A UNIQUENESS RESULT RELATED TO CERTAIN NON-LINEAR DIFFERENTIAL POLYNOMIALS

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(Received on: 02-04-12; Accepted on: 19-04-12)

ABSTRACT

In this paper, we deal with some uniqueness question of meromorphic functions whose certain non-linear differential polynomials have a nonzero finite value, and obtain some results, which improve and generalize the related results due to I. Lahiri and R. Pal[4], X. M. Li and H. X. Yi[6] and A. Banerjee and P. Bhattacharjee[1].

1. INTRODUCTION

In this paper, by meromorphic function we will always mean meromorphic function in complex plane. We adopt the standard notations of Nevanlinna theory of meromorphic function as explained in [2], [7] and [8]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by T(r, h) the Nevanlinna characteristic of h and by S(r, h) any quantity satisfying S(r, h)=o{T(r, h)}, as r→∞ and r∈E.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share a CM, provided that f and g have the same a-points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [8]). We say that a is a small function of f, if a is a meromorphic function satisfying T(r, a) = S(r, f) (see [8]). Let l be a positive integer or ∞. Next we denote by E₁₁(a; f) the set of those a-points of f in the complex plane, where each point is of multiplicity ≤ l and counted according to its multiplicity. By E₂₂(a; f) we denote the reduced form of E₁₁(a; f). If E₂₂(a; f) = E₂₂(a; g), we say that a is a l-order pseudo common value of f and g (see[3]).

Obviously, if E₃₃(a; f) = E₃₃(a; g) (E₄₄(a; f) = E₄₄(a; g)), resp. then f and g share a CM (IM, resp.).


Theorem A: Let f and g be two non-constant meromorphic functions, and let n ≥ 14 be positive integer.

If E₃₃(1; fₙ( f³ − 1) f) = E₃₃(1; gₙ( g³ − 1) g), then f ≡ g.

Theorem B: Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying n>3k+11 and max {χ₁, χ₂} < 0, where

χ₁ = \[ \frac{2}{n - 2k + 1} + \frac{2}{n + 2k + 1} + \frac{2k + 1}{n + k + 1} + 1 - \theta₁(1, f) - \theta₁(1, g) \]

and

χ₂ = \[ \frac{2}{n - 2k + 1} + \frac{2}{n + 2k + 1} + \frac{2k + 1}{n + k + 1} + 1 - \theta₁(1, g) - \theta₁(1, g) \]

If 0 > 2/n and if \( fₙ( f - 1) \)\(^{(k)}\) = P and \( gₙ( g - 1) \)\(^{(k)}\) = P share 0CM, where P is nonzero polynomial, then f ≡ g.

Theorem C: Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying n>9k+20 and where max {χ₁, χ₂} < 0, where χ₁, χ₂ are defined as in Theorem B.

If 0 > 2/n and if \( fₙ( f - 1) \)\(^{(k)}\) = P and \( gₙ( g - 1) \)\(^{(k)}\) = P share 0 IM, where P is nonzero polynomial, then f ≡ g.
In 2011, A. Banerjee and P. Bhattacharjee [1] proved the following theorem.

**Theorem D:** Let \( f \) and \( g \) be two transcendental meromorphic functions, and let \( n, k \geq 1 \) and \( m \geq 2 \) be three positive integers. Suppose for two nonzero constants \( a \) and \( b \), \( E_{ij}(1;\{f^n(a f^m + b)\}^{(k)}) = E_{ij}(1;\{g^n(a g^m + b)\}^{(k)}) \). Then \( f \equiv g \) or \( f \equiv -g \) or \( \{f^n(a f^m + b)\}^{(k)} \{g^n(a g^m + b)\}^{(k)} \equiv 1 \) provided one of the following holds:

(i) when \( l \geq 3 \) and \( n > 3k+m+8 \);
(ii) when \( l = 2 \) and \( n > 4k+\frac{3m}{2}+9 \);
(iii) when \( l = 1 \) and \( n > 7k+3m+12 \).

When \( k=1 \) the possibility \( \{f^n(a f^m + b)\}^{(k)} \{g^n(a g^m + b)\}^{(k)} \equiv 1 \) does not occur. Also the possibility \( f \equiv -g \) arises only if \( n \) and \( m \) are both even.

**Question:** What can be said about the relationship between two meromorphic functions \( f \) and \( g \), if the condition \( E_{ij}(1;\{f^n(a f^m + b)\}^{(k)}) = E_{ij}(1;\{g^n(a g^m + b)\}^{(k)}) \) in Theorem B is replaced with the condition \( E_{ij}(1;\{f^n(a f^m + b)\}^{(k)}) = E_{ij}(1;\{g^n(a g^m + b)\}^{(k)}) \), where \( l \geq 3 \) is an integer, then either \( f \equiv g \) or \( f \equiv -g \) or \( \{f^n(a f^m + b)\}^{(k)} \{g^n(a g^m + b)\}^{(k)} \equiv 1 \).

