International Journal of Mathematical Archive-2(2), Feb. - 2011, Page: 280-284
Available online through www.ijma.info ISSN 2229 – 5046

# FIXED POINT THEOREMS FOR CONTRACTION MAPPINGS SATISFYING ASYMPTOTICALLY REGULARITY IN INTEGRAL SETTING

**Mantu Saha[1] and Anamika Ganguly[2]**

*[1]Department of Mathematics, The University of Burdwan, Burdwan-713104, West Bengal*
*[2]Burdwan Railway Balika Vidyapith High School, Khalasipara, Burdwan-713101, West Bengal, India*
*E-mail: mantusaha@yahoo.com, anamika.simi@gmail.com*

*(Received on: 25-01-11; Accepted on: 04-02-11)*

---

## ABSTRACT

*In the present paper, we prove analogues of some fixed pointic results for contraction type mappings having asymptotic regularity property in integral setting.*

***2000 Mathematics Subject Classification:*** *Primary 47H10; Secondary 54H25.*

***Key words and phrases:*** *fixed point, general contractive condition, integral type.*

---

## 1. INTRODUCTION AND PRELIMINARIES:

Banach's contraction mapping principle [1] is one of the pivotal results of non- linear analysis. It has been the source of metric fixed point theory and its significance rests in its vast applicability in different branches of mathematics. In the general setting of complete metric space, this theorem runs as follows (see Theorem 2.1, [5] or, Theorem 1.2.2, [13]).

**Theorem: 1.1.** (Banach's contraction principle) *Let (X, d) be a complete metric space,* $c \in (0,1)$ *and* $f : X \to X$ *be a mapping such that for each* $x, y \in X$,

$$d\left(fx, fy\right) \le cd\left(x, y\right) \tag{1.1}$$

*then* $f$ *has a unique fixed point* $a \in X$ *, such that for each* $x \in X, \lim_{n\to\infty} f^n x = a$ .

After this classical result, Kannan [6] gave a substantially new contractive mapping to prove the fixed point theorem. Since then, a number of researchers have been proving the existence of fixed point theorems by using more generalised contractive conditions ( see [2], [9], [10] [11], [12]). On the otherhand, asymptotically regularity (a.r.) property for a self-map has been investigated by F. E. Browder and W.V. Petryshyn [4] in 1966.

**Definition: 1.2.** A self-mapping $T$ on a metric space $(X, d)$ is said to be asymp- totically regular at a point $x$ in $X$, i.e. if

$$d\left(T^n x, T^{n+1} x\right) = 0 \text{ as } n \to \infty \tag{1.2}$$

where $T^n x$ denotes the $n$-th iterate of $T$ at $x$.

Motivated by this concept researchers have been interested to use this notion for various contraction mappings as an additional sufficient condition to force the map extracting fixed point. They have verified by examples that without this property (a.r.) contractive mappings are not sufficient enough to produce fixed point.

In 2002, A. Branciari [3] for the first time analyzed the existence of fixed point for mapping $f$ defined on a complete metric space $(X, d)$ satisfying a general contractive condition of integral type in the following theorem.

**Theorem: 1.3.** (Branciari) *Let (X, d) be a complete metric space,* $c \in (0,1)$ *and let* $f : X \to X$ *be a mapping such that for each* $x, y \in X$ ,

---

***Corresponding author: Mantu Saha[1], *E-mail: mantusaha@yahoo.com**

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le c\int_0^{d(x,y)} \varphi(t)\,dt \tag{1.3}$$

*where* $\varphi:[0,+\infty)\to[0,+\infty)$ *is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,+\infty)$, *nonnegative, and such that for each* $\in>0, \int_0^{\in}\varphi(t)\,dt>0$, *then* $f$ *has a unique fixed point* $a\in X$ *such that for each* $x\in X, \lim_{n\to\infty} f^n x = a$.

