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A BRIEF STUDY ON GOMPERTZ-MAKEHAM MODEL AND SOME ASPECTS ON AGRICULTURAL GROWTH OF ASSAM

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ABSTRACT

Here we study some aspects of Gompertz-Makeham model that is used as an effective one in solving some socio-economic development problems in Assam so to say mainly the agriculture. Fortran programme has been developed for the testing of validity of the model. Chi-Square goodness of fit test reveals what we have proposed for. It is expected that with the help of the method and the software that we have developed prediction related to growth of Agriculture of Assam and others could be made.

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1. INTRODUCTION

The Gompertz model is, $\mu(x)=Ae^{\beta x}$ where A and β are the parameters of estimation. For our purpose we use here a redefined one, the so called Gompertz-Makeham model, $\mu(x) =A+Be^{\beta x}$ where μ(x) is the force of mortality with extra parameter A responsible for age independent component of the same. Interestingly enough this method fits well to empirical mortality distributions. It is claimed that nearly all subsequent models of the age pattern mortality have been extensions of the Gompertz-Makeham model.

The objectives of our study may be recorded as follows.
i). Fitting of the three parameters of the Gompertz-Makeham model.
ii). Estimation of the three parameters using various methods of estimations.
iii). Select the best fit model based on Chi-Square goodness of fit test.

Here we find that the method of three equidistant points used to estimate the three parameters present in the above model fits the data well compared to the method of three partial sums.

2. PRELIMINARIES

PARAMETER ESTIMATION

2.1 DEFINITION AND METHODS

Parameter estimation is the process of calculating model parameters based on a data set. This data set can be the result of time course or steady-state experiments or both.

In the case of linear models the estimation of parameters can be done using the method of least squares. But for non-linear models the parameters cannot be estimated using the method of least squares.

Hence we will have to go for other methods with the help of which we will be able to estimate the parameters of our desired model.

2.2 Some of the well known methods of parameter estimation are

(i) Method of three equidistant points.
(ii) Method of three partial sums.
(iii) Method of four equidistant points.
(iv) Method of four partial sums.
(v) Method of sum of reciprocals.
(vi) Composite method.

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For the Gompertz Makeham model the following three methods of parameter estimation have been considered:

2.2.i Method of three equidistant points.
2.2.ii Method of three partial sums.

2.2. i Method of three equidistant points

In this method we determine three unknown constants A, B, and λ by selecting three equidistant points such that the whole range of observation is more or less evenly covered.

Let us take three points t_1, t_2, t_3 such that

$$\lambda x_1 = t_1 \,, \quad \lambda x_2 = t_2 \text{, and} \quad \lambda x_3 = t_3$$

Also let, $$t_2 - t_1 = t_3 - t_2 = m$$

Let,

$$\begin{Bmatrix} \mu_{x_1} = A + Be^{\lambda x1} = A + Be^{t_1} \\ \mu_{x_2} = A + Be^{\lambda x2} = A + Be^{t_2} \\ \mu_{x_3} = A + Be^{\lambda x3} = A + Be^{t_3} \end{Bmatrix} \tag{2.2.i.1}$$

Now,

$$\begin{aligned} \mu_{x_2} - \mu_{x_1} &= B\left(e^{t_2} - e^{t_1}\right) \\ &= B\left(e^{m+t_1} - e^{t_1}\right) \\ &= Be^{t_1}(e^m - 1) \end{aligned} \tag{2.2.i.2}$$

Similarly,

$$\begin{aligned} \mu_{x_3} - \mu_{x_2} &= B\left(e^{t_3} - e^{t_2}\right) \\ &= B(e^{2m+t_1} - e^{m+t_1}) \\ &= Be^{m+t_1}(e^m - 1) \end{aligned} \tag{2.2.i.3}$$

Dividing (2.2.3) by (2.2.2),

$$\frac{\mu_{x_3} - \mu_{x_2}}{\mu_{x_2} - \mu_{x_1}} = e^m$$

$$\Rightarrow \frac{\log\left(\mu_{x_3} - \mu_{x_2}\right)}{\mu_{x_2} - \mu_{x_1}} = m = t_2 - t_1$$

$$\Rightarrow \hat{\lambda} = \frac{1}{x_2 - x_1}\log\left(\frac{\mu_{x3} - \mu_{x2}}{\mu_{x2} - \mu_{x1}}\right) \tag{2.2.i.4}$$

