International Journal of Mathematical Archive-3(6), 2012, 2358-2373
IJMA Available online through www.ijma.info ISSN 2229 – 5046

On Gαo-kernel in the digital plane

M. Vigneshwaran* & R. Devi
Department of Mathematics, Kongunadu Arts and Science College (Autonomous), Coimbatore–641 029

(Received on: 08-05-12; Revised & Accepted on: 31-05-12)

ABSTRACT

*We introduce the concept of *gα-closed sets in a topological space and characterize it using its Gαo-kernel. Moreover we investigate new seperation axioms and new functions in topological spaces. For the digital plane, we have explicite forms of Gαo-kernel and α-kernel of a subset in the plane.*

Key words: **gα-closed sets, $_\alpha T_{1/2}{}^{**}$ spaces, $T_c{}^{**}$ spaces, $_\alpha T_c{}^{**}$ spaces, $^{**}{}_\alpha T_{1/2}$ spaces, *gα-continuous, *gα-irresolute maps, *gαc- homeomorphism, Gαo-kernel, α-kernel and digtal plane.*

AMS Subject Classification: *54A05, 54C05, 54C10, 68U05, 68U10.*

1. INTRODUCTION

Levine [14] and Njastad [19] introduced semi-open sets and α-sets respectively. The complement of a semi-open (resp. α-open) set is called a semi-closed [3] (resp.α-closed [19]) set. Levine [13] introduced g-closed sets and studied their most fundamental properties. S.P. Arya and T. Nour [1], H. Maki et.al. [16, 17] introduced gs-closed sets, αg-closed sets and gα-closed sets respectively. Dontchev [9] and Gnanambal [10] introduced gsp-closed sets and gpr-closed sets respectively.

In this paper, we introduce a new class of sets, namely *gα-closed sets by generalizing gα-open sets. This new class is properly placed between the class of closed sets and the class of g-closed sets. Applying *gα-closed sets, we introduce and study some new spaces, namely $_\alpha T_{1/2}{}^{**}$ spaces, $T_c{}^{**}$ spaces, $_\alpha T_c{}^{**}$ spaces and $^{**}{}_\alpha T_{1/2}$ spaces. In the fifth chapter we introduce and study *gα-continuous, *gα-irresolute maps and its group structure., In the sixth chapter we investigate *gαc- homeomorphism and its properties. In the seventh chapter, we investigate the explicit form in the digital plane of Gαo-kernel and α-kernel which are used for charaterization of *gα-closed sets and gα-closed sets, respectively. The digital plane is a mathematical model of the computer screen (cf.[5],[11],[12]).

2. PRELIMINARIES

Throughout this paper (X, τ), (Y,σ) and (Z,η) represent topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), cl(A), int(A) and C(A) denote the closure of A, the interior of A and the complement of A in X respectively.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1: A subset A of a space (X, τ) is called
1. a *semi-open* set [14] if A $\subseteq$ cl(int(A)) and a *semi-closed* set if int(cl(A)) $\subseteq$ A,
2. an *α-open* set [19] if A $\subseteq$ int(cl(int((A))) and an *α-closed* set if cl(int(cl(A))) $\subseteq$ A and

Definition 2.2: A subset A of a space (X, τ) is called
1. a *generalized closed* (briefly *g-closed*) set [13] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of a g- closed set is called a g-open set,
2. a *generalized semi-closed* (briefly *gs-closed*) set [1] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ),
3. an *α-generalized closed* (briefly *αg-closed*) set [16] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of an αg-closed set is called an αg-open set,
4. a *generalized α-closed* (briefly *gα-closed*) set [17] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is α-open in (X, τ),
5. a *generalized preclosed* (briefly *gp-closed*) set [18] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ),
6. a *generalized semi-preclosed* (briefly *gsp-closed*) set [9] if spcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X , τ),
7. a *generalized preregular closed* (briefly *gpr-closed*) set [10] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and is regular open in (X, τ),

Definition 2.3: A function f: $(X, \tau) \to (Y,\sigma)$ is called

1. semi-continuous [14] if $f^{-1}(V)$ is semi-open in (X, τ) for every closed set V of (Y,σ),
2. α-continuous [15] if $f^{-1}(V)$ is α-closed in (X, τ) for every closed set V of (Y,σ),
3. g-continuous [2] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y,σ),
4. gs-continuous [7] if $f^{-1}(V)$ is gs-closed in (X, τ) for every closed set V of (Y,σ),
5. αg-continuous [4] if $f^{-1}(V)$ is αg-closed in (X, τ) for every closed set V of (Y,σ),
6. gα-continuous [17] if $f^{-1}(V)$ is gα-closed in (X, τ) for every closed set V of (Y,σ),
7. gsp-continuous [9] if $f^{-1}(V)$ is gsp-closed in (X, τ) for every closed set V of (Y,σ),
8. gpr-continuous [10] if $f^{-1}(V)$ is gpr-closed in (X, τ) for every closed set V of (Y,σ),
9. gc-irresolute [2] if $f^{-1}(V)$ is g-closed in (X, τ) for every g-closed set V of (Y,σ),
10. gs-irresolute [7] if $f^{-1}(V)$ is gs-closed in (X, τ) for every gs-closed set V of (Y,σ),
11. αg -irresolute [4] if $f^{-1}(V)$ is αg -closed in (X, τ) for every αg -closed set V of (Y,σ) and
12. gα-irresolute [17] if $f^{-1}(V)$ is gα -closed in (X, τ) for every gα -closed set V of (Y,σ).

Definition 2.4: A space (X, τ) is called

1. a $T_{1/2}$ *space* [13] if every g-closed set is closed,
2. a T_b *space* [6] if every gs-closed set is closed,
3. a T_d *space* [6] if every gs-closed set is g-closed,
4. an ${}_{\alpha}T_b$ *space* [4] if every αg-closed set is closed,
5. an ${}_{\alpha}T_d$ *space* [4] if every αg-closed set is g-closed.

Notation 2.5: For a space (X, τ), $C(X, \tau)$ (resp.$SC(X, \tau)$, $\alpha C(X, \tau)$, $G\alpha C(X, \tau)$,$GC(X, \tau)$,$GSC(X, \tau)$, $\alpha GC(X, \tau)$) denote the class of all closed (resp.semi-closed, α-closed, gα-closed, g-closed, gs-closed, αg-closed) subsets of (X, τ).

3. BASIC PROPERTIES OF *gα-CLOSED SETS

We introduce the following definition.

Definition 3.1: A subset A of (X, τ) is called a *gα-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gα-open in (X, τ).

The class of *gα-closed subsets of (X, τ) is denoted by $^*G\alpha C(X, \tau)$.

Theorem 3.2: Every closed set is a *gα -closed set.

Proof: Let $A \subseteq U$, where U is gα-open set in X. Since A is closed, $cl(A) = A \subseteq U$. Therefore $cl(A) \subseteq U$.

Hence A is *gα-closed.

Following example shows that the above implication is not reversible.

Example 3.3: Let X = {a, b, c} and τ = {X, ϕ, {a, b}}. $^*G\alpha C(X, \tau)$ = {X, ϕ, {c}, {b, c}, {a, c}}.

Here{b, c}is a *gα-closed set of (X, τ) but it is not a closed set of (X, τ).

Theorem 3.4: Every *gα -closed set is g-closed set.

Proof: Let $A \subseteq U$, where U is an open set in X. Since every open set is gα-open, U is gα-open .Since A is *gα-closed, $cl(A) \subseteq U$. Hence A is g-closed.

Following example shows that the above implication is not reversible.

Example 3.5: Let X = {a, b, c} and τ = {X, ϕ, {a}, {b, c}}. $GC(X, \tau)$ = {X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}. $^*G\alpha C(X, \tau)$ = {X, ϕ, {a}, {b, c}}.

Here {b} is a g-closed set of (X, τ) it is not a *gα-closed set of (X, τ).

Theorem 3.6: Every *gα-closed set is gα-closed set.

Proof: Let $A \subseteq U$, where U is an α-open set in X. Since every α-open set is gα-open, U is gα-open. Since A is *gα-closed, $cl(A) \subseteq U$. But $\alpha cl(A) \subseteq cl(A) \subseteq U$. Therefore A is g$\alpha$-closed.

Corresponding author: *M. Vigneshwaran**
Department of Mathematics, Kongunadu Arts and Science College (Autonomous), Coimbatore–641 029

Following example shows that the above implication is not reversible.

Example 3.7: Let X and τ be as in the example 3.5. Let A = {a, c}. A is a gα-closed set of (X, τ). But A is not a *gα-closed set of (X, τ).

Theorem 3.8: Every *gα-closed set is gp-closed set.

Proof: Let A⊆U, where U is an open set in X. Since every open set is gα-open, U is gα-open. Since A is *gα-closed, cl(A) ⊆U. But pcl(A) ⊆ cl(A) ⊆U. Therefore A is gp-closed set.

Following example shows that the above implication is not reversible.

