A NOTE ON SĂLĂGEAN CARLSON-SHAFFER OPERATOR

L. Dileep* and S. Latha**

*Department of mathematics Vidyavardhaka College of Engineering Mysore
  Email: dileep84@gmail.com
**Department of mathematics Yuvaraja’s College University of Mysore, Mysore
  Email: drlatha@gmail.com

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ABSTRACT

In the present work, using Sălăgean and Carlson-Shafer operator we introduce a linear operator \( SL_\lambda \). The objective is to define the classes \( VS_\lambda^\alpha(a, c, n, \beta) \) and \( VS_\lambda^\alpha(a, c, n) \) using the above linear operator and for functions belonging to these classes we obtain coefficient estimates and many more properties like extreme points, integral means, unified radii results etc.

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1. INTRODUCTION:

Let \( U = \{ z \in C : |z| < 1 \} \) be the open unit disk and \( A \) denote the class of functions normalized by

\[
f(z) = z + \sum_{m=2}^{\infty} a_m z^m
\]

which are analytic in the open unit disk \( U \) satisfying the conditions \( f(0) = f'(0) - 1 = 0 \).

The class \( A \) is closed under the convolution or Hadamard product

\[
(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m, \quad z \in U,
\]

where \( f \) is given by (1.1) and \( g(z) = z + \sum_{m=2}^{\infty} b_m z^m \).

For \( n \in N_0, \lambda \geq 0, a, c \in R \setminus Z \), we introduce a linear operator \( SL_\lambda : A \to A \) defined by

\[
SL_\lambda f(z) = (1 - \lambda)[(k * k * \cdots * k) * f](z) + \lambda [\phi(a,c) * f](z), \quad z \in U,
\]

where \( k(z) = z(1-z)^{-2} \) is Koebe function and

\[
\phi(a,c;z) = \sum_{m=2}^{\infty} \frac{(a)_m (c)_m}{(m-1)!} z^m, \quad |z| < 1, a, c \neq 0,-1,-2, \ldots,
\]

is the incomplete beta function.
For functions \( f \in A \) of the form (1.1), we have
\[
SL_\lambda f(z) = z + \sum_{m=2}^{\infty} B_\lambda(a,c,m,n)a_m z^m,
\]
(1.4)
where
\[
B_\lambda(a,c,m,n) = \left[(1-\lambda)m^n + \lambda(c)_{m-1} \right].
\]
(1.5)
Here \( (a)_m \) is the Pochhammer symbol defined in terms of the Gamma function by
\[
(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1, & \text{for } m = 0 \\ a(a+1)(a+2) \cdots (a+m-1), & \text{for } m \in \mathbb{N}. \end{cases}
\]

Now using the linear operator \( SL_\lambda \) we define the class \( SL_\lambda^\alpha(a,c,n) \) consisting functions of the form (1.1) satisfying the condition
\[
R\left\{ \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - \alpha \right\} > \left| \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right|\alpha.
\]
(1.6)

Silverman [9] defined the class \( V(\theta_m) \) as the class of all functions in \( A \) such that \( \arg a_m = \theta_m \) for all \( m \). If further there exists a real number \( t \) such that \( \theta_m + (m-1)t = \pi(\mod 2\pi) \), then \( f \) is said to be in the class \( V(\theta_m, t) \). The union of \( V(\theta_m, t) \) taken overall possible sequences \( \{\theta_m\} \) and all possible real numbers \( t \) is denoted by \( V \).

Further, we define \( VS_\lambda^\alpha(a,c,n,\beta) = S_\lambda^\alpha(a,c,n,\beta) \cap V \).

**Definition 1.1:** A function \( f \in V \) of the form (1.1) is in \( VS_\lambda^\alpha(a,c,n,\beta) \) if \( f \) satisfies the analytic condition
\[
R\left\{ \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} \right\} > \beta \left| \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right| + \alpha,
\]
(1.7)
where \( \alpha, \beta \geq 0 \) and \( z \in U \).

These classes stem essentially from the classes studied earlier by Vijaya and Murugusundaramoorthy [10].