The possibility \( \{f^n(a f^m + b)\}^{(k)} \{g^n(a g^m + b)\}^{(k)} \equiv 1 \) does not arise for \( k=1 \) and the possibility \( f \equiv -g \) does not arise if \( n \) and \( m \) are both odd or if \( n \) is even and \( m \) is odd or if \( n \) is odd and \( m \) is even.

**Theorem 1.2:** Let \( f \) and \( g \) be two transcendental meromorphic functions, and let \( n, k \geq 1 \) and \( m \geq 2 \) be three positive integers with \( n > \frac{2k+m+6}{3} \) and \( a \) and \( b \) be nonzero constants. If \( E_1(1;\{f^n(a f^m + b)\}^{(k)}) = E_1(1;\{g^n(a g^m + b)\}^{(k)}) \) and \( E_2(1;\{f^n(a f^m + b)\}^{(k)}) = E_2(1;\{g^n(a g^m + b)\}^{(k)}) \), where \( l \geq 4 \) is an integer, then the conclusions of Theorem 1.1 still holds.

**Remark 1:** Theorem 1.2 is an improvement of Theorem A and Theorem D.

**Remark 2:** Theorem 1.2 is an improvement of Theorem C for \( m = 1 \), \( a = 1 \) and \( b = -1 \).

2. LEMMAS

In this section, we present some lemmas which are needed in the sequel.

**Lemma 2.1:** ([7]) Let \( f \) be a nonconstant meromorphic function and

\[ P(f) = a_0 + a_1 f + \cdots + a_n f^n, \]

where \( a_0, a_1, \ldots, a_n \) are constants and \( a_n \neq 0 \). Then

\[ T(r, P(f)) = nT(r, f) + S(r, f). \]

**Lemma 2.2:** ([5]) Let \( E_1(1;\{F^*\}^{(k)}) = E_{ij}(1;\{G^*\}^{(k)}) \), \( E_1(1;\{F^*\}^{(k)}) = E_1(1;\{G^*\}^{(k)}) \) and \( H^* \neq 0 \), where \( l \geq 3 \).

Then

\[ T(r, F^*) \leq \frac{k + 2k}{3} N(r, \infty; F^*) + \frac{5}{3} N(r, 0; F^*) + \frac{2}{3} N_h(r, \infty; G^*) + N_{k+1}(r, \infty; F^*) \]

\[ + (k + 2)N(r, 0; F^*) + N(r, 0; G^*) + N_{k+1}(r, 0; G^*) + S(r, F^*) + S(r, G^*) \]

Where

\[ H^* := \left[ \frac{(F^*)^{(k+2)}}{(F^*)^{(k+1)}} - \frac{2(F^*)^{(k+1)}}{(F^*)^{(k)}} \right] - \left[ \frac{(G^*)^{(k+2)}}{(G^*)^{(k+1)}} - \frac{2(G^*)^{(k+1)}}{(G^*)^{(k)}} \right]. \]
Lemma 2.3: \((5\)\) Let \(E_{ij}(1; [F*]^{(k)}) = E_{ij}(1; [G*]^{(k)})\) and \(E_{ij}(1; [F*]^{(k)}) = E_{ij}(1; [G*]^{(k)})\), where \(l \geq 3\).

If \(\Delta_{ij} = \left(\frac{8}{3} + \frac{2}{3}k\right)\theta(\infty, F*) + (k + 2)\theta(\infty, G*) + \frac{5}{3}\theta(0, F*) + \theta(0, G*) + \delta_{k+1}(0, F*) + \delta_{k+1}(0, G*) + \frac{2}{3}\delta_{k}(0, F*)\)

\(\Delta_{ij} > \frac{5}{3}k + 9\), then either \([F*]^{(k)}[G*]^{(k)} \equiv 1\) or \(F* = G*\).

Lemma 2.4: \((5\)\) Let \(E_{ij}(1; [F*]^{(k)}) = E_{ij}(1; [G*]^{(k)})\), \(E_{2j}(1; [F*]^{(k)}) = E_{2j}(1; [G*]^{(k)})\) and \(H* \neq 0\), where \(l \geq 4\).

Then

\[T(r, F*) + T(r, G*) \leq (k + 4)N(r, \infty; F*) + 2N(r, 0; F*) + 2N_{k+1}(r, \alpha; F*) + \frac{2}{3}\delta_{k}(0, F*) \]

Where \(H*\) is defined as Lemma 2.2.

Lemma 2.5: \((5\)\) Let \(E_{ij}(1; [F*]^{(k)}) = E_{ij}(1; [G*]^{(k)})\) and \(E_{2j}(1; [F*]^{(k)}) = E_{2j}(1; [G*]^{(k)})\), where \(l \geq 4\).

If \(\Delta_{2i} = \left(2 + \frac{1}{2}k\right)\theta(\infty, F*) + \left(\frac{k}{2} + 2\right)\theta(\infty, G*) + \theta(0, F*) + \theta(0, G*) + \delta_{k+1}(0, F*) + \delta_{k+1}(0, G*)\)

\(\Delta_{2i} > k + 5\), then either \([F*]^{(k)}[G*]^{(k)} \equiv 1\) or \(F* = G*\).