Example 3.9: Let X and τ be as in the example 3.5. Let B= {a, b}. B is a gp-closed set of (X, τ). But B is not a *gα-closed set of (X, τ).

Thus the class of *gα-closed sets are contained in the class of g-closed sets, gα-closed sets, αg-closed sets, gs-closed sets, gsp-closed sets, gpr-closed sets and gp-closed sets. The class of *gα-closed sets contains the class of closed sets.

Remark 3.10: *gα-closedness is independent of semi-closedness and α-closedness.

Proof: It can be seen by the following example.

Example 3.11: Let X = {a, b, c} and τ = {X, ϕ, {a}}. SC(X, τ) = {X, ϕ, {b}, {c}, {b, c}} = αC(X, τ)
*GαC(X, τ) = {X, ϕ, {b, c}}.

Here {b} is semi-closed set and α-closed set of (X, τ). But it is not a *gα-closed set of (X, τ).

Example 3.12: Let X and τ be as in the example 3.3. Here {b, c} is not a semi-closed and α-closed set of (X, τ). But it is a *gα-closed set of (X, τ).

Theorem 3.13: The intersection of two gα-closed sets is again in gα-closed set.

Proof: Let A and B are gα-closed sets. Let A∩B ⊆ U, U is α-open. Since A and B are gα-closed sets, αcl(A) ⊆ U and αcl(B) ⊆ U. This implies that αcl(A∩B) = αcl(A) ∩ αcl(B) ⊆ U ⇒ αcl(A∩B) ⊆ U. Therefore A∩B is gα-closed.

Theorem 3.14: Let A be an open set and B be an gα-open set, then A∪B is gα-open set.

Proof: Suppose that A is an open set and B is an gα-open set. Since every open set is gα-open set, A is gα-open set. Then A∪B is gα-open set, since union of two gα-open set is again gα-open set.

Theorem 3.15:
1. Let A be a *gα-closed set of (X, τ) if and only if cl(A)-A does not contain any non empty gα-closed set.

2. If A is a *gα-closed and A ⊆ B ⊆ cl(A), then B is *gα-closed.

Proof:
1. Necessity part- Suppose that A is *gα-closed and let F be a non empty gα-closed set with F ⊆ cl(A)-A. Then A ⊆ X-F and so cl(A) ⊆ X-F. Hence F ⊆ X-cl(A), a contradiction.

Sufficient part - Suppose A is a subset of (X, τ) such that cl(A)-A does not contain any non-empty gα-closed set. Let U be a gα-open set of (X,τ) such that A ⊆ U. If cl(A)⊆ U, then cl(A)∩C(U) ≠ ϕ.Then ϕ ≠ cl(A)∩C(U) is a gα-closed set of (X, τ), since the intersection of two gα-closed sets is again gα-closed set.

2. Let U be a gα-open set of (X, τ) such that B ⊆ U. Then A ⊆ U. Since A is *gα-closed, cl(A) ⊆ U. Now cl(B) ⊆ cl(cl(A))=cl(A) ⊆ U. Therefore B is also a *gα-closed set of (X, τ).

Theorem 3.16: Let X be a topological space. A subset A of (X, τ) is *gα-open if and only if U⊆Int(A), whenever U is gα-closed set and U⊆ A.

Proof: Let A be a *gα-open set and U is gα-closed set such that U⊆ A implies X-U ⊇ X-A and X-A is *gα-closed set. So cl(X-A) ⊆ X-U implies (X- cl(X-A)) ⊇ (X-(X-U)) = U. But (X- cl(X-A)) = Int(A). Thus U⊆Int(A).

Conversely, suppose A is subset such that $U \subseteq Int(A)$. Whenever U is gα-closed and $U \subseteq A$. We show that X-A is *gα-closed set. Let $X\text{-}A \subseteq U$, where U is gα-open. Since $X\text{-}A \subseteq U$ implies $X\text{-}U \subseteq A$. By assumption that we must have $X\text{-}U \subseteq Int(A)$ or $X\text{-}Int(A) \subseteq U$. Now $X\text{-}Int(A) = cl(X\text{-}A)$ which implies that $cl(X\text{-}A) \subseteq U$ and X-A is *gα-closed set.

Theorem 3.17: The union of two *gα-closed sets is a *gα-closed set.

Proof: Let A and B are *gα-closed sets. Let $A \cup B \subseteq U$, U is gα-open. Since A and B are *gα-closed sets, $cl(A) \subseteq U$ and $cl(B) \subseteq U$. This implies that $cl(A \cup B) = cl(A) \cup cl(B) \subseteq U \Rightarrow cl(A \cup B) \subseteq U$. Therefore $A \cup B$ is *gα-closed.

Remark 3.18: The intersection of two *gα-closed sets is again in *gα-closed set.
(i) The intersection of two *gα-closed sets is again in *gα-closed set.
(ii) The intersection of an open and a *gα-open sets is a *gα-open set.
(iii) The union of an open and a *gα-open sets is a *gα-open set.

We prepare the following notations:

For a subset A of (X, τ),
$G\alpha O(X, \tau) = \{U / U \text{ is } g\alpha\text{-open in } (X, \tau)\}$;
$ker(A) = \cap\{U / U \in \tau \text{ and } A \subseteq U\}$;
$\alpha\text{-}ker(A) = \cap\{U / U \text{ is } \alpha\text{-open set and } A \subseteq U\}$;
$G\alpha O\text{-}ker(A) = \cap\{U / U \in G\alpha O(X, \tau) \text{ and } A \subseteq U\}$.
$X_{g\alpha c} = \{x \in X / \{x\} \text{ is } g\alpha\text{-closed in } (X, \tau)\}$ and
$X_{*g\alpha o} = \{x \in X / \{x\} \text{ is } ^*g\alpha\text{-open in } (X, \tau)\}$.

Theorem 3.19: Any subset A is gα-closed set if and only if $\alpha cl(A) \subseteq \alpha\text{-}ker(A)$ holds.

Proof: Necessary: We know that $A \subseteq \alpha\text{-}ker(A)$. Since A is gα-closed, then $\alpha cl(A) \subseteq \alpha\text{-}ker(A)$.

Sufficiency: Let $A \subseteq U$ and U is α-open. Given that $\alpha cl(A) \subseteq \alpha\text{-}ker(A)$. If $U \subseteq \alpha cl(A)$, then $\alpha\text{-}ker(A) \subseteq U \subseteq \alpha cl(A)$, which is a contradiction to the hypothesis. Therefore $\alpha cl(A) \subseteq U$. Hence A is gα-closed.

Lemma 3.20: For any space (X, τ), $X = X_{g\alpha c} \cup X_{*g\alpha o}$ holds.

Proof: Let $x \in X$. Suppose that $\{x\}$ is not *gα-closed set in (X, τ). Then X is a unique gα-open set containing $X/\{x\}$. Thus $X/\{x\}$ is *gα-closed in (X, τ) and so $\{x\}$ is *gα-open. Therefore $x \in X_{g\alpha c} \cup X_{*g\alpha o.}$

Theorem 3.21: For a subset A of (X, τ), the following conditions are equivalent:
1. A is *gα-closed in (X, τ).
2. $cl(A) \subseteq G\alpha O\text{-}ker(A)$ holds.
3. (i) $cl(A) \cap X_{g\alpha c} \subseteq A$ and (ii) $cl(A) \cap X_{*g\alpha o} \subseteq G\alpha O\text{-}ker(A)$ holds.

Proof:
(1)⇒(2): Let $x \notin G\alpha O\text{-}ker(A)$. Then there exists a set $U \in G\alpha O(X, \tau)$ such that $x \notin U$ and $A \subseteq U$.

Since A is *gα-closed, $cl(A) \subseteq U$ and $x \notin cl(A)$. This is a contradiction.

(2)⇒(3):

(i): It follows from (2) that $cl(A) \cap X_{g\alpha c} \subseteq G\alpha O\text{-}ker(A) \cap X_{g\alpha c.}$ We claim that $G\alpha O\text{-}ker(A) \cap X_{g\alpha c} \subseteq A$.Suppose $x \in G\alpha O\text{-}ker(A) \cap X_{g\alpha c}$ and assume that $x \notin A$.Since the set $X/\{x\} \in G\alpha O(X, \tau)$ and $A \subseteq X/\{x\}$. Then we have that $x \in X/\{x\}$ and so this is a contradiction. Thus we show that $cl(A) \cap X_{g\alpha c} \subseteq A$. by using (2) $cl(A) \cap X_{g\alpha c} \subseteq G\alpha O\text{-}ker(A) \cap X_{g\alpha c} \subseteq A$.

(ii): It is obtained by (2).

