  $mC_n$ , path union of some  $K_3$ , some  $C_n$  graphs and duplication of vertices by an edge to some star graphs and crown graphs are square difference graphs. Also proved a path is an Odd square difference graph and star is a perfect square difference graph.

**Keywords:** Square difference labeling, square difference graph, Odd square difference graph, Perfect square graph, Path union of  $C_m$  the comb graph.

### INTRODUCTION

We begin with simple, finite, connected and undirected graph  $G = (V(G), E(G))$  with  $p$  vertices and  $q$  edges. For all other terminology and notations I follow Harary [2]. Here brief summary of definitions which is used for the present investigations are given.

If the vertices of the graph are assigned values subject to certain conditions is known as **graph labeling**. A dynamic survey on graph labeling is regularly updated by Gallian [3] and it is published by Electronic Journal of Combinatory. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in past three decades.

Previously K.A. Germina, S. Arumugam and V. Ajitha defined Square sum labeling [1] and myself proved square difference labeling for some graphs [9] and also some references [5], [6] and [7] are used. Some concepts are used from B. West [8] and Gary Chartrand, Ping Zhang [4]

**Definition 1.1:** Let  $G = (V(G), E(G))$  be a graph.  $G$  is said to be square difference labeling if there exist a bijection  $f: V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$  such that the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  given by

$$f^*(uv) = |f(u)^2 - f(v)^2| \text{ is injective.}$$

**Definition 1.2:** A square difference graph with weights as odd numbers are called Odd square difference graph.

**Definition 1.3:** Let  $G_1, G_2, \dots, G_n$ ,  $n \geq 2$  be  $n$  copies of a fixed graph  $G$ . The graph  $G$  obtained by adding an edge between  $G_i$  and  $G_{i+1}$  for  $i = 1, 2, \dots, n-1$  is called path union of  $G$ .

**Definition 1.4:** A square difference graph with weights as perfect square numbers are called Perfect square difference graph.

**Definition 1.5:** Let  $G$  be a graph with fixed vertex  $v$ . The comb of  $G$  is the graph  $(P_m; G)$  obtained from  $m$  copies of  $G$  and the path  $P_m: u_1, u_2, \dots, u_m$  by joining  $u_i$  with the vertex  $v$  of the  $i^{\text{th}}$  copy of  $G$  by means of an edge, for  $1 \leq i \leq m$ .

**Definition 1.6:** Duplication of a vertex  $v_k$  by an edge  $e = v_k'v_k''$  in a graph  $G$  produces a new graph  $G_1$  such that  $N(v_k') \cap N(v_k'') = v_k$

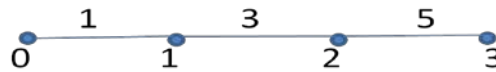
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## 2. MAIN RESULTS

**Theorem 2.1:** The path  $P_n$  is an Odd square difference graph.

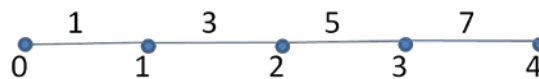
**Proof:** Let the graph  $G$  be a path  $P_n$ . Let  $|V(G)| = n$  and  $|E(G)| = n-1$ . The mapping  $f: V(G) \rightarrow \{0, 1, \dots, n-1\}$  is defined by  $f(u_i) = i-1$ ,  $1 \leq i \leq n-1$  and the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  defined by  $f^*(xy) = |[f(x)]^2 - [f(y)]^2|$ . The edges of the path have distinct values in increasing order hence a one to one mapping. In this labeling one interesting thing is the weights are odd numbers, starting from 1 it goes on increasing as 1, 3, 5, ...  $2n-3$ . Hence this graph can also be called as Odd square difference graph.