After the paper of Branciari, a lot of research works have been carried out on generalising contractive conditions of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [9] extending the result of Branciari by replacing the condition (1.3) by the following

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le c\int_0^{\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{[d(x,fy)+d(y,fx)]}{2}\right\}} \varphi(t)\,dt \tag{1.4}$$

The aim of this paper is to prove analogues of some fixed pointic results for mixed type of contraction mappings having asymptotic regularity property in integral setting which in turn generalize certain results due to Panja and Baisnab [7] in integral setting.

## 2. MAIN RESULTS:

**Theorem: 2.1.** *Let T be a self mapping of a complete metric space (X, d) satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \alpha\int_0^{[d(x,Tx)+d(y,Ty)]} \varphi(t)\,dt + \beta\int_0^{d(x,y)} \varphi(t)\,dt + \gamma\int_0^{\max\{d(x,Ty),d(y,Tx)\}} \varphi(t)\,dt \tag{2.1}$$

*for each* $x,y\in X$ *with* $\alpha,\beta,\gamma\ge 0$ *and* $\max\{\alpha,\beta\}+\gamma<\frac{1}{2}$. *where* $\varphi:[0,+\infty)\to[0,+\infty)$ *is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,+\infty)$, *nonnegative, and such that*

$$\text{for each } \in>0, \int_0^{\in}\varphi(t)\,dt>0 \tag{2.2}$$

*If T also satisfies*

$$\int_0^{d(T^n x,T^{n+1}x)} \varphi(t)\,dt = 0 \text{ as } n\to\infty, \tag{2.3}$$

*then* $T$ *has a unique fixed point* $z\in X$.

**Proof:** Let $x_0\in X$ be arbitrary. Then for positive integers *m, n,*

$$\begin{aligned}\int_0^{d(T^m x_0,T^n x_0)} \varphi(t)\,dt &= \int_0^{d\left(T(T^{m-1}x_0),T(T^{n-1}x_0)\right)} \varphi(t)\,dt\\
&\le \alpha\int_0^{\left[d(T^{m-1}x_0,T^m x_0)+d(T^{n-1}x_0T^n x_0)\right]} \varphi(t)\,dt + \beta\int_0^{d(T^{m-1}x_0,T^{n-1}x_0)} \varphi(t)\,dt\\
&+\gamma\int_0^{\max\left\{d(T^{m-1}x_0,T^n x_0),d(T^m x_0,T^{n-1}x_0)\right\}} \varphi(t)\,dt\\
&\le \alpha\int_0^{\left[d(T^{m-1}x_0,T^m x_0)+d(T^{n-1}x_0,T^n x_0)\right]} \varphi(t)\,dt + \beta\int_0^{d(T^{m-1}x_0,T^{n-1}x_0)} \varphi(t)\,dt\\
&+\gamma\int_0^{d(T^{m-1}x_0,T^n x_0)} \varphi(t)\,dt + \gamma\int_0^{d(T^m x_0,T^{n-1}x_0)} \varphi(t)\,dt\end{aligned}$$

Applying some routine calculation, we have

$$\int_0^{d(T^m x_0,T^n x_0)} \varphi(t)\,dt \le \frac{\alpha+\beta+\gamma}{1-\beta-2\gamma}\left[\int_0^{d(T^{m-1}x_0,T^m x_0)} \varphi(t)\,dt + \int_0^{d(T^{n-1}x_0,T^n x_0)} \varphi(t)\,dt\right] \tag{2.4}$$

where $\max\{\alpha,\beta\}+\gamma<\frac{1}{2}$. Now since $T$ satisfies (2.3), then by (2.4)

$$\int_0^{d(T^m x_0, T^n x_0)}\varphi(t)\,dt \to 0 \text{ as } m,n\to\infty$$

Therefore, $\{T^n x_0\}$ is Cauchy, hence convergent. Call the limit $z\in X$ i.e. $\lim_n T^n x_0 = z$