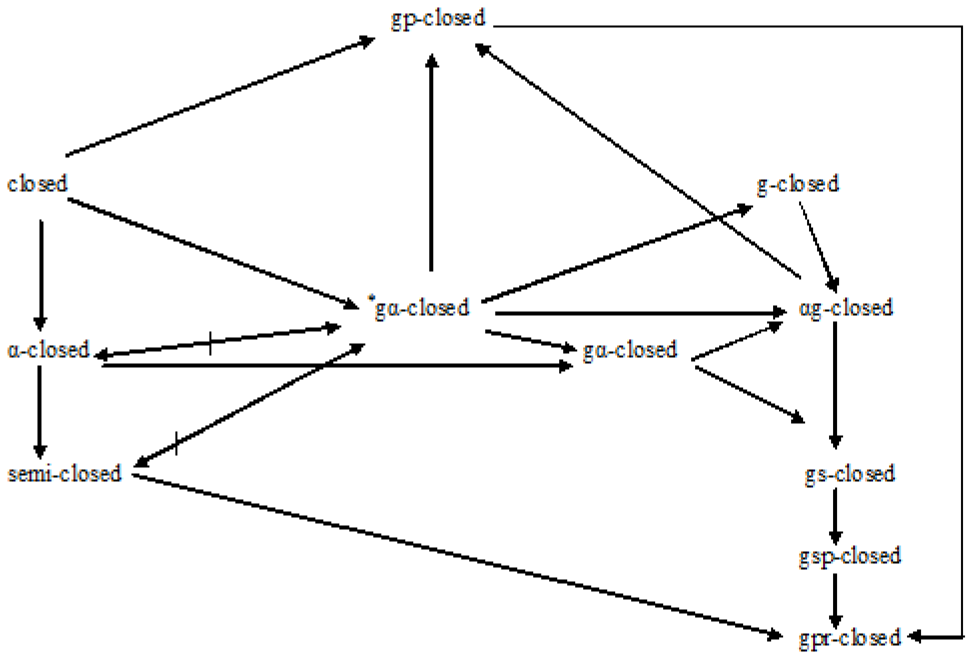
(3) ⇒(2): By Remark 3.8 and (3),

$$\begin{aligned} cl(A) = cl(A) \cap X &= cl(A) \cap (X_{g\alpha c} \cup X_{*g\alpha o}) \\ &= (cl(A) \cap X_{g\alpha c}) \cup (cl(A) \cap X_{*g\alpha o}) \\ &\subseteq A \cup G\alpha O\text{-}ker(A) \\ &= G\alpha O\text{-}ker(A). \end{aligned}$$

That is cl(A) $\subseteq$ GαO-ker(A) holds.

(2) $\Rightarrow$ (1): Let U$\in$ GαO(X, τ) such that A $\subseteq$ U. Then we have that GαO-ker(A) $\subseteq$ U and so by (2) cl(A) $\subseteq$ U. Therefore A is *gα-closed.

Remark 3.22: The following diagram shows the relationships established between *gα-closed sets and some other sets in theorem 3.2, 3.4, 3.6, 3.8, remark3.10 and reference [22], [21]. A→B (A $\not\leftrightarrow$ B) represents A implies B but not conversely(A and B are independent each other).



4. APPLICATIONS OF *gα-CLOSED SETS

We introduce the following definition.

Definition 4.1: A space (X, τ) is called an $_{\alpha}T_{1/2}^{**}$ space if every *gα-closed set is closed.

The following theorem gives a characterization of $_{\alpha}T_{1/2}^{**}$ spaces.

Theorem 4.2: If (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space, then every singleton of X is either gα-closed or open.

Proof: Let x$\in$X and suppose that {x} is not a gα-closed set of (X, τ). Then X/{x} is not gα-open. This implies that X is the only gα-open set containing X/{x}, so X/{x} is a *gα-closed set of (X, τ).Since (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space, X/{x} is closed or equivalently {x} is open in (X, τ).

Theorem 4.3: Every $T_{1/2}$ space is an $_{\alpha}T_{1/2}^{**}$ space.

Proof: Let A be a *gα-closed set of (X, τ). Since every *gα-closed set is g-closed, A is g-closed. Since (X, τ) is a $T_{1/2}$ space, A is closed. Therefore (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space.

The space in the following example is an $_{\alpha}T_{1/2}^{**}$ space but not a $T_{1/2}$ space.

Example 4.4: Let X={a, b, c}with τ ={X, ϕ,{a},{a, b}}.
*GαC(X, τ)={X, ϕ,{c}, {b, c}}
GC(X, τ) = {X, ϕ,{c}, {b, c}, {a, c}}.

Here (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space but not a $T_{1/2}$ space. Since {a, c} is a g-closed set but not a closed set.

Theorem 4.5: Every T_b space is an $_{\alpha}T_{1/2}^{**}$ space.

Proof: Let A be a $^*g\alpha$-closed set of (X, τ). Since every $^*g\alpha$-closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_b space, A is closed. Therefore (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space.

The space in the following example is an ${}_\alpha T_{1/2}^{**}$ space but not a T_b space.

Example 4.6: Let X= {a, b, c} with $\tau = \{X, \phi, \{b\}, \{b, c\}\}$. $^*G\alpha C(X, \tau) = \{X, \phi, \{a\}, \{a, c\}\}$
$GSC(X, \tau) = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$.

Here (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space but not a T_b space. Since {a, b} is a gs-closed set but not a closed set.

Theorem 4.7: Every ${}_\alpha T_b$ space is an ${}_\alpha T_{1/2}^{**}$ space.

Proof: Let A be a $^*g\alpha$-closed set of (X, τ). Since every $^*g\alpha$-closed set is αg-closed, A is αg-closed. Since (X, τ) is an ${}_\alpha T_b$ space, A is closed. Therefore (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space.

The space in the following example is an ${}_\alpha T_{1/2}^{**}$ space but not an ${}_\alpha T_b$ space.

Example 4.8: Let X and τ be as in example 4.6. Here (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space but not an ${}_\alpha T_b$ space. Since {c}is an αg-closed set but not a closed set.

Definition 4.9: A space (X, τ) is called a T_c^{**} if every gs-closed set is $^*g\alpha$-closed.

The following theorem gives a characterization of T_c^{**} spaces.

Theorem 4.10: If (X, τ) is a T_c^{**} space, then every singleton of X is either closed or $^*g\alpha$-open.

Proof: Let $x \in X$ and suppose that {x} is not a closed set of (X, τ). Then X/{x} is not open. This implies X is the only open set containing X/{x}. So X/{x} is a gs-closed set of (X, τ). Since (X, τ) is a T_c^{**} space, X/{x} is a $^*g\alpha$-closed set or equivalently {x} is $^*g\alpha$-open in (X, τ).

The converse of the above theorem is not true as can be seen by the following example.

Example 4.11: Let X = {a, b, c} with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$.
$^*g\alpha$-open sets of (X, τ) are $X, \phi, \{a\}, \{b\}, \{a, b\}$.
$GSC(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$.
$^*G\alpha C(X, \tau) = \{X, \phi, \{c\}, \{b, c\}, \{a, c\}\}$.

Here {a} and {b} are $^*g\alpha$-open sets and {c} is a closed set but (X, τ) is not a T_c^{**} space. Since {b} is a gs-closed set but not a $^*g\alpha$-closed set of (X, τ).

Theorem 4.12: Every T_b space is a T_c^{**} space.

Proof: Let A be a gs-closed set of (X, τ). Since (X, τ) is a T_b space, A is closed. Since every closed set is $^*g\alpha$-closed, A is $^*g\alpha$-closed set. Therefore (X, τ) is a T_c^{**} space.

The space in the following example is a T_c^{**} space but not a T_b space.

Example 4.13: Let X and τ be as in example 3.3. Here (X, τ) is a T_c^{**} space but not a T_b space. Since {a, c} is a gs-closed set but not a closed set.

Theorem 4.14: Every T_c^{**} space is a T_d space.

Proof: Let A be a gs-closed set of (X, τ). Since (X, τ) is a T_c^{**} space, A is $^*g\alpha$- closed. Since every $^*g\alpha$ -closed set is g-closed, A is g-closed set. Therefore (X, τ) is a T_d space.

The space in the following example is a T_d space but not a T_c^{**} space.

Example 4.15: Let X and τ be as in example 3.5. Here (X, τ) is a T_d space but not a T_c^{**} space. Since {b} is a gs-closed set but not $^*g\alpha$-closed set.

Theorem 4.16: Every T_c^{**} space is an ${}_\alpha T_d$ space.

Proof: Let A be a αg -closed set of (X, τ). Since every αg -closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_c^{**} space, A is *gα- closed. Since every *gα -closed set is g-closed, A is g-closed set. Therefore (X, τ) is an $_\alpha T_d$ space.

The space in the following example is an $_\alpha T_d$ space but not a T_c^{**} space.

Example 4.17: Let X and τ be as in example 3.5. Here (X, τ) is an $_\alpha T_d$ space but not a T_c^{**} space. Since {a, c} is a gs-closed set but not *gα-closed set.

Theorem 4.18: The space (X, τ) is a T_b space if and only if it is a T_c^{**} space and an $_\alpha T_{1/2}^{**}$ space.

Proof: Necessity part: By theorem 4.12 and 4.5.