**Example: 2.2:** The path  $P_4$  is an Odd square difference graph.



**Fig-1:** Odd square difference graph of the path  $P_4$

**Example: 2.3:** The path  $P_5$  is an Odd square difference graph.

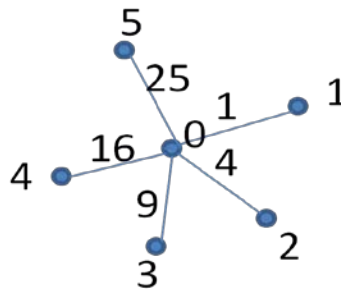


**Fig-2:** Odd square difference graph of the path  $P_5$

**Theorem 2.4:** Star graphs  $K_{1,n}$  are perfect square difference graphs.

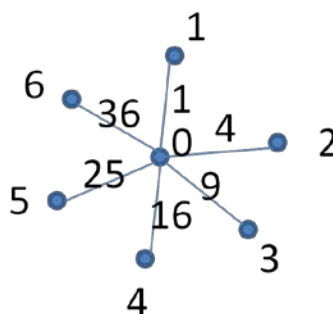
**Proof:** Let  $v_1, v_2, \dots, v_n$  be the pendent vertices of the star graph  $K_{1,n}$  and  $c$  be the apex vertex. Let  $|V(G)| = n+1$  and  $|E(G)| = n$ . The mapping  $f: V(G) \rightarrow \{0, 1, \dots, n\}$  is defined by  $f(u_i) = i$ ,  $1 \leq i \leq n$  and  $f(c) = 0$ ,  $1 \leq i \leq n-1$  and the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  defined by  $f^*(xy) = |[f(x)]^2 - [f(y)]^2|$  is an increasing function and also one to one. The weights of the edges are distinct also perfect squares as 1, 4, 9, 16, ...  $n^2$ . Hence the Star graphs are perfect square difference graphs.