From (2.1) we get

$$\int_0^{d(z,Tz)}\varphi(t)\,dt \le \int_0^{d(z,T^n x_0)}\varphi(t)\,dt + \int_0^{d(T^n x_0,Tz)}\varphi(t)\,dt$$

$$\le \int_0^{d(z,T^n x_0)}\varphi(t)\,dt + \alpha\int_0^{[d(T^{n-1}x_0,T^n x_0)+d(z,Tz)]}\varphi(t)\,dt$$

$$+\beta\int_0^{d(T^{n-1}x_0,z)}\varphi(t)\,dt + \gamma\int_0^{\max\{d(T^{n-1}x_0,Tz),d(z,T^n x_0)\}}\varphi(t)\,dt$$

Taking limit as $n\to\infty$, we get

$$\int_0^{d(z,Tz)}\varphi(t)\,dt \le (\alpha+\gamma)\int_0^{d(z,Tz)}\varphi(t)\,dt$$

which, from (2.2), implies that $d(z, Tz) = 0$ or, $Tz = z$.

Next suppose that $w(\ne z)$ be another fixed point of $T$. Then from (2.1) we have

$$\int_0^{d(z,w)}\varphi(t)\,dt = \int_0^{d(Tz,Tw)}\varphi(t)\,dt$$

$$\le \alpha\int_0^{[d(z,Tz)+d(w,Tw)]}\varphi(t)\,dt + \beta\int_0^{d(z,w)}\varphi(t)\,dt + \gamma\int_0^{\max\{d(z,Tw),d(w,Tz)\}}\varphi(t)\,dt$$

which implies

$$\int_0^{d(z,w)}\varphi(t)\,dt \le (\beta+\gamma)\int_0^{d(z,w)}\varphi(t)\,dt$$

gives a contradiction. Hence $z = w$ and so the fixed point is unique.

**Theorem: 2.2.** *Let $T$ be a self mapping of a complete metric space $(X, d)$ satisfying* (2.1) *and* (2.2). *If $T$ satisfies* (2.3) *and the sequence of iterates $\{T^n(x)\}$ has a subsequence converging to a point $z\in X$, then $z$ is the unique fixed point of $T$ and $\{T^n(x)\}$ also converges to $z$.*

**Proof:** Let $\lim_k T^{n_k}(x) = z$. Then using (2.1) we get,

$$\int_0^{d(z,Tz)}\varphi(t)\,dt \le \int_0^{d(z,T^{n_k+1}x)}\varphi(t)\,dt + \int_0^{d(T^{n_k+1}x,Tz)}\varphi(t)\,dt$$

$$\le \int_0^{d(z,T^{n_k+1}x)}\varphi(t)\,dt + \alpha\int_0^{[d(T^{n_k}x,T^{n_k+1}x)+d(z,Tz)]}\varphi(t)\,dt$$

$$+\beta\int_0^{d(T^{n_k}x,z)}\varphi(t)\,dt + \gamma\int_0^{\max\{d(T^{n_k}x,Tz),d(z,T^{n_k+1}x)\}}\varphi(t)\,dt$$

Taking limit as $k\to\infty$, we get

$$\int_0^{d(z,Tz)}\varphi(t)\,dt \le (\alpha+\gamma)\int_0^{d(z,Tz)}\varphi(t)\,dt$$

 

which, from (2.2), implies that $d(z, Tz) = 0$ or, $Tz = z$. Also uniqueness of $z$ follows very immediate.