Sufficient part: Let A be a gs-closed sets of (X, τ). Since (X, τ) is a T_c^{**} space, A is *gα-closed set. Since (X, τ) is an $_\alpha T_{1/2}^{**}$ space, A is closed. Therefore (X, τ) is an T_b space.

Remark 4.19: T_c^{**}space and $_\alpha T_{1/2}^{**}$ space are independent of each other.

It can be seen by the following examples.

Example 4.20: Let X and τ be as in example 3.5. Here (X, τ) is an $_\alpha T_{1/2}^{**}$ space but not a T_c^{**}space. Since {b}is gs-closed set but not *gα-closed set.

Example 4.21: Let X and τ be as in example 3.3. Here (X, τ) is a T_c^{**} space but not an $_\alpha T_{1/2}^{**}$ space. Since {b, c}is *gα -closed set but not closed set.

Definition 4.22: A space (X, τ) is called an $_\alpha T_c^{**}$space if every αg-closed set is *gα-closed.

Theorem 4.23: Every T_b space is an $_\alpha T_c^{**}$space.

Proof: Let A be a αg-closed set of (X, τ). Since every αg-closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_b space, A is closed. Since every closed set is *gα-closed, A is *gα-closed set. Therefore (X, τ) is a $_\alpha T_c^{**}$space.

The space in the following example is an $_\alpha T_c^{**}$space but not a T_b space.

Example 4.24: Let X and τ be as in example 3.3. Here (X, τ) is an $_\alpha T_c^{**}$ space but not a T_b space. Since {b, c} is a gs-closed set but not closed set.

Theorem 4.25: Every $_\alpha T_b$ space is an $_\alpha T_c^{**}$space.

Proof: Let A be a αg-closed set of (X, τ). Since (X, τ) is a $_\alpha T_b$ space, A is closed. Since every closed set is *gα-closed, A is *gα-closed set. Therefore (X, τ) is an $_\alpha T_c^{**}$space.

The space in the following example is an $_\alpha T_c^{**}$space but not an $_\alpha T_b$ space.

Example 4.26: Let X and τ be as in example 3.3. Here (X, τ) is an $_\alpha T_c^{**}$space but not an $_\alpha T_b$ space. Since {a, c} is a αg-closed set but not closed set.

Theorem 4.27: Every $_\alpha T_c^{**}$space is an $_\alpha T_d$ space.

Proof: Let A be a αg-closed set of (X, τ). Since (X, τ) is a $_\alpha T_c^{**}$ space, A is *gα -closed. Since every *gα -closed set is g-closed, A is g-closed set. Therefore (X, τ) is an $_\alpha T_d$ space.

The space in the following example is an $_\alpha T_d$ space but not an $_\alpha T_c^{**}$space.

Example 4.28: Let X and τ be as in example 3.5. Here (X, τ) is an $_\alpha T_d$ space but not an $_\alpha T_c^{**}$space. Since {c} is a αg-closed set but not *gα-closed set.

Theorem 4.29: Every T_c^{**}space is an $_\alpha T_c^{**}$ space.

Proof: Let A be a αg-closed set of (X, τ). Since every αg-closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_c^{**} space, A is *gα-closed. Therefore (X, τ) is an $_\alpha T_c^{**}$space.

The space in the following example is an ${}_{\alpha}T_c^{**}$space but not a T_c^{**} space.

Example 4.30: Let X and τ be as in example 4.11. Here (X, τ) is an ${}_{\alpha}T_c^{**}$space but not a T_c^{**} space. Since{a} is a gs-closed set but not *gα-closed set.

Theorem 4.31: The space (X, τ) is an ${}_{\alpha}T_b$ space if and only if it is a ${}_{\alpha}T_c^{**}$space and an ${}_{\alpha}T_{1/2}^{**}$space.

Proof: Necessity part: By theorem 4.25 and 4.7.

Sufficient part: Let A be a αg-closed set of (X, τ). Since (X, τ) is an ${}_{\alpha}T_c^{**}$ space, A is *gα-closed. Since (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$, A is closed set. Therefore (X, τ) is an ${}_{\alpha}T_b$ space.

Remark 4.32: ${}_{\alpha}T_c^{**}$ space and ${}_{\alpha}T_{1/2}^{**}$ space are independent of each other.

It can be seen by the following examples.

Example 4.33: Let X and τ be as in example 3.5. Here (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$ space but not an ${}_{\alpha}T_c^{**}$ space. Since {b}is αg-closed set but not *gα-closed set.

Example 4.34: Let X and τ be as in example 3.3.Here (X, τ) is an ${}_{\alpha}T_c^{**}$ space. But not an ${}_{\alpha}T_{1/2}^{**}$ space. Since {b, c}is *gα -closed set but not closed set.

Definition 4.35: A space (X, τ) is called a ${}^{**}{}_{\alpha}T_{1/2}$ space if every g-closed set is *gα-closed set.

Theorem 4.36: Every $T_{1/2}$ space is a ${}^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since (X, τ) is a $T_{1/2}$ space, A is closed. Since every closed set is *gα-closed, A is *gα-closed. Therefore (X, τ) is an ${}^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a ${}^{**}{}_{\alpha}T_{1/2}$ space but not a $T_{1/2}$ space.

Example 4.37: Let X and τ be as in example 3.3. Here (X, τ) is a ${}^{**}{}_{\alpha}T_{1/2}$ space but not a $T_{1/2}$ space. Since {b, c} is a g-closed set but not closed set.

Theorem 4.38: Every T_b space is a ${}^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since every g-closed set is gs-closed, A is gs-closed set. Since (X, τ) is an T_b space, A is closed. Since every closed set is *gα-closed, A is *gα-closed. Therefore (X, τ) is an ${}^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a $*^{*}{}_{\alpha}T_{1/2}$ space but not a T_b space.

Example 4.39: Let X and τ be as in example 3.3. Here (X, τ) is a ${}^{**}{}_{\alpha}T_{1/2}$ space but not a T_b space. Since a, c} is a gs-closed set but not closed set.

Theorem 4.40: Every ${}_{\alpha}T_b$ space is a ${}^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since every g-closed set is αg-closed, A is αg-closed set. Since (X, τ) is an ${}_{\alpha}T_b$ space, A is closed. Since every closed set is *gα-closed, A is *gα-closed. Therefore (X, τ) is an ${}^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a ${}^{**}{}_{\alpha}T_{1/2}$ space but not an ${}_{\alpha}T_b$ space.

Example 4.41: Let X and τ be as in example 3.3.Here (X, τ) is a ${}^{**}{}_{\alpha}T_{1/2}$ space but not an ${}_{\alpha}T_b$ space. Since {a, c} is a αg-closed set but not closed set.

Theorem 4.42: Every T_c^{**} space is a ${}^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since every g-closed set is gs-closed, A is gs-closed set. Since (X, τ) is a T_c^{**} space, A is *gα-closed. Therefore (X, τ) is an ${}^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a ${}^{**}{}_{\alpha}T_{1/2}$ space but not a T_c^{**} space.

Example 4.43: Let X and τ be as in example 4.11. Here (X, τ) is an ${}^{**}{}_{\alpha}T_{1/2}$ space but not a T_c^{**} space. Since {a} is a gs-closed set but not a *gα-closed set.

Theorem 4.44: The space (X, τ) is a $T_{1/2}$ space if and only if it is a ${}^{**}{}_{\alpha}T_{1/2}$ space and an ${}_{\alpha}T_{1/2}{}^{**}$ space.

Proof: Necessity part: By theorem 4.36 and 4.3.

Sufficient part: Let A be a g-closed set of (X, τ). Since (X, τ) is a ${}^{**}{}_{\alpha}T_{1/2}$ space, A is *gα-closed. Since (X, τ) is an ${}_{\alpha}T_{1/2}{}^{**}$ space, A is closed. Therefore (X, τ) is an $T_{1/2}$ space.

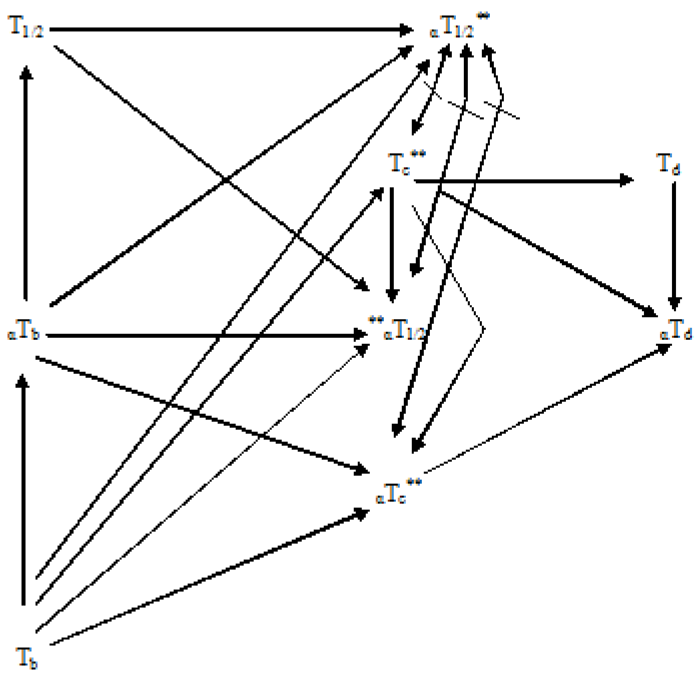
Remark 4.45: ${}^{**}{}_{\alpha}T_{1/2}$ space and ${}_{\alpha}T_{1/2}{}^{**}$ space are independent of each other.