Again,

$$\begin{aligned} e^m - 1 &= \frac{\mu_{x_3} - \mu_{x_2}}{\mu_{x_2} - \mu_{x_1}} - 1 \\ &= \frac{\mu_{x_3} - 2\mu_{x_1} + \mu_{x_1}}{\mu_{x_2} - \mu_{x_1}} \end{aligned} \tag{2.2.i.5}$$

So from,(2.2.i.2) and (2.2.i.5) we have,

$$\mu_{x_2} - \mu_{x_1} = Be^{\lambda x_1}\left(\frac{\mu_{x_3} - 2\mu_{x_1} + \mu_{x_1}}{\mu_{x_2} - \mu_{x_1}}\right)$$

$$\Rightarrow \hat{B} = \frac{\left(\mu_{x_2} - \mu_{x_1}\right)^2}{\left[\!\left[(\mu\right]\!\right]_{x_3} - 2\mu_{x_1} + \mu_{x_1}\right) e^{x_1}} \left(\frac{\mu_{x_3} - \mu_{x_1}}{\mu_{x_2} - \mu_{x_1}}\right)^{\frac{1}{x_2 - x_1}} \tag{2.2.i.6}$$

And,

$$\hat{A} = \mu_{x_1} - \hat{B} e^{\hat{\lambda} x_1} \tag{2.2.i.7}$$

putting the value of $\widehat{\boldsymbol{B}}$ and $\widehat{\lambda}$ we get the estimated value of $\hat{A}$. So the estimated values of the parameters $\hat{A}$, $\widehat{\boldsymbol{B}}$, $\widehat{\lambda}$ using the three equidistant points are given by (2.2.i.7), (2.2.i.6) and (2.2.i.4).

2.2. ii Method of three partial sums

In this method the number of observations i.e. the set of data that we will take should be multiple of three.

If N is the total number of observations then $\frac{N}{3} = m$ (say). So the three equidistant points will be:

$$N_1 = m$$
$$N_2 = N_1 + m = 2m$$
$$N_3 = N_2 + m = 3m$$

Let us take,

S_1=sum of the first ‘m’ observations.
S_2=sum of the second ‘m’ observations.
S_3=sum of the third ‘m’ observations.

Therefore,

$$S_1 = mA + \frac{Be^{\lambda}\left(e^{m\lambda} - 1\right)}{e^{\lambda} - 1} \tag{2.2.ii.1}$$

$$S_2 = mA + \frac{Be^{(m+1)\lambda}\left(e^{m\lambda} - 1\right)}{e^{\lambda} - 1} \tag{2.2.ii.2}$$

$$S_2 = mA + \frac{Be^{(2m+1)\lambda}\left(e^{m\lambda} - 1\right)}{e^{\lambda} - 1} \tag{2.2.ii.3}$$

Now,

$$S_2 - S_1 = \frac{Be^{\lambda}\left(e^{m\lambda} - 1\right)^2}{e^{\lambda} - 1} \tag{2.2.ii.4}$$

$$S_3 - S_2 = \frac{Be^{(m+1)\lambda}\left(e^{m\lambda} - 1\right)^2}{e^{\lambda} - 1} \tag{2.2.ii.5}$$

Dividing (2.2.ii.5) by (2.2.ii.4) we get,

$$\frac{S_3 - S_2}{S_2 - S_1} = e^{m\lambda}$$

$$\Rightarrow \hat{\lambda} = \frac{1}{m} log \frac{S_3 - S_2}{S_2 - S_1} \tag{2.2.ii.6}$$

And,

$$e^{m\lambda} - 1 = \frac{S_3 - S_2}{S_2 - S_1} - 1$$
$$= \frac{S_3 - 2S_2 + S_1}{S_2 - S_1} \tag{2.2.ii.7}$$

Now putting the value of $e^{m\lambda} - 1$ in equation (2.2.ii.4) we have,

$$S_2 - S_1 = \frac{Be^{\lambda}}{(e^{\lambda} - 1)\left(\frac{S_3 - 2S_2 + S_1}{S_2 - S_1}\right)^2}$$

Therefore,

$$\hat{B} = \frac{(S_2 - S_1)^3 \left(e^{\lambda} - 1\right)}{(S_3 - 2S_2 + S_1)^2 e^{\lambda}} \tag{2.2.ii.8}$$

Now putting the value of $\hat{B}$ and $e^{m\lambda} - 1$ in equation (2.2.ii.1) we have,

$$mA = S_1 - \frac{Be^{\lambda}(e^{m\lambda} - 1)}{e^{\lambda} - 1}$$

$$\Rightarrow mA = S_1 - \frac{(S_2 - S_1)^2}{S_3 - 2S_2 + S_1}$$

$$= \frac{S_3 S_1 - S_2^{\ 2}}{S_3 - 2S_2 + S_1}$$

Therefore,

$$\hat{A} = \frac{S_3 S_1 - S_2^{\ 2}}{m(S_3 - 2S_2 + S_1)} \tag{2.2.ii.9}$$

Hence, by the method of three partial sums the parameters $\hat{A}$, $\hat{B}$, $\hat{\lambda}$ of the Gompertz Makeham model are given by (2.2.ii.9), (2.2.ii.8) and (2.2.ii.6).