Next,

$$\int_0^{d(z,T^n x)} \varphi(t)\,dt = \int_0^{d(Tz,T^n x)} \varphi(t)\,dt$$
$$\le \alpha \int_0^{\left[d(z,Tz)+d\left(T^{n-1}x,T^n x\right)\right]} \varphi(t)\,dt + \beta \int_0^{d\left(z,T^{n-1}x\right)} \varphi(t)\,dt + \gamma \int_0^{\max\left\{d\left(z,T^n x\right),d\left(T^{n-1}x,Tz\right)\right\}} \varphi(t)\,dt$$

Excercising some routine calculation, we have

$$\int_0^{d(z,T^n x)} \varphi(t)\,dt \le \left(\frac{\alpha+\beta+\gamma}{1-\beta-2\gamma}\right) \int_0^{d\left(T^n x,T^{n-1}x\right)} \varphi(t)\,dt$$

Where $\max\{\alpha,\beta\}+\gamma<\frac{1}{2}$. As $T$ satisfies (2.3),

$$\int_0^{d(z,T^n x)} \varphi(t)\,dt \to 0 \text{ as } n \to \infty$$

which completes the proof.

**Theorem: 2.3.** *Let (X, d) be a complete metric space* $\{T_n\}$ *be a sequence of mappings from X into itself satisfying* (2.1) *and* (2.2) *with same constants* $\alpha,\beta,\gamma$ *and possessing fixed points* $u_n (n = 1, 2, ...)$. *Suppose that* $T(x)=\lim_{n\to\infty} T_n(x)$ *for all* $x \in X$ *and T satisfies* (2.3). *Then T has a unique fixed point u if and only if* $u = \lim_n u_n$.

**Proof:** The proof is trivial.

**Theorem: 2.4.** *Let (X, d) be a complete metric space and* $\{T_j\}_{j\in N}$ *be a sequence of mappings from X into itself*

$$\int_0^{d(T_j x,T_j y)} \varphi(t)\,dt \le \alpha \int_0^{\left[d(x,T_j x)+d(y,T_j y)\right]} \varphi(t)\,dt + \beta \int_0^{d(x,y)} \varphi(t)\,dt + \gamma \int_0^{\max\left\{d(x,T_j y),d(y,T_j x)\right\}} \varphi(t)\,dt \tag{2.5}$$

*for each* $x, y \in X$ *with* $\alpha,\beta,\gamma \ge 0$ *and* $\max\{\alpha,\beta\}+\gamma<\frac{1}{2}$. *where* $\varphi:[0,+\infty)\to[0,+\infty)$ *is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,+\infty)$, *nonnegative, and such that*

*for each* $\in > 0$, $\int_0^{\varepsilon} \varphi(t)\,dt > 0$ (2.6)

*Suppose* $T^n(x) = \lim_{j\to\infty} T_j^n(x)$ *for all* $x \in X$. *Then T has a unique fixed point in X if T satisfies (2.3).*

**Proof:** Let

$$T^n x_0 = \lim_{j\to\infty} T_j^n x_0 \text{ for } x_0 \in X.$$

Then for positive integers *m, n*

$$\int_0^{d\left(T_j^m x_0,T_j^n x_0\right)} \varphi(t)\,dt = \int_0^{d\left(T_j\left(T_j^{m-1}x_0\right),T_j\left(T_j^{n-1}x_0\right)\right)} \varphi(t)\,dt$$
$$\le \alpha \int_0^{\left[d\left(T_j^{m-1}x_0,T_j^m x_0\right)+d\left(T_j^{n-1}x_0,T_j^n x_0\right)\right]} \varphi(t)\,dt + \beta \int_0^{d\left(T_j^{m-1}x_0,T_j^{n-1}x_0\right)} \varphi(t)\,dt$$
$$+\gamma \int_0^{\max\left\{d\left(T_j^{m-1}x_0,T_j^n x_0\right),d\left(T_j^m x_0,T_j^{n-1}x_0\right)\right\}} \varphi(t)\,dt$$
$$\le \alpha \int_0^{\left[d\left(T_j^{m-1}x_0,T_j^m x_0\right)+d\left(T_j^{n-1}x_0,T_j^n x_0\right)\right]} \varphi(t)\,dt + \beta \int_0^{d\left(T_j^{m-1}x_0,T_j^{n-1}x_0\right)} \varphi(t)\,dt$$

$$+\gamma\int_0^{d\left(T_j^{m-1}x_0,T_j^n x_0\right)}\varphi(t)\,dt+\gamma\int_0^{d\left(T_j^{m}x_0,T_j^{n-1}x_0\right)}\varphi(t)\,dt$$