Lemma 2.6: \((1\)\) Let \(f\) and \(g\) be two nonconstant meromorphic functions and \(a\) and \(b\) be nonzero constants. Then \([f^n(af^m + b)]^1[g^n(ag^m + b)]^1 \neq 1\), where \(n, m \geq 2\) be two positive integers and \(n \geq (m+3)\).

3. PROOF OF THE THEOREM

Proof of Theorem 1.1: Let \(F* = f^n(af^m + b)\), \(G* = g^n(ag^m + b)\).

By Lemma 2.1, we get

\[(1.1) \quad \theta(0, F*) = 1 - \lim_{r \to \infty} \sup \frac{N(r, 0; F*)}{T(r, F*)} \geq \frac{n-1}{n+m}\]

Similarly

\[(1.2) \quad \theta(0, G*) \geq \frac{n-1}{n+m}\]

\[(1.3) \quad \theta(\infty, F*) = 1 - \lim_{r \to \infty} \sup \frac{N(r, \infty; F*)}{T(r, F*)} \geq \frac{n+m-1}{n+m}\]

Similarly

\[(1.4) \quad \theta(\infty, G*) \geq \frac{n+m-1}{n+m}\]

\[(1.5) \quad \delta_{k+1}(0, F*) = 1 - \lim_{r \to \infty} \sup \frac{N_{k+1}(r, 0; F*)}{T(r, F*)} \geq \frac{n-k-1}{n+m}\]

Similarly

\[(1.6) \quad \delta_{k+1}(0, G*) \geq \frac{n-k-1}{n+m}, \quad \delta_{k}(0, F*) \geq \frac{n-k}{n+m}, \text{ and } \delta_{k}(0, G*) \geq \frac{n-k}{n+m}\]

From the condition of Theorem 1.1, we have

\[E_{ij}(1; [f^n(af^m + b)]^{(k)}) = E_{ij}(1; [g^n(ag^m + b)]^{(k)}) \quad \text{and} \quad E_{ij}(1; [f^n(af^m + b)]^{(k)}) = E_{ij}(1; [g^n(ag^m + b)]^{(k)}),\]

where \(l \geq 3\).
From (3.1) - (3.6) and Lemma 2.3, we have
\[ \Delta_{1l} = \left( \frac{14}{3} + \frac{5}{3}k \right) \frac{n + m - 1}{n + m} + \frac{8n - 1}{3n + m} + 2 \frac{n - k - 1}{n + m} + \frac{2n - k}{3n + m} \]

It is easily verified that if \( n > \frac{13k + 13m + 28}{3} \), then \( \Delta_{1l} > \frac{5}{3}k + 9 \). So by Lemma 2.3, we have \( [F^*]^{(k)}[G^*]^{(k)} \equiv 1 \) or \( F^* \equiv G^* \). Also by Lemma 2.6 the case \( [F^*]^{(k)}[G^*]^{(k)} \equiv 1 \) does not arise for \( k = 1 \) and \( m \geq 2 \).

Let \( F^* \equiv G^* \), i.e.,
\[ f^n(a^m + b) \equiv g^n(a^m + b) \]

Clearly if \( n \) and \( m \) are both odd or if \( n \) is even and \( m \) is odd or if \( n \) is odd and \( m \) is even, then \( f \equiv -g \) contradicts \( F^* \equiv G^* \). Let neither \( f \equiv g \) nor \( f \equiv -g \). We put \( h = \frac{g}{f} \). Then \( h \neq 1 \) and \( h \neq -1 \). Also \( F^* \equiv G^* \) implies
\[ f^{m - 1} = \frac{b}{a} \frac{h^{n-1}}{h^{n+m-1}}. \]

Since \( f \) is non-constant it follows that \( h \) is non-constant. Again since \( f^{m} \) has no simple pole \( -u_r \) has no simple zero, where \( u_r = \exp \left( \frac{2\pi ir}{n+m} \right) \) and \( r = 1, 2...n+m-1 \). Therefore either \( f \equiv g \) or \( f \equiv -g \). This proves the theorem.

**Proof of Theorem 1.2:** From the condition of Theorem 1.2,
we have \( E_{1l}(1; [f^n(a^m + b)]^{(k)}) = E_{1l}(1; [g^n(a^m + b)]^{(k)}) \)
and \( E_{2l}(1; [f^n(a^m + b)]^{(k)}) = E_{2l}(1; [g^n(a^m + b)]^{(k)}) \)
where \( l \geq 4 \).

From (3.1)-(3.6) and Lemma 2.5, we have
\[ \Delta_{2l} = (k + 4) \frac{n + m - 1}{n + m} + 2 \frac{n - 1}{n + m} + 2 \frac{n - k - 1}{n + m} \]

It is easily verified that if \( n > \frac{3k + m + 8}{3} \), then \( \Delta_{2l} > k + 5 \). So by Lemma 2.5, we have \( [F^*]^{(k)}[G^*]^{(k)} \equiv 1 \) or \( F^* \equiv G^* \).

Proceeding as in the proof of Theorem 1.1, we can get the conclusion of Theorem 1.2. Thus, we complete the proof of Theorem 1.2.

**REFERENCES**


Source of support: Nil, Conflict of interest: None Declared