It can be seen by the following examples.

Example 4.46: Let X and τ be as in example 3.5. Here (X, τ) is an ${}_{\alpha}T_{1/2}{}^{**}$ space but not an ${}^{**}{}_{\alpha}T_{1/2}$ space. Since {b} is g-closed set but not *gα-closed set.

Example 4.47: Let X and τ be as in example 3.3. Here (X, τ) is an ${}^{**}{}_{\alpha}T_{1/2}$ space but not an ${}_{\alpha}T_{1/2}{}^{**}$ space. Since {b, c} is *gα -closed set but not closed set.

Remark 4.48: The following diagram shows them relationship among the separation axioms considered in this paper and reference [18], [19]. A → B (A↮ B) represents A implies B but B need not imply A always (A and B are independent of each other).



5. *gα – CONTINUITY AND *gα – IRRESOLUTNESS:

We introduce the following definition

Definition 5.1: A function f: (X, τ) →(Y,σ) is called *gα – continuous if $f^{-1}(V)$ is a *gα – closed set of (X, τ) for every closed set V of (Y, σ).

Theorem 5.2: Every continuous map is *gα – continuous.

Proof: Let V be a closed set of (Y,σ). Since f is continuous $f^{-1}(V)$ is closed in (X, τ). But every closed set is *gα-closed set. Hence $f^{-1}(V)$ is *gα-closed set in (X, τ). Thus f is *gα – continuous.

The converse of the above theorem need not be true by the following example.

Example 5.3: Let X= {a, b, c} =Y with τ={X, ϕ, {a, b}} and σ={Y, ϕ, {a}}.

Define f: (X, τ) →(Y, σ) by f(a)=b, f(b)=a, f(c)=c.
*Gα C(X, τ) ={X, ϕ, {c}, {b, c}, {a, c}}.

Here $f^{-1}(\{b, c\}) = \{a, c\}$ is not a closed set in (X, τ). Therefore f is not continuous. However f is ${}^{*}g\alpha$ – continuous.

Theorem 5.4: Every ${}^{*}g\alpha$ – continuous map is g–continuous.

Proof: Let V be a closed set of (Y,σ).Since f is ${}^{*}g\alpha$ – continuous, $f^{-1}(V)$ is ${}^{*}g\alpha$ –closed in (X, τ). But every ${}^{*}g\alpha$ –closed set is g-closed set. Hence $f^{-1}(V)$ is g-closed set in (X, τ). Thus f is g – continuous.

The converse of the above theorem need not be true by the following example.

Example 5.5: Let X= {a, b, c} =Y with $\tau=\{X, \phi, \{a\},\{a, c\}\}$ and $\sigma=\{Y, \phi, \{a\},\{a, b\}\}$.
Define f: $(X, \tau) \to (Y,\sigma)$ **by** f(a)=b, f(b)=c, f(c)=a.
${}^{*}G\alpha\, C(X, \tau) =\{X, \phi, \{b\},\{b, c\}\}$.
$GC(X, \tau) =\{X,\phi, \{b\}, \{a, b\},\{b, c\}\}$.

Here $f^{-1}(\{b, c\}) = \{a, b\}$ is not a ${}^{*}g\alpha$ -closed set in (X, τ). Therefore f is not ${}^{*}g\alpha$ -continuous. However f is g– continuous.

Theorem 5.6: Every ${}^{*}g\alpha$ – continuous map is gα-continuous.

Proof: Let V be a closed set of (Y,σ). Since f is ${}^{*}g\alpha$ – continuous $f^{-1}(V)$ is ${}^{*}g\alpha$ –closed in (X, τ). But every ${}^{*}g\alpha$ –closed set is gα-closed set in (X, τ). Hence $f^{-1}(V)$ is gα-closed set in (X, τ). Thus f is gα-continuous.

.
The converse of the above theorem need not be true by the following example.

Example 5.7: Let X= {a, b, c} =Y with $\tau =\{ X, \phi, \{a\},\{b, c\}\}$ and $\sigma=\{Y, \phi, \{a\},\{a, c\}\}$.
Define f: $(X, \tau) \to (Y,\sigma)$ by f (a) =b, f (b) =c, f(c) =a.
${}^{*}G\alpha\, C(X, \tau) =\{X, \phi, \{a\},\{b, c\}\}$.
$G\alpha C(X, \tau) =\{X,\phi, \{a\},\{b\}, \{c\}, \{a, b\},\{b, c\}, \{a, c\}\}$.

Here $f^{-1}(\{b, c\}) = \{a, b\}$ is not a ${}^{*}g\alpha$ -closed set in (X, τ). Therefore f is not ${}^{*}g\alpha$ -continuous. However f is gα-continuous.

Remark 5.8: Every ${}^{*}g\alpha$ – continuous map is αg-continuous, gs-continuous, gsp-continuous and gpr-continuous.

Theorem 5.9: Every ${}^{*}g\alpha$ – continuous map is gp-continuous.

Proof: Let V be a closed set of (Y, σ). Since f is ${}^{*}g\alpha$ – continuous $f^{-1}(V)$ is ${}^{*}g\alpha$ –closed in (X, τ). But every ${}^{*}g\alpha$ –closed set is gp-closed set in (X, τ). Hence $f^{-1}(V)$ is gp-closed set in (X, τ). Thus f is gp-continuous.

The converse of the above theorem need not be true by the following example.

Example 5.10: Let (X, τ) and (Y,σ) be as in example 5.7. Here $f^{-1}(\{b, c\}) = \{a, b\}$ is not a ${}^{*}g\alpha$ -closed set in (X, τ). Therefore f is not ${}^{*}g\alpha$ -continuous. However f is gp-continuous.

Remark 5.11: ${}^{*}g\alpha$ –continuity is independent of semi-continuity and α-continuity.

The proof follows from the following example.

Example 5.12: Let X= {a, b, c} =Y with $\tau =\{X, \phi, \{a\}\}$ and $\sigma=\{Y, \phi, \{a\},\{a, b\}\}$.
Define f: $(X, \tau) \to (Y,\sigma)$ by f (a) =a, f (b) =b, f(c) =c.
${}^{*}G\alpha\, C(X, \tau) =\{X, \phi, \{b, c\}\}$.
$SC(X, \tau) =\{X,\phi, \{b\}, \{c\}, \{b, c\}\}= \alpha C(X, \tau)$

Here $f^{-1}(\{b\}) = \{b\}$ is not a ${}^{*}g\alpha$ -closed set in (X, τ). Therefore f is not ${}^{*}g\alpha$ -continuous. However f is semi–continuous and α-continuous.

Example 5.13: Let X={a, b, c}=Y with $\tau =\{ X, \phi, \{a, b\}\}$ and $\sigma=\{Y, \phi, \{b\},\{b, c\}\}$.
Define f: $(X, \tau) \to (Y, \sigma)$ by f(a)=c, f(b)=b, f(c)=a.
${}^{*}G\alpha C(X, \tau) =\{X,\phi, \{c\},\{b, c\},\{a, c\}\}$. $SC(X, \tau) =\{X,\phi, \{c\}\}= \alpha C(X, \tau)$

Here $f^{-1}(\{a, c\}) = \{a, c\}$ is not a semi -closed set and α-closed set in (X, τ). Therefore f is not semi-continuous and α-continuous. However f is ${}^{*}g\alpha$ -continuous.

Remark 5.14: The composition of two $^*g\alpha$ –continuous map need not be a $^*g\alpha$ –continuous.

The proof follows from the example.

Example 5.15: Let X={a, b, c}=Y =Z with τ={ X, ϕ, {a},{a, b}} ,σ={Y, ϕ, {a, b}} and η={ Z, ϕ, {b},{b, c}}
Define f: (X, τ) → (Y, σ) by f(a)=a, f(b)=b, f(c)=c.
Define g: (Y, σ) → (Z, η) by g(a)=c, g(b)=b, g(c)=a.
$^*G\alpha$ C(X, τ)={X, ϕ, {c},{b, c}}.
$^*G\alpha$ C(Y, σ)={Y, ϕ, {c},{b, c},{a, c}}.

Clearly f and g are $^*g\alpha$ –continuous.

Here {a, c} is a closed set in (Z,η). But $(gof)^{-1}$({a, c}) = {a, c} is not a $^*g\alpha$ –closed set in (X, τ).

Therefore gof is not *gα -continuous.

We introduce the following definition.