**Example: 2.5:** The Star graphs  $K_{1,5}$  is a perfect square difference graph.



**Fig-3:** perfect square difference graph of the Star graphs  $K_{1,5}$

**Example: 2.6:** The Star graphs  $K_{1,6}$  is a perfect square difference graph.

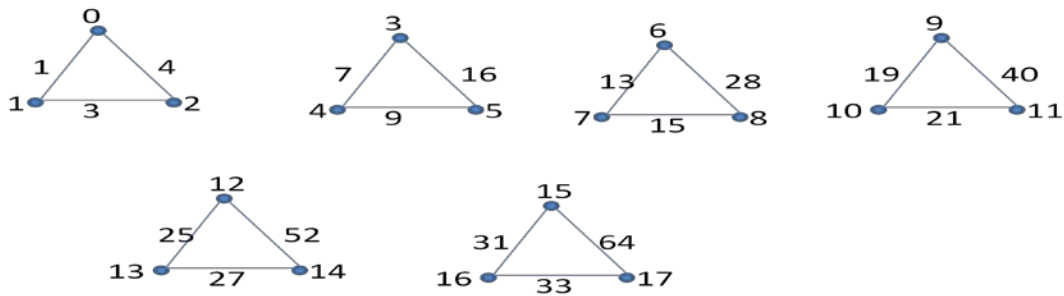


**Fig-4:** perfect square difference graph of the Star graphs  $K_{1,6}$

**Theorem 2.6:**  $mK_3$  is a square difference graph.

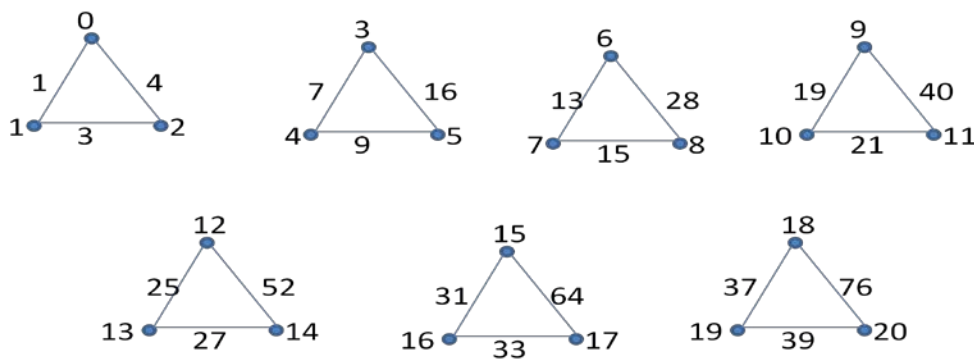
**Proof:** Let  $G_1, G_2, \dots, G_n$  be  $m$  copies of  $K_3$ . Let  $v_1, v_2, v_3$  be the vertices of  $G_1$ ,  $v_4, v_5, v_6$  be the vertices of  $G_2$ ,  $v_7, v_8, v_9$  be the vertices of  $G_3 \dots, v_{3n-2}, v_{3n-1}, v_{3n}$  be the vertices of  $G_n$ . Let  $|V(G)| = 3n$ ,  $|E(G)| = 3n$  and let us define  $f: V(G) \rightarrow \{0, 1, \dots, 3n-1\}$  by  $f(u_i) = i-1$ ,  $1 \leq i \leq 3n$  and the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  defined by  $f^*(xy) = |[f(x)]^2 - [f(y)]^2|$  is an increasing function, hence  $f^*$  is an one to one function. Each edges having distinct weights such that  $f^*(e_i) \neq f^*(e_j)$  for every  $e_i \neq e_j$ . Here one interesting factor is in each  $G_i$ ,  $i = 1, 2, \dots, n$ . is the weight of one edge of  $K_3$  is equal to sum of the other two.. Also the weights of one edge of  $K_3$  is  $(m+3)$  multiple of 4,  $m > 0$ . This can be verified in the examples 2.7 and 2.8

**Example 2.7:** The graph  $6K_3$  is a square difference graph.



**Fig-5:** square difference graph of  $6K_3$

**Example 2.8:** The graph  $7K_3$  is a square difference graph.

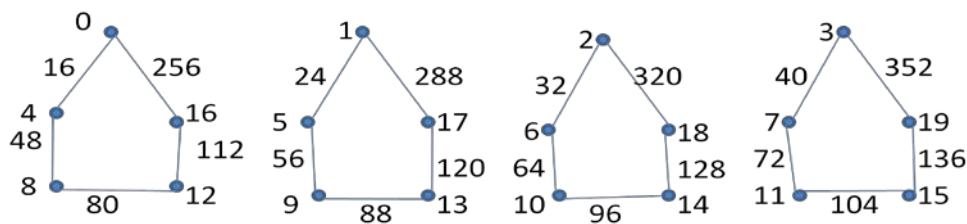


**Fig-6:** square difference graph of  $7K_3$

**Theorem 2.9:**  $mC_5$  is a square difference graph.

**Proof:** Let  $G_1, G_2, \dots, G_n$  be  $m$  copies of  $C_5$ . Let  $v_1, v_2, v_3, v_4, v_5$  be the vertices of  $G_1$ ,  $v_6, v_7, v_8, v_9, v_{10}$  be the vertices of  $G_2$ ,  $\dots, v_{mn-4}, v_{mn-3}, v_{mn-2}, v_{mn-1}, v_{mn}$  be the vertices of  $G_n$ . Let  $|V(G)| = mn$ ,  $|E(G)| = mn$  and let us define  $f: V(G) \rightarrow \{0, 1, \dots, mn-1\}$  by  $f(u_i) = i-1$ ,  $1 \leq i \leq mn$  and the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  defined by  $f^*(xy) = |[f(x)]^2 - [f(y)]^2|$  is an increasing function, hence  $f^*$  is an one to one function. Each edges having distinct weights such that  $f^*(e_i) \neq f^*(e_j)$  for every  $e_i \neq e_j$ . Hence  $mC_5$  is a square difference graph.