Again applying some routine calculation, we have

$$\int_0^{d\left(T_j^{m}x_0,T_j^n x_0\right)}\varphi(t)\,dt\le\frac{\alpha+\beta+\gamma}{1-\beta-2\gamma}\left[\int_0^{d\left(T_j^{m-1}x_0,T_j^m x_0\right)}\varphi(t)\,dt+\int_0^{d\left(T_j^{n-1}x_0,T_j^n x_0\right)}\varphi(t)\,dt\right] \tag{2.7}$$

where $\max\{\alpha,\beta\}+\gamma<\frac{1}{2}$.

Letting $j\to\infty$, we obtain

$$\int_0^{d\left(T^{m}x_0,T^n x_0\right)}\varphi(t)\,dt\le\frac{\alpha+\beta+\gamma}{1-\beta-2\gamma}\left[\int_0^{d\left(T^{m-1}x_0,T^m x_0\right)}\varphi(t)\,dt+\int_0^{d\left(T^{n-1}x_0,T^n x_0\right)}\varphi(t)\,dt\right] \tag{2.8}$$

Where $\max\{\alpha,\beta\}+\gamma<1$. Since $T$ satisfies (2.3), then by (2.8)

$$\int_0^{d\left(T^{m}x_0,T^n x_0\right)}\varphi(t)\,dt\to 0 \text{ as } m,n\to\infty$$

Therefore, $\{T^n x_0\}$ is Cauchy sequence. Then by completeness of *X*, $\lim_n T^n x_0 = z\in X$.

Then it follows from Theorem 2.1 that $z$ is the unique fixed point of *T*.

**Remark: 2.5** On setting $\varphi(t)=1$ over $[0,+\infty)$, the contractive condition of integral type transforms into a general contractive condition not involving integrals.

## REFERENCES:


[1] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, Fund. Math. 3, Fund. Math. 3, (1922)133181 (French).

[2] R. Bianchini, Su un problema di S. Reich riguardante la teori dei punti fissi Boll. Un. Math. Ital. 5 (1972), 103-108.

[3] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci, 29 (2002), no.9, 531 - 536.

[4] F. E. Browder and W. V. Petrysyn, The solution by iteration of nonlinear functional equation in Banach spaces, Bull. Amer. Math. Soc. 72 (1966), 571- 576.

[5] K. Goebel and W. A. Kirk, Topics in metric fixed point theory, Cambridge University Press, Newyork, (1990).

[6] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., 60 (1968), 71-76.

[7] C. Panja and A. P. Baisnab, Asymptotic regularity and fixed point theorems, The Mathe- matics Student, 46 1(1978), 54-59.

[8] S. Reich, Kannan's fixed point theorem Boll. Un. Math. Ital. 4 (1971), 1-11.

[9] B. E. Rhoades, Two fixed point theorems for mappings satisfying a general contractive condi- tion of integral type, International Journal of Mathematics and Mathematical Sciences,63 (2003), 4007 - 4013.

[10] B. E. Rhoades, A comparison of various definitions of contractive type mappings, Trans. Amer. Math. Soc., 226 (1977), 257-290.

[11] B. E. Rhoades, Contractive definitions revisited, Topological methods in nonlinear functional analysis, (Toronto, Ont.,1982), Contemp. Math., Vol. 21, American Mathematical Society, Rhoade Island, (1983), 189-203.

[12] B. E. Rhoades, Contractive definitions, Nonlinear Analysis, World Science Publishing, Sin-gapore (1987), 513-526.

[13] O. R. Smart, Fixed Point Theorems, Cambridge University Press, London, 1974.


 