Definition 5.16: A function f: (X, τ) → (Y, σ) is called $^*g\alpha$ –irresolute if $f^{-1}(V)$ is a $^*g\alpha$ –closed set of (X, τ) for every $^*g\alpha$ –closed set of (Y, σ).

Theorem 5.17: Every $^*g\alpha$ –irresolute function is $^*g\alpha$ -continuous.

Proof: Let V be a closed set of (Y,σ). Since every closed set is $^*g\alpha$ -closed set. Therefore V is $^*g\alpha$ -closed set of (Y,σ). Since f is $^*g\alpha$ – irresolute $f^{-1}(V)$ is $^*g\alpha$ –closed in (X, τ). Therefore f is $^*g\alpha$ -continuous.

The converse of the above theorem need not be true by the following example.

Example 5.18: Let X={a, b, c}=Y with τ={ X, ϕ, {b},{b, c}} and σ={Y, ϕ, {a, b}}.
Define f: (X, τ) →(Y, σ) by f(a)=c, f(b)=a, f(c)=b.
$^*G\alpha$ C(X, τ) ={X, ϕ, {a},{b, c}}.
$^*G\alpha$ C(Y, σ) ={Y, ϕ, {c},{b, c},{b, c}}.

Here f is $^*g\alpha$ –continuous but f is not $^*g\alpha$ –irresolute. Since {a, c} is $^*g\alpha$ –closed set in (Y, σ) but f^{-1}({a, c})={a, b} is not $^*g\alpha$ –closed set in (X, τ).

Theorem 5.19: Let f: (X, τ) → (Y, σ) and g: (Y, σ) →(Z, η) be any two functions. Then
(i) gof: (X, τ) →(Z, η) is $^*g\alpha$ –continuous if g is continuous and f is $^*g\alpha$ –continuous.
(ii) gof: (X, τ) → (Z, η) is $^*g\alpha$ –irresolute if both g and f are $^*g\alpha$ – irresolute.
(iii) gof: (X, τ) → (Z, η) is $^*g\alpha$ –continuous if g is $^*g\alpha$ –continuous and f is $^*g\alpha$ – irresolute.

Proof:
(i) Let V be a closed set in (Z, η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is $^*g\alpha$ –continuous, $f^{-1}(g^{-1}(V))= (gof)^{-1}(V)$ is $^*g\alpha$ –closed in (X, τ). Therefore gof is $^*g\alpha$ –continuous.

Similarly we can prove (ii) and (iii).

Theorem 5.20: Let f: (X, τ)→ (Y, σ) be a $^*g\alpha$ –continuous(resp.gs-continuous, αg-continuous, g-continuous) map. If (X, τ) is an $_{\alpha}T_{1/2}^{**}$ (resp. T_c^{**} , $_{\alpha}T_c^{**}$, $^{**}{}_{\alpha}T_{1/2}$) space, then f is continuous ($^*g\alpha$ –continuous, $^*g\alpha$ –continuous, $^*g\alpha$ –continuous).

Proof: Let V be a closed set of (Y, σ).Since f is $^*g\alpha$ –continuous (resp.gs-continuous, αg-continuous, g-continuous), $f^{-1}(V)$ is $^*g\alpha$ –closed (resp.gs-closed, αg-closed, g-closed) in (X, τ). Since (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space(resp. T_c^{**} , $_{\alpha}T_c^{**}$, $^{**}{}_{\alpha}T_{1/2}$ space), $f^{-1}(V)$ is closed ($^*g\alpha$ –closed) in (X, τ). Therefore f is continuous (*gα –continuous, $^*g\alpha$ –continuous, $^*g\alpha$ –continuous).

Theorem 5.21: Let f: (X, τ) → (Y, σ) be a surjective, gα-irresolute and a closed map. Then f(A) is $^*g\alpha$ –closed set of (Y, σ) for every $^*g\alpha$ –closed set A of (X, τ).

Proof: Let A be a $^*g\alpha$ –closed set of (X, τ). Let U be a gα-open set of (Y, σ) such that f(A)⊆U. Since f is surjective and gα-irresolute, $f^{-1}(U)$ is a gα-open set of (X, τ). Since A⊆ $f^{-1}(U)$ and A is $^*g\alpha$ –closed set of (X, τ),cl(A) ⊆ $f^{-1}(U)$.

Then $f(cl(A)) \subseteq f(f^{-1}(U)) = U$. Since f is closed, $f(cl(A))=cl(f(cl(A)))$. This implies $cl(f(A)) \subseteq cl(f(cl(A)))= f(cl(A)) \subseteq U$. Therefore f(A) is a *gα –closed set of (Y, σ).

Theorem 5.22: Let f: (X, τ) →(Y,σ) be a surjective, *gα-irresolute and a closed map. If (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$ space, then (Y,σ) is also an ${}_{\alpha}T_{1/2}^{**}$ space.

Proof: Let A be a *gα –closed set of (Y,σ). Since f is *gα-irresolute, $f^{-1}(A)$ is a *gα-closed set of (X, τ). Since (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$ space, $f^{-1}(A)$ is a closed set of (X, τ). Then $f(f^{-1}(A)) = A$ is closed in (Y, σ). Thus A is a closed set of (Y,σ). Therefore (Y,σ) is a ${}_{\alpha}T_{1/2}^{**}$ space.

Definition 5.23: A function f: (X, τ) →(Y, σ) is called pre-*gα-closed if f(A) is a *gα-closed set of (Y, σ) for every *gα-closed set A of (X, τ).

Theorem 5.24: Let f: (X, τ) → (Y,σ) be a surjective, gs-irresolute and a pre-*gα-closed map. If (X, τ) is an T_c^{**} space, then (Y,σ) is also an T_c^{**} space.

Proof: Let A be a gs –closed set of (Y,σ). Since f is gs-irresolute, $f^{-1}(A)$ is a gs-closed set in (X, τ). Since (X, τ) is a T_c^{**} space, $f^{-1}(A)$ is a *gα-closed set in (X, τ). Since f is pre-*gα-closed map, $f(f^{-1}(A))$ is *gα-closed in (Y, σ) for every *gα-closed set $f^{-1}(A)$ of (X, τ). Since f is surjection, $A=f(f^{-1}(A))$. Thus A is a *gα-closed set of (Y,σ). Therefore (Y, σ) is a T_c^{**} space.

Theorem 5.25 Let f: (X, τ) →(Y,σ) be a surjective, αg-irresolute and a pre-*gα-closed map. If (X, τ) is an ${}_{\alpha}T_c^{**}$ space, then (Y, σ) is also an ${}_{\alpha}T_c^{**}$ space.

Proof: Let A be a αg–closed set of (Y, σ). Since f is αg -irresolute, $f^{-1}(A)$ is a αg -closed set in (X, τ). Since (X, τ) is a ${}_{\alpha}T_c^{**}$ space, $f^{-1}(A)$ is a *gα-closed set in (X, τ). Since f is pre-*gα-closed map, $f(f^{-1}(A))$ is *gα-closed in (Y, σ) for every *gα-closed set $f^{-1}(A)$ of (X, τ). Since f is surjection, $A=f(f^{-1}(A))$. Thus A is a *gα-closed set of (Y, σ). Therefore (Y,σ) is a ${}_{\alpha}T_c^{**}$ space.

Theorem 5.26: Let f: (X, τ) → (Y,σ) be a surjective, gc-irresolute and a pre-*gα-closed map. If (X, τ) is an ${}^{**}{}_{\alpha}T_{1/2}$ space, then (Y, σ) is also an ${}^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g–closed set of (Y,σ). Since f is gc-irresolute, $f^{-1}(A)$ is a g -closed set in(X, τ). Since (X, τ) is a ${}^{**}{}_{\alpha}T_{1/2}$ space, $f^{-1}(A)$ is a *gα-closed set in (X, τ). Since f is pre-*gα-closed map, $f(f^{-1}(A))$ is *gα-closed in (Y, σ) for every *gα-closed set $f^{-1}(A)$ of (X, τ). Since f is surjection, $A=f(f^{-1}(A))$. Thus A is a *gα-closed set of (Y, σ). Therefore (Y, σ) is a ${}^{**}{}_{\alpha}T_{1/2}$ space.

6.*Generalized αc - homeomorphism and their group structure

Definition 6.1: A function f: (X, τ) →(Y,σ) is said to be *gα-open if the image f (U) is *gα-open in (Y, σ) for every open set U of (X, τ).

Definition 6.2: A function f: (X, τ) → (Y,σ) is said to be *gα-closed if the image f (U) is *gα-closed in (Y,σ) for every closed set U of (X, τ).

Definition 6.3: A function f: (X, τ) →(Y,σ) is said to be *gαc- homeomorphism (resp.*gα- homoeomorphism) if f is bijective and f and f^{-1} are *gα-irresolute (resp. *gα-continuous).