**Example 2.10:** The graph  $4C_5$  is a square difference graph.

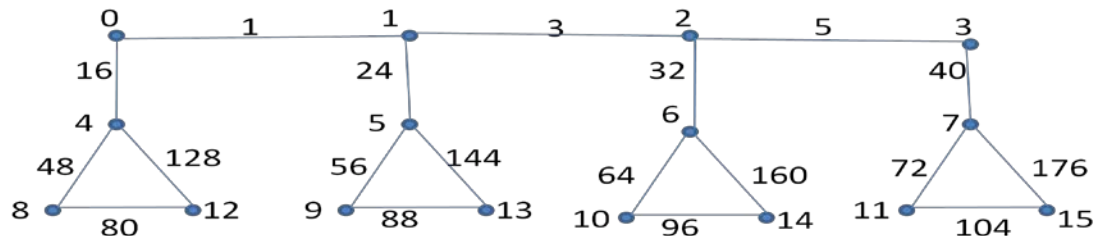


**Fig-7:** square difference graph of  $4C_5$

**Theorem 2.11:** The comb of the cycle is the graph  $(P_m; C_n)$  is a square difference graph.

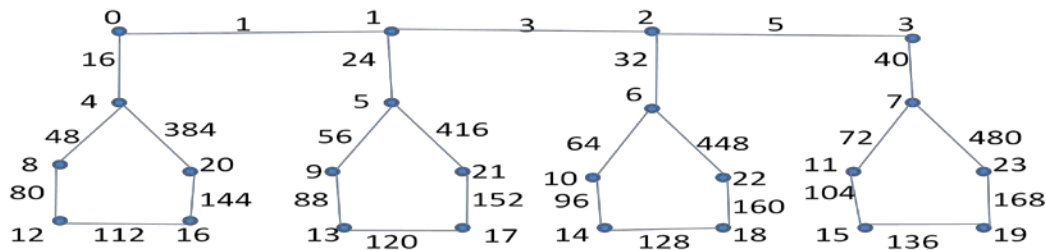
**Proof:** Let  $G_1, G_2, \dots, G_n$  be  $n$  copies of  $C_3$  and  $P_m: u_1, u_2, \dots, u_n$  be a path. The comb graph  $(P_m, C_n)$  has  $(m + mn)$  vertices and  $(mn + 2n + 1)$  edges. Let us define  $f: V(G) \rightarrow \{0, 1, 2, \dots, (m + mn) - 1\}$  by  $f(u_i) = i - 1, 1 \leq i \leq (m + mn) - 1$  such that the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  given by  $f^*(uv) = |[f(u)]^2 - [f(v)]^2|$  is injective and the images are distinct. This graph is the combination of the above two graphs one is odd square difference graph and the sum of two edges of  $C_3$  is equal to the third edge. The edges of  $C_3$  and the edge connecting the path and  $C_3$ 's are the multiples of  $m + n + 1$ .

**Example: 2.10:** The graph  $(P_4; C_3)$  is a square difference graph.



**Fig-8:** square difference graph  $(P_4; C_3)$

**Example: 2.11:** The graph  $(P_4; C_5)$  is a square difference graph.

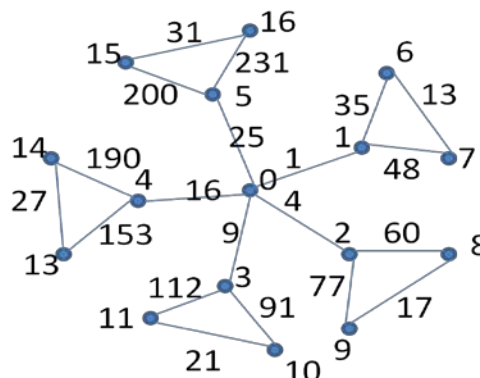


**Fig-9:** square difference graph  $(P_4; C_5)$

**Theorem 2.12:** The graph obtained by duplication of an arbitrary vertex by a new edge in star  $K_{1,n}$  admits a square difference labeling.

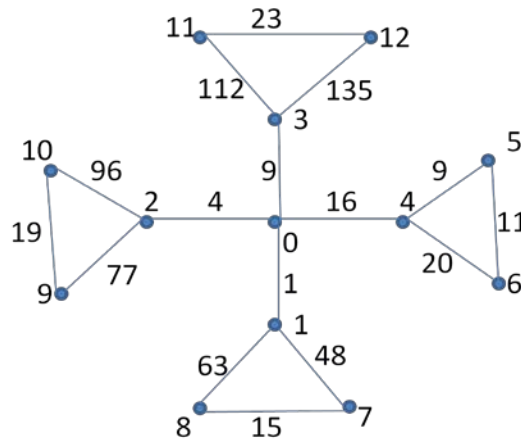
**Proof:** Let  $v_1, v_2, \dots, v_n$  be the pendent vertices of the star graph  $K_{1,n}$  and  $c$  be the apex vertex. Let  $G$  be the graph obtained by duplicating an arbitrary vertex  $v_1$  by a new edge  $e = v_1' v_1''$ . Let  $|V(G)| = 3n+1$ ,  $|E(G)| = 4n$  and define  $f: V(G) \rightarrow \{0, 1, \dots, 3n+1\}$  by  $f(u_i) = i$ ,  $1 \leq i \leq 3n$  and  $f(c) = 0$  and the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  defined by  $f^*(xy) = |f(x)|^2 - |f(y)|^2$  is an increasing function. i.e  $f^*(e_1) < f^*(e_2) < \dots < f^*(e_{4n})$ . The weights of all the edges of the graph  $G$  are distinct. Hence the graph  $G$  admits square difference labeling. Therefore  $G$  is a square difference graph. This graph is the combination of perfect square difference graph and sum of two sides of  $K_3$  is equal to the third side.