3. MAIN OBSERVATION

3.1 GOODNESS OF FIT TEST

A very powerful test for testing the significance of the discrepancy between theory and experiment was given by Prof. Karl Pearson in 1900 and is known as “Chi-Square goodness of fit”. It enables is to find whether the deviation of the experiment from theory is just by chance or is it really due to the inadequacy of the theory to fit the observed data. In this test we compare observed values with theoretical or expected values.

If O_i(i=1,2,3,............,n) is the observed(experimental) frequencies and E_i (i=1,2,3,............,n) is the corresponding set of expected(theoretical or hypothetical) frequencies, then Karl Pearson’s Chi-square, is given by,

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

with (n-1) degrees of freedom.

This is an approximation test for large values of n. For the validity of ‘Chi square test of ‘goodness of fit’ between theory and experiment, the following conditions must be satisfied:

(a) The sample observations should be independent.
(b) Constraints, if any, should be linear.
(c) N, the total frequency should be reasonably large.

It may be noted that the χ^2 test depends only on the set of observed and expected frequencies and on degrees of freedom. It does not make assumptions regarding the parent population from which the observations are taken. χ^2 follows Chi-square distribution with (n-1) degrees of freedom. If calculated χ^2 >tabulated value at a particular level of significance, then χ^2 test is significant and does not fit well to the distribution at a particular level of significance. If calculated χ^2 <tabulated value at a particular level of significance, then χ^2 test is not significant and fits well to the distribution at a particular level of significance. Using goodness of fit test the χ^2 values of the following table and the estimated values of the parameters using:

(a) Estimation using three equidistant points
(b) Estimation using three partial sums, are given.

3.2 Fitting of Gompertz- Makeham model:

TABLE 1: Agricultural production in Assam

A=	-281.318	92.869
B=	305.792	-68.886
C=	0.020	-0.147

year	Observed data	(a)	(b)
1988	30.700	30.700	33.425
1989	39.900	37.053	41.572
1990	53.000	43.536	48.603
1991	56.900	50.151	54.670
1992	66.200	56.900	59.906
1993	65.600	63.787	64.424
1994	71.900	70.814	68.323
1995	77.100	77.984	71.687
1996	77.100	85.300	74.590
1997	85.300	92.765	77.096
	Chi square=	5.274	1.981

TABLE 2: Net domestic product at factor cost by economic activity in agriculture

A=	94.420	37.879
B=	-88.853	-32.583
C=	-0.020	-0.062

year	Observed values	(a)	(b)
1	55.670	55.670	68.163
2	91.090	81.738	92.356
3	115.620	103.687	112.574
4	133.600	122.168	129.469
5	147.350	137.729	143.589
6	158.210	150.832	155.389
7	167.000	161.864	165.250
8	174.260	171.153	173.490
9	180.360	178.974	180.377
10	185.560	185.560	186.132
11	190.030	191.105	190.941
12	193.940	195.774	194.941
13	197.360	199.705	198.319
14	200.400	203.015	201.126
15	203.110	205.125	203.472
16	205.540	208.149	205.432
17	207.730	210.125	207.071
18	209.720	211.789	208.440
19	211.530	213.190	209.584
20	213.190	214.370	210.540
	Chi square=	4.996	2.774

TABLE 3: General consumer price index number for agricultural labour in Assam

A=	4417.010	7484.693
B=	-3745.901	-6780.670
C=	-0.036	-0.017

Year	Observed data	(a)	(b)
1990	805	805.000	821.192
1991	925	934.105	936.337
1992	1077	1058.596	1049.492
1993	1142	1178.637	1160.691
1994	1260	1294.387	1269.969
1995	1406	1406.000	1377.359
1996	1471	1513.623	1482.893
1997	1551	1617.400	1586.603
1998	1736	1717.468	1688.522
1999	1868	1813.958	1788.679
2000	1907	1907.000	1887.105
	Chi square=	8.197	8.109