2. MAIN RESULTS

**Theorem 2.1:** A function \( f \) of the form (1.1) is in \( VS_\lambda^\alpha(a,c,n) \) if and only if
\[
\sum_{m=2}^{\infty} (2m-1-\alpha)B_\lambda(a,c,m,n)a_m \leq 1 - \alpha.
\]
(2.1)

**Proof:** From (1.6), it suffices to show that
\[
\left| \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right| \leq R\left\{ \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - \alpha \right\}.
\]
That is
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\[
\left| \frac{z (SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right| - R \left( \left| \frac{z (SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right| \right) \\
\leq 2 \left| \frac{z (SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right| \\
\leq 2 \sum_{m=2}^{\infty} (m-1) B_\lambda (a,c,m,n) \left| a_m \right| z^{-m-1} \\
1 - \sum_{m=2}^{\infty} B_\lambda (a,c,m,n) \left| a_m \right| z^{-m-1}.
\]

Now the last expression is bounded by \((1 - \alpha)\) if

\[
\sum_{m=2}^{\infty} (2m-1-\alpha) B_\lambda (a,c,m,n) \left| a_m \right| \leq 1 - \alpha.
\]

Conversely, if \(f \in VS_\lambda^\alpha (a,c,n)\) then by definition

\[
\left| \frac{z + \sum_{m=2}^{\infty} m B_\lambda (a,c,m,n) a_m z^{-m}}{z + \sum_{m=2}^{\infty} B_\lambda (a,c,m,n) a_m z^{-m}} - 1 \right| \leq R \left( \left| \frac{z + \sum_{m=2}^{\infty} m B_\lambda (a,c,m,n) a_m z^{-m}}{z + \sum_{m=2}^{\infty} B_\lambda (a,c,m,n) a_m z^{-m}} - \alpha \right| \right).
\]

That is

\[
\sum_{m=2}^{\infty} (m-1) B_\lambda (a,c,m,n) a_m z^{-m-1} \leq R \left( \left| \frac{1 - \alpha + \sum_{m=2}^{\infty} (m-\alpha) B_\lambda (a,c,m,n) a_m z^{-m-1}}{1 + \sum_{m=2}^{\infty} B_\lambda (a,c,m,n) a_m z^{-m-1}} \right| \right).
\]

Since \(f \in V\) and \(f\) lies in \(V(\theta_m, t)\) for some sequence \(\theta_m\) and a real number \(t\) such that \(\theta_m + (m-1)t \equiv \pi (\mod 2\pi)\) set \(z = re^{i\theta}\) in the above inequality

\[
\sum_{m=2}^{\infty} (m-1) B_\lambda (a,c,m,n) a_m r^{-m-1} \leq R \left( \left| \frac{1 - \alpha + \sum_{m=2}^{\infty} (m-\alpha) B_\lambda (a,c,m,n) a_m r^{-m-1}}{1 + \sum_{m=2}^{\infty} B_\lambda (a,c,m,n) a_m r^{-m-1}} \right| \right).
\]

Letting \(r \to 1\), leads the desired inequality

\[
\sum_{m=2}^{\infty} (2m-1-\alpha) B_\lambda (a,c,m,n) \left| a_m \right| \leq 1 - \alpha.
\]

**Corollary 2.2:** If \(f \in VS_\lambda^\alpha (a,c,n)\) then

\[
\left| a_m \right| \leq \frac{1 - \alpha}{(2m-1-\alpha) B_\lambda (a,c,m,n)}, \quad \text{for } m \geq 2.
\]

The sharpness follows for the function
\[ f(z) = z + \sum_{m=2}^{\infty} \frac{1 - \alpha}{(2m - 1 - \alpha)B_{\lambda}(a, c, m, n)}, \quad \text{for} \quad m \geq 2, \quad z \in U. \]

Similar to the proof of Theorem 2.1 we get the following result:

**Theorem 2.3:** A function \( f \) of the form (1.1) is in \( VS_{\alpha}^a(a, c, n, \beta) \) if and only if

\[
\sum_{m=2}^{\infty} E_m B_{\lambda}(a, c, m, n) |a_m| \leq 1 - \alpha, \tag{2.2}
\]

where \( E_m = m(\beta + 1) - (\alpha + \beta) \).