Theorem 6.4:

(i) Suppose that f is bijection. Then the following conditions are equivalent:
(1) f is *gα- homoeomorphism.
(2) f is *gα-open and *gα-continuous.
(3) f is *gα-closed and *gα-continuous.
(ii) If f is a homeomorphism, then f and f^{-1} are *gα-irresolute.
(iii) Every *gαc- homeomorphism is a *gα- homoeomorphism.

Proof:
(ii) First we prove that f^{-1} is *gα-irresolute. Let A be a *gα-closed set of (X, τ). To show $(f^{-1})^{-1}(A) = f(A)$ is *gα-closed in (Y, σ). Let U be a gα-open set such that $f(A) \subseteq U$. Then $A = (f^{-1}(f(A)) \subseteq f^{-1}(U)$ is gα-open. Since A is *gα-closed, $cl(A) \subseteq f^{-1}(U)$.We have $cl(f(A)) \subseteq f(cl(A)) \subseteq f(f^{-1}(U)) = U$ and so f(A) is *gα-closed. Thus f^{-1} is *gα-irresolute. Since f^{-1} is also a homeomorphism $(f^{-1})^{-1} = f$ is *gα-irresolute.

(iii) Let f is bijective. Since f is *gαc- homeomorphism, f and f^{-1} are *gα-irresolute. Since every *gα-irresolute map is *gα-continuous, then f and f^{-1} are *gα-continuous. Therefore f is *gα- homoeomorphism.

Definition 6.5: For a topological space (X, τ) we define the following three collections of functions:
(i) *gαch (X, τ) = {f/ f: (X, τ) → (X, τ) is a *gαc- homeomorphism}.
(ii) *gαh (X, τ) = {f/ f: (X, τ) → (X, τ) is a *gα- homeomorphism}.
(iii) h (X, τ) = {f/ f: (X, τ)→ (X, τ) is a homeomorphism}.

Corollary 6.6: For a space (X, τ) the following properties hold.
(i) h (X, τ) ⊆ *gαch (X, τ) ⊆ *gαh (X, τ).
(ii) The set *gαch (X, τ) forms a group under composition of functions.
(iii) The group h (X, τ) is a subgroup of *gαch (X, τ).
(iv) If f: (X, τ) →(Y,σ) is a *gαc- homeomorphism then it induces an isomorphism f_*: *gαch (X, τ) → *gαch(Y, σ).

Proof:
(i) These implications are obtained by theorem 6.4(ii), (iii).
(ii) By theorem 5.19.
(iii) By (i).
(iv) We define f_*: *gαch (X, τ) → *gαch(Y, σ) by $f_*(h) = fohof^{-1}$. Then using 5.19 we have that $f_*(h)\in$ *gαch (X, τ). It is shown that f_* is the required group isomorphism.

Remark 6.7: The following example shown that the converse of the above theorem (iv) is not true.

Example 6.8: Let X = {a, b, c}= Y with with τ = { X, ϕ, {a},{b, c}} and σ = {Y, ϕ, {a, b}}.
Define f: (X, τ) →(Y, σ) by f(a)=c, f(b)=a, f(c)=b.
*Gα C(X, τ)={X, ϕ, {a},{b, c}}.
*Gα C(Y, σ) ={Y, ϕ, {c},{b, c},{a,c}}.

Also define three functions h_a, h_b, h_c: (X, τ)→(X, τ) by
$h_a(a)=a$, $h_a(b)=c$, $h_a(c)=b$
$h_b(a)=a$, $h_b(b)=b$, $h_b(c)=c$
$h_c(a)=b$, $h_c(b)=a$, $h_c(c)=c$

Then it is shown that *gαch (X, τ) = {1_x, h_a}, *gαch(Y,σ) = {1_y, h_c} and f_*: *gαch (X, τ) → *gαch(Y,σ) is an isomorphism such that $f_*(h_a) = h_b$. However f is not *gαc- homeomorphism.

7. EXAMPLES IN THE DIGITAL PLANE

In the digital plane, we investigate explicite forms of GαO-kernel α-kerenl and of a subset. The digital line or the so called Khalimsky line is the set of the integers Z, equipped with the topology k having {{2n+1,2n,2n –1}/n∈Z} as a subbase. This is denoted by (Z, k). Thus, a subset U is open in (Z, k) if and only if whenever x∈U is an even integer, then x-1,x+1∈U. Let (Z^2,k^2) be the topological product of two digital lines (Z, k), where Z^2 =Z×Z and k^2=k×k. This space is called the digital plane in the present paper(cf.[5],[11],[12]). We note that for each point x∈ Z^2 there exists the smallest open set containing x, say U(x). For the case of x = (2n+1,2m+1), U(x)={2n+1}× {2m+1}; for the case of x = (2n,2m), U(x) = {2n-1,2n,2n+1}× {2m-1,2m,2m+1}; for the case of x = (2n,2m+1), U(x) = {2n-1,2n,2n+1}× {2m+1}; for the case of x = (2n+1,2m), U(x) = {2n+1}× 2m-1,2m,2m+1}, where n,m∈Z. For a subset E of (Z^2,k^2), we define the following three subsets as follows: E_F = {x∈E/ x is closed in (Z^2,k^2)}; E_{k^2} = {x∈E/ x is open in (Z^2,k^2)}; E_{mix}= E\($E_F \cup E_{k2}$). Then it is shown that E_F = {(2n, 2m)∈E/ n,m ∈ Z},E_{k^2}= {(2n+1, 2m+1)∈E/ n,m∈ Z} and E_{mix}={(2n, 2m+1) ∈E/ n,m ∈ Z}∪ {(2n+1, 2m) ∈E/ n,m ∈ Z}.

Theorem 7.1: Let A and E be subsets of (z^2, k^2).
(i) If E is non - empty α–closed set, then $E_F \neq \phi$[8].
(ii) If E is α– closed and $E \subseteq B_{mix} \cup B_{k^2}$ holds for some subset B of (z^2, k^2) then E = ϕ[8].
(iii) The set U (A_F) ∪ $A_{mix} \cup A_{k^2}$ is a gα-open set containing A.

Proof:
(iii): We claim that $A_{mix} \cup A_{k^2}$ is a gα-open set . Let F be any non –empty α– closed set such that $F \subseteq A_{mix} \cup A_{k^2}$. Then by (ii) , F = ϕ. Thus, we have that F ⊆ α - Int ($A_{mix} \cup A_{k^2}$) then $A_{mix} \cup A_{k^2}$ is gα - open . But we know that U (A_F) is a open set. Then U (A_F) ∪ $A_{mix} \cup A_{k^2}$ is gα-open by theorem 3.14. But $A = A_F \cup A_{mix} \cup A_{k^2}$. $A \subseteq U(A_F) \cup A_{mix} \cup A_{k^2}$. This implies that gα-open set contains A.

Theorem 7.2: Let A be a subset of (Z^2, k^2). The Gαo– kernel of A and the α-kernel of A are obtained precisely as follows:
(i) Gαo–ker (A) = $U(A_F) \cup A_{mix} \cup A_{k^2}$, where $U(A_F) = \cup \{ U(x) \mid x \in A_F\}$.
(ii) α-ker (A) = U(A) , where $U(A) = \cup \{ U(x) \mid x \in A\}$[8].

Proof:
(i): Let $U_A = U(A_F) \cup A_{mix} \cup A_{k^2}$. By Lemma 7.1 (iii), Gαo - ker (A) $\subseteq U_A$.

To prove $U_A \subseteq$ Gαo-ker (A), it is claimed that (*)if there exists a gα-open set V such that $A \subseteq V \subset U_A$ then $V = U_A$. Indeed, let x be any point of U_A. There are three cases for the point x.

Case (1): $x \in (U_A)_F$. we note that $(U_A)_F = (U(A_F))_F \cup (A_{mix} \cup A_{k^2})_F = A_F$.

Then we have that $x \in A_F \subseteq A \subseteq V$.

Case (2): $X \in (U_A)_{k^2}$. We note that

$(U_A)_{k^2} = (U(A_F)_{k^2}) \cup (A_{mix})_{k^2} \cup (A_{k^2})_{k^2} = (U(A_F))_{k^2} \cup A_{k^2}$.

Firstly suppose that $x \in U(A_F)$ Then $x \in U(y)$ for some $y \in A_F$. Since $y \in A_F \subseteq A \subseteq V$ and V is gα-open, we have $\{y\} \subseteq$ α-Int (V). Then $U(y) \subseteq$ α-Int (V), because α-Int (V) is α-open. Thus we have that $x \in V$.

Secondly, suppose $x \in A_{k^2}$, then we have $x \in V$, because $x \in A_{k^2} \subseteq A \subseteq V$.

Case (3): $x \in (U_A)_{mix}$. We note that

$$
\begin{aligned}
(U_A)_{mix} &= (U(A_F))_{mix} \cup (A_{k^2})_{mix} \cup (A_{mix})_{mix} \\
&= (U(A_F))_{mix} \cup A_{mix}
\end{aligned}
$$

Firstly suppose that $x \in U(A_F)$. Then $x \in U(y)$ for some $y \in A_F$. Then y be a α–closed point since every closed point is α–closed point. Since $y \in A_F \subseteq A \subseteq V$, {y} is α–closed and V is gα - open set, we have $\{y\} \subseteq$ α - Int (V). Then U (y) $\subseteq$ α - Int (V) and so $x \in V$.