TABLE 4: Food group consumer price index number for agricultural labour in Assam

A=	3245.292	3058.433
B=	16.681	186.650
C=	0.401	0.104

Year	Observed data	(a)	(b)
1991	3270.200	3270.200	3265.524
1992	3297.200	3282.483	3288.205
1993	3299.700	3300.824	3313.370
1994	3361.100	3328.209	3341.291
1995	3390.100	3369.100	3372.269
1996	3428.200	3430.156	3406.640
1997	3482.200	3521.322	3444.776
1998	3554.800	3657.446	3487.088
1999	3554.800	3860.700	3534.035
2000	3860.700	4184.189	3586.123
	Chi square=	58.610	21.520

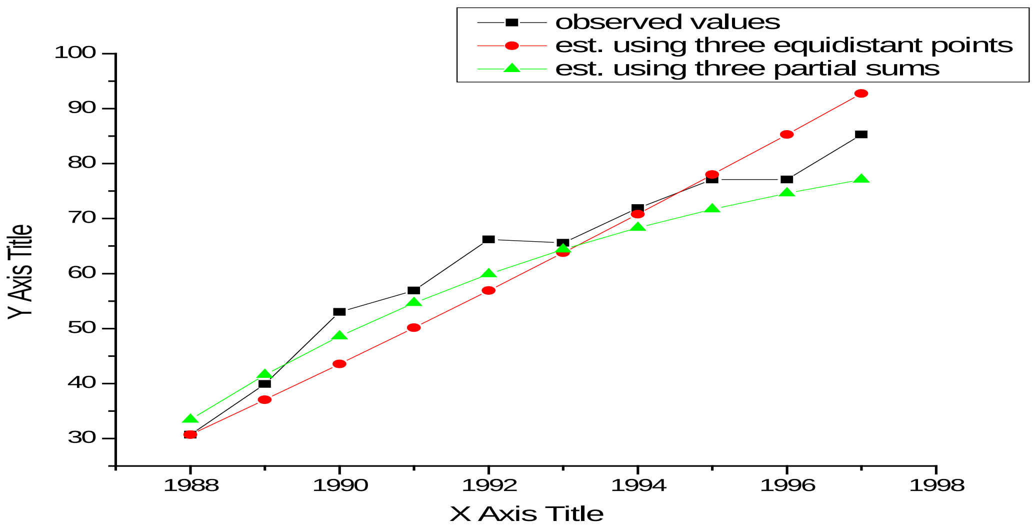

Figure 1(Table 1)

Table 1

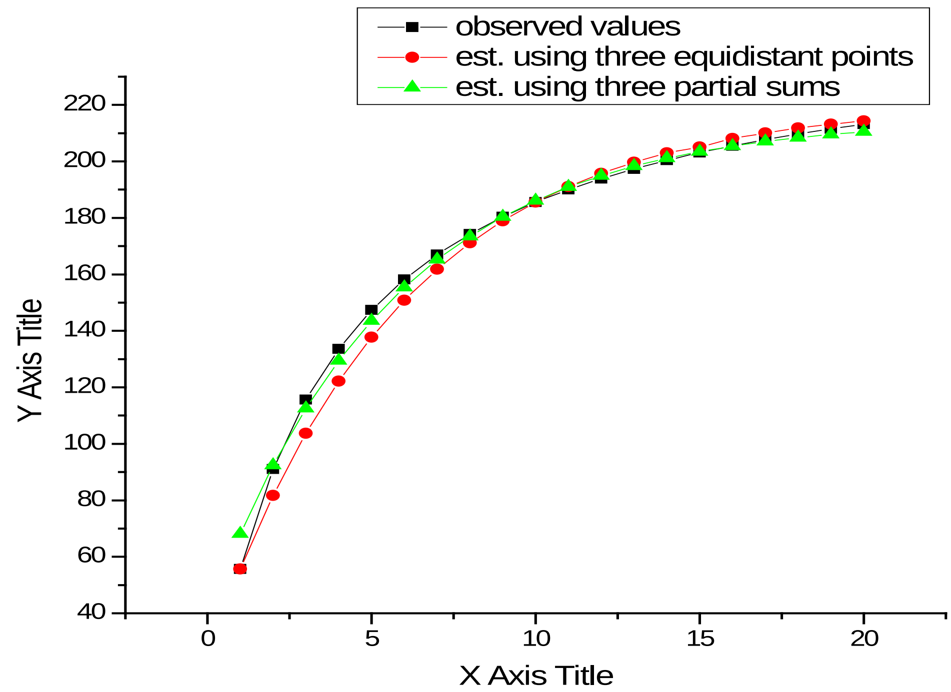

Table 2