The result obtained in our next Theorem unifies the radii results concerning close-to-convexity, starlikeness etc.

**Theorem 2.4:** Let \( f \in VS_{\alpha}^a(a, c, n, \beta) \). Then \( \frac{f * \Phi}{f * \Psi} - 1 < 1 - \delta \), in \( |z| r \) with \( \Phi(z) = z + \sum_{m=2}^{\infty} \gamma_m z^m \), and \( \Psi(z) = z + \sum_{m=2}^{\infty} \mu_m z^m \), are analytic in \( U \) with the conditions \( \gamma_m, \mu_m \geq 0, \gamma_m \geq \mu_m \), for \( m \geq 2 \) and \( f * \Psi \neq 0 \), where

\[
r = \inf_{m} \left[ \frac{E_m B_{\lambda}(a, c, m, n)(1 - \delta)}{(1 - \alpha)(\lambda_m - \mu_m + \mu_m (1 - \delta))} \right]^{1-\frac{1}{m-1}}, \quad m \geq 2. \tag{2.3}
\]

**Proof:** Consider,

\[
\left| \frac{f * \Phi}{f * \Psi} - 1 \right| = \left| \frac{z - \sum_{m=2}^{\infty} \gamma_m a_m z^m}{z - \sum_{m=2}^{\infty} \mu_m a_m z^m} - 1 \right| \leq \left| \frac{z - \sum_{m=2}^{\infty} \gamma_m a_m z^m - z + \sum_{m=2}^{\infty} \mu_m a_m z^m}{z - \sum_{m=2}^{\infty} \mu_m a_m z^m} \right| \leq \frac{\sum_{m=2}^{\infty} a_m (|\gamma_m - \mu_m| |z|^{m-1})}{1 - \sum_{m=2}^{\infty} \mu_m a_m |z|^{m-1}} < 1 - \delta.
\]

\[
\sum_{m=2}^{\infty} a_m [(|\gamma_m - \mu_m| + (1 - \delta) \mu_m) \leq 1 - \delta, \quad (|z| < r, 0 \leq \delta < 1), \tag{2.4}
\]

where \( r \) is given by (2.3). From Theorem 2.3, (2.4) will be true if,

\[
\frac{[(|\gamma_m - \mu_m| + (1 - \delta) \mu_m)] |z|^{m-1}}{1 - \delta} < \frac{E_m B_{\lambda}(a, c, m, n)(1 - \delta)}{(1 - \alpha)(|\gamma_m - \mu_m| + (1 - \delta) \mu_m)},
\]
that is, if

$$|z| = \left[ \frac{E_m B_2(a,c,m,n)(1-\delta)}{(1-\alpha)(\beta_m - \mu_m) + (1-\delta)\mu_m} \right]^{\frac{1}{m-1}}. \tag{2.5}$$

As corollaries to the above Theorem we get the following result:

By choosing \( \Phi(z) = \frac{z}{(1-z)^2} \) and \( \Psi(z) = z \), we have

**Corollary 2.5:** Let the function \( f \) defined by (1.1) belongs to \( VS_2^\alpha(a,c,n,\beta) \). Then \( f \) is close-to-convex of order \( \delta(0 \leq \delta < 1) \), hence univalent in the disc \(|z| < r_1\), where

$$r_1 = \inf_m \left[ \frac{E_m B_2(a,c,m,n)(1-\delta)}{(1-\alpha)(m + \delta)} \right]^{\frac{1}{m-1}}, \quad m \geq 2. \tag{2.6}$$

The result is sharp.

For \( \Phi(z) = \frac{z}{(1-z)^2} \) and \( \Psi(z) = \frac{z}{1-z} \), we have

**Corollary 2.6:** Let the function \( f \) defined by (1.1) belongs to \( VS_2^\alpha(a,c,n,\beta) \). Then \( f \) is starlike of order \( \delta(0 \leq \delta < 1) \), hence univalent in the disc \(|z| < r_2\), where

$$r_2 = \inf_m \left[ \frac{E_m B_2(a,c,m,n)(1-\delta)}{(1-\alpha)(m + \delta)} \right]^{\frac{1}{m-1}}, \quad m \geq 2. \tag{2.7}$$

The result is sharp.