**Example: 2.13:** The graph obtained by duplication of an arbitrary vertex by a new edge in star  $K_{1,5}$  is a square difference graph.



**Fig-10:** Duplication of a vertex by an edge in a graph  $K_{1,5}$  and its square difference labeling

**Example: 2.14:** The graph obtained by duplication of an arbitrary vertex by a new edge in star  $K_{1,4}$  is a square difference graph.

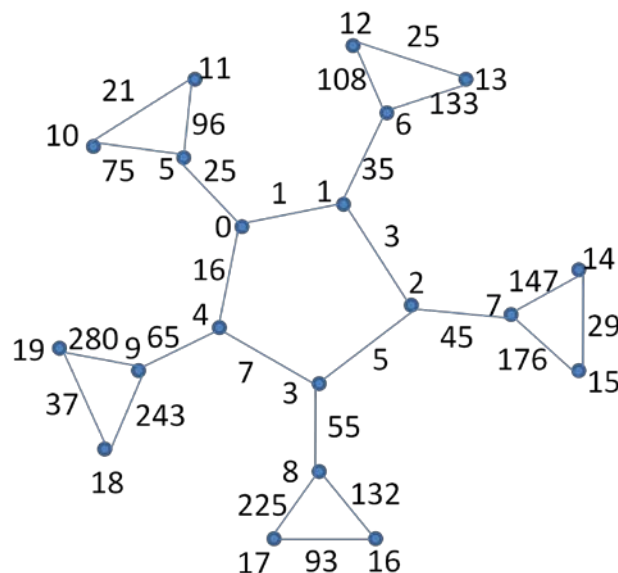


**Fig-11:** Duplication of a vertex by an edge in a graph  $K_{1,4}$  and its square difference labeling

**Theorem 2.15:** The graph obtained by duplication of an arbitrary vertex by a new edge in a crown graph  $C_n \odot K_1$  admits a square difference labeling.

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the crown graph  $C_n \odot K_1$ . Let  $G$  be the graph obtained by duplicating an arbitrary vertex  $v_1$  by a new edge  $e = v_1' v_1''$ . Let  $|V(G)| = 4n$ ,  $|E(G)| = 5n$  and define  $f: V(G) \rightarrow \{0, 1, \dots, 4n-1\}$  by  $f(u_i) = i$ ,  $1 \leq i \leq 4n-1$  and the induced function  $f^*: E(G) \rightarrow \mathbb{N}$  defined by  $f^*(xy) = |f(x)^2 - f(y)^2|$  is an increasing function. i.e  $f^*(e_1) < f^*(e_2) < \dots < f^*(e_{5n})$ . The weights of all the edges of the graph  $G$  are distinct. Hence the crown graph  $C_n \odot K_1$  admits a square difference labeling.

**Example: 2.16:** The graph obtained by duplication of an arbitrary vertex by a new edge in a crown graph admits a square difference graph.



**Fig-12:** Duplication of a vertex by an edge in a graph  $C_5 \odot K_1$  and its square difference labeling

In this paper some special types of Square difference graphs are discussed. Duplication of a vertex by an edge generates some new graphs. And I proved some graphs generated using duplication concepts are Square difference graphs. Similarly one can verify this Square difference labeling for many more graphs. Some other labeling is also verified for these graphs which I used in this paper.

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