Secondly, suppose that $x \in A_{mix}$. Then $x \in A_{mix} \subseteq A \subseteq V$ implies $x \in V$.

For all cases we assume that $x \in U_A$ then we show that $x \in V$, then $U_A \subseteq V$. But we know that $V \subseteq U_A$. From the above cases we conclude that $V = U_A$. Thus we shown (*).

Let Gαo(A) be the family of all gα-open sets containing A. Then, we have that $U_A \subseteq W$ for each $W \in$ Gαo(A), using (*) above and properties that $A \subseteq W \cap U_A \subseteq U_A$ and $W \cap U_A$ is gα -open set. Hence, we show that $U_A \subseteq \cap\{W / W \in$ Gαo(A)} = Gαo–ker (A).

That is $U_A \subseteq$ Gαo–ker (A). Therefore Gαo–ker (A) = U_A.

Theorem 7.3: Let E be a subset of (Z^2, k^2).
(i) If E is a non-empty gα-closed set, then $E_F \neq \phi$.
(ii) If E is gα-closed set and $E \subseteq B_{mix} \cup B_{k^2}$ holds for some subset B of (Z^2, k^2), then $E = \phi$.

Proof:
(i): We recall that a subset E is gα-closed if and only if αcl(E) $\subseteq$ α-ker (E).Let y be a point of E.

We consider the following three cases for the point y.

Case 1: $y \in E_{k^2}$. Let y = (2n+1, 2m+1) for some $n,m \in Z$. Then αcl(y)= {2n, 2n+1, 2n+2} × {2m, 2m+1, 2m+2} $\subseteq$ αcl(E) $\subseteq$ α-ker (E). Thus there exists a point (2n, 2m) $\in$ α-ker (E), say y_1 = (2n, 2m). Using theorem 7.2(ii), we have that $y_1 \in U(z)$ for some $z \in E$.

If $z \in E_{mix}$, say z = (2s+1, 2t) for some $s,t \in Z$, then U(z) = {2s+1}× {2t-1,2t,2t+1} and $y_1 \notin U(Z)$. This is a contradiction.

Next if $z \in E_k^2$, say $z = (2s+1, 2t+1)$ for some $s,t \in Z$, then $U(z) = \{(2s+1, 2t+1)\}$ and $y_1 \notin U(z)$. This is also a contradiction.

Thus we have that $z \in E_F$ and hence $E_F \neq \phi$ for case1.

Case 2: $y \in E_{mix}$. Let $y = (2n+1, 2m)$ for some $n,m \in Z$. Then $\alpha cl(y) = \{2n, 2n+1, 2n+2\} \times \{2m\} \subseteq \alpha cl(E) \subseteq \alpha\text{-ker}(E)$. Thus there exists a point $(2n, 2m) \in \alpha\text{-ker}(E)$, say $y_1 = (2n, 2m)$. Using theorem 7.2(ii), we have that $y_1 \in U(z)$ for some $z \in E$.

If $z \in E_{mix}$, say $z = (2s+1, 2t)$ for some $s,t \in Z$, then $U(z) = \{2s+1\} \times \{2t-1, 2t, 2t+1\}$ and $y_1 \notin U(z)$. This is a contradiction.

Next if $z \in E_k^2$, say $z = (2s+1, 2t+1)$ for some $s,t \in Z$, then $U(z) = \{(2s+1, 2t+1)\}$ and $y_1 \notin U(z)$. This is also a contradiction.

Thus we have that $z \in E_F$ and hence $E_F \neq \phi$ for case2.

Case 3: $y \in E_F$. Then $E_F \neq \phi$.

We shown that $E_F \neq \phi$ for all cases.

(ii): Suppose that $E \neq \phi$. By (i) we have that $E_F \neq \phi$. It follows from assumption and definition that $E_F \subseteq (B_{mix} \cup B_k^2)_F = \phi$. We have a contradiction.

Theorem 7.4: Let A be a subset in (Z^2, k^2).
(i) If $(Z^2)_F \subseteq A$ holds, then A is $^*g\alpha$-closed.
(ii) If $(Z^2)_F \subseteq A$ holds and there exists a point $x \in A_k^2$ such that $cl\{x\} \subseteq A$, then A is $^*g\alpha$-closed set which is not α-closed.

Proof:
(i) Using theorem 7.2, we have $G\alpha o\text{-ker}(A) = U(A_F) = Z^2$. Then, A is $^*g\alpha$-closed set by theorem 3.21.
(ii) By(i), A is $^*g\alpha$-closed set. Since $\{x\} \subseteq A_k^2 \subseteq A$ and $Int(cl(\{x\})) = \{x\}$, we have that $cl(\{x\}) \subseteq cl(Int(cl(A)))$ and so $cl(\{x\}) \subseteq \alpha cl(A)$. Suppose that A is α-closed. Then, we have that $cl(\{x\}) \subseteq A$. This is a contradiction.

Example 7.5: The converse of the theorem 7.3(i) is not true in general. A set $A=\{x, y, z\}$ where $x = (3, 3)$, $y = (3, 2)$ and $z = (4, 2)$ is not gα-closed but $A_F \neq \phi$.

REFERENCES

[1] S.P. Arya and T. Nour, Characterizations of s-normal spaces, Indian J. Pure. Appl. math., **21**(8) (1990), 717-719.

[2] K. Balachandran, P. Sundram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi Univ. Ser A. Math, **12**(1991), 5-13.

[3] N. Biswas, On charaterization of semi-continuous functions, Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. natur. **48**(8) (1970), 399-402.

[4] R. Devi, K. Balachandran and H. Maki, Generalized α-closed maps and α-generalized closed maps, Indian J. Pure. Appl. Math, **29**(1) (1998), 37-49.

[5] R. Devi, K. Bhuvaneshwari and H. Maki, Weak form on gρ-closed sets, where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$ and the digital plane, Mem. Fac. Sci. Kochi Univ. (Math.), Vol.25 (2004), 37-54.

[6] R. Devi, H. Maki and K. Balachandran, Semi-generalized closed maps and generalized semi-closed maps, Mem. Fac. Sci. Kochi Univ. Ser. A, Math **14**(1993),41-54.

[7] R.Devi, H.Maki and K.Balachandran, Semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces, Indian J.Pure. Appl.Math, **26**(3) (1995), 271-284.

[8] R. Devi and K. Mohaneshwari, On μs-closed sets and digital plane, Bulletin of pure and Applied Sciences, Vol. 25E(No.2) 2006: P.1-7.

[9] J. Dontchev, On generalizing semi-preopen sets, Mem. Fac. Sci. Kochi. Univ. Ser A. Math, **16**(1995), 35-48.

[10] Y.Ganambal, On generalized preregular closed sets in topological spaces, Indian J. Pure Appl. Math, **28**(3)(1997),351-360.

[11] E.D. Khalimsky, R. Kopperman and P.R. Meyer, Computer graphics and connected topologies in finite ordered sets, Topology Appl.. **36**(1990), 1-17.

[12] T.Y. Kong, R. Kopperman and P.R. Meyer, A topological approach to digital topology, Amer. Math. Monthly, **98**(1991), 901-907.

[13] N.Levine. Generalized closed sets in topology, Rend. Circ. Math. Palermo, **19**(2) (1970), 89-96.

[14] N.Levine, Semi-open sets and semi-continuity in topological spaces, Amer.Math. Monthly, **70**(1963), 36-41.

[15] A.S. Mahhour, I.A. Hasanein and S.N. El-Deeb,α-continuous and α-open mappings, Acta Math. hung, **41**(3-4) (1983), 213-218.

[16] H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi. Univ. Ser. A, Math, **15**(1994), 51-63.

[17] H. Maki, R. Devi and K. Balachandran, Generalized α-closed sets in topology, Bull. Fukuoka Univ. Ed. Part III, 42(1993), 13-21.

[18] H. Maki, J. Umehara and T. Noiri, Every topological space is pre-T1/2, Mem. Fac Sci. Kochi Univ. Ser. A, Math, **17**(1996), 33-42.

[19] O.Njastad, On some classes of nearly open sets, Pacific J. Math, **15**(1965),961- 970.

[20] P. Sundram, M. Sheik John, Weakly closed sets and weakly continuous maps in topological spaces, Proc.82nd Indian Sci. Cong., Calcutta, (1995)49.

[21] M.K.R.S.Veerakumar, Between closed sets and g-closed sets, Mem. Fac. Sci. Kochi. Univ.Ser. A, Math., **21**(2000), 1-19.

[22] M.K.R.S. Veerakumar, Between g^* - closed sets and g-closed sets, Antartica J. Math., **3**(1)(2006), 43-65.

Source of support: Nil, Conflict of interest: None Declared