If \( \Phi(z) = \frac{z + z^2}{(1-z)^3} \) and \( \Psi(z) = \frac{z}{(1-z)^2} \), we have

**Corollary 2.7:** Let the function \( f \) defined by (1.1) belongs to \( VS_2^\alpha(a,c,n,\beta) \). Then \( f \) is convex of order \( \delta(0 \leq \delta < 1) \), hence univalent in the disc \(|z| < r_3\), where

$$r_3 = \inf_m \left[ \frac{E_m B_2(a,c,m,n)(1-\delta)}{m(1-\alpha)(m + \delta)} \right]^{\frac{1}{m-1}}, \quad m \geq 2. \tag{2.8}$$

The result is sharp.

Using the coefficient inequality proved above we can easily prove the following growth and distortion theorem.

**Theorem 2.8:** Let \( f \) of the form (1.1) to be in \( VS_2^\alpha(a,c,n,\beta) \). Then

$$r - \frac{1-\alpha}{E_2 B_2(a,c,2,n)} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{E_2 B_2(a,c,2,n)} r^2$$

and

$$1 - \frac{2(1-\alpha)}{E_2 B_2(a,c,2,n)} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{E_2 B_2(a,c,2,n)} r.$$
The result is sharp.

**Proof:** Let \( f \) of the form (1.1) belongs to \( VS_\Delta^\alpha (a,c,n,\beta) \).

\[
|f(z)| = \left| z + \sum_{m=2}^{\infty} a_m z^m \right| \leq |z| + |z|^2 \sum_{m=2}^{\infty} |a_m|,
\]

since \( f \in VS_\Delta^\alpha (a,c,n,\beta) \) and by Theorem 2.3, we have

\[
E_2 B_\Delta (a,c,2,n) \sum_{m=2}^{\infty} |a_m| \leq \sum_{m=2}^{\infty} E_m B_\Delta (a,c,m,n) |a_m| \leq 1 - \alpha.
\]

Thus \( |f(z)| \leq |z| + \frac{1-\alpha}{E_2 B_\Delta (a,c,2,n)} |z|^2 \).

That is

\[
|f(z)| \leq r + \frac{1-\alpha}{E_2 B_\Delta (a,c,2,n)} r^2,
\]

similarly, we get

\[
|f(z)| \leq r - \frac{1-\alpha}{E_2 B_\Delta (a,c,2,n)} r^2.
\]

On the other hand \( f'(z) = 1 + \sum_{m=2}^{\infty} ma_m z^{m-1} \), and

\[
|f'(z)| = 1 + \sum_{m=2}^{\infty} m |a_m| |z|^{m-1} \leq 1 + |z| \sum_{m=2}^{\infty} m |a_m|,
\]

since \( f \in VS_\Delta^\alpha (a,c,n,\beta) \).

Then by Theorem 2.3 we have \( \sum_{m=2}^{\infty} m |a_m| \leq \frac{2(1-\alpha)}{E_2 B_\Delta (a,c,2,n)} \). Thus

\[
|f'(z)| \leq 1 + \frac{2(1-\alpha)}{E_2 B_\Delta (a,c,2,n)} r.
\]

Similarly we get

\[
|f'(z)| \geq 1 - \frac{2(1-\alpha)}{E_2 B_\Delta (a,c,2,n)} r.
\]

This completes the proof.

**Theorem 2.9:** A function \( f \) of the form (1.1) belongs to \( VS_\Delta^\alpha (a,c,n,\beta) \) with \( \arg a_m = \theta_m \) where \( [\theta_m + (m-1)t] = \pi \mod 2\pi \). Define and \( \tilde{f}_m(z) = z \) and

\[
f_m(z) = z + \frac{1-\alpha}{E_m B_\Delta (a,c,m,n)} e^{i\theta_m} z^m, \quad m \geq 2, \quad z \in U.
\]

Then \( f \in VS_\Delta^\alpha (a,c,n,\beta) \) if and only if \( f \) expressed in the form \( f(z) = \sum_{m=2}^{\infty} \mu_m f_m(z) \) where

\[
\mu_m \geq 0 \text{ and } \sum_{m=2}^{\infty} \mu_m = 1.
\]
Proof: If \( f(z) = \sum_{m=2}^{\infty} \mu_m f_m(z) \) with \( \sum_{m=2}^{\infty} \mu_m = 1 \) and \( \mu_m \geq 0 \) then
\[
\sum_{m=2}^{\infty} E_m B_\lambda(a,c,m,n) \frac{1 - \alpha}{E_m B_\lambda(a,c,m,n)} \mu_m \\
= \sum_{m=2}^{\infty} \mu_m (1 - \alpha) = (1 - \mu_1)(1 - \alpha) \geq 1 - \alpha.
\]
Hence \( f \in VS_\lambda^\alpha(a,c,n,\beta) \).

Conversely, let the function \( f \) defined by (1.1) be in the class \( VS_\lambda^\alpha(a,c,n,\beta) \), since
\[
|a_m| \leq \frac{1 - \alpha}{E_m B_\lambda(a,c,m,n)}, \quad m = 2, 3, \ldots.
\]
We may set \( \mu_m = \frac{E_m B_\lambda(a,c,m,n) |a_m|}{1 - \alpha}, \quad m \geq 2 \) and \( \mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m \).

Then \( f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z) \), this completes the proof.

Lemma 2.10: [3] If for the functions \( f \) and \( g \) are analytic in \( U \) with \( g < f \), then for \( k > 0 \) and \( 0 < r < 1 \)
\[
\int_0^{2\pi} |g(re^{i\theta})|^k \, d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^k \, d\theta.
\]
In [6] Silverman found that the function \( f_2(z) = z - \frac{z^2}{2} \) is often extremal over the family \( T \). He applied this function to resolve the integral means inequality, conjectured in [7] and [8], such that
\[
\int_0^{2\pi} |f(re^{i\theta})|^k \, d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^k \, d\theta, \quad \text{for all} \ f \in V, k > 0 \text{ and } 0 < r < 1.
\]
In [8] he also proved his conjecture for the subclasses \( T^\ast(\beta) \) and \( C(\beta) \) of \( T \).

Theorem 2.11: Let \( f \) of the form (1.1) belongs to \( VS_\lambda^\alpha(a,c,n,\beta) \) and \( f_2 \) is defined by
\[
f_2(z) = z - \frac{1 - \alpha}{E_2 B_\lambda(a,c,2,n)} z^2 \quad \text{then for} \ z = re^{i\theta}, 0 < r < 1, \text{we have}
\]
\[
\int_0^{2\pi} |f(z)|^k \, d\theta \leq \int_0^{2\pi} |f_2(z)|^k \, d\theta.
\]
Proof: For \( f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m \), (2.9) is equivalent to prove that
\[
\int_0^{2\pi} \left| \sum_{m=2}^{\infty} |a_m| z^m \right|^k \, d\theta \leq \int_0^{2\pi} \left| \frac{1 - \alpha}{E_2 B_\lambda(a,c,2,n)} z \right|^k \, d\theta.
\]
By Lemma 2.10 it suffices to show that

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1 - \sum_{m=2}^{\infty} |a_m| z^{-m-1} < 1 - \frac{1 - \alpha}{E_2 B_\lambda(a,c,2,n)} z \quad \text{setting}

1 - \sum_{m=2}^{\infty} |a_m| z^{-m-1} = 1 - \frac{1 - \alpha}{E_2 B_\lambda(a,c,2,n)} \omega(z)

And using (2.2) we obtain

\omega(z) = \left| \sum_{m=2}^{\infty} E_m B_\lambda(a,c,m,n) |a_m| z^{-m-1} \right|

\leq |z| \left| \sum_{m=2}^{\infty} E_m B_\lambda(a,c,m,n) |a_m| \right| \leq |z|.

This completes the proof.

In Theorem 2.4, 2.8, 2.9 and 2.11 if we substitute \( \beta = 1 \) we get the result for the class \( VS_\lambda^\alpha(a,c,n) \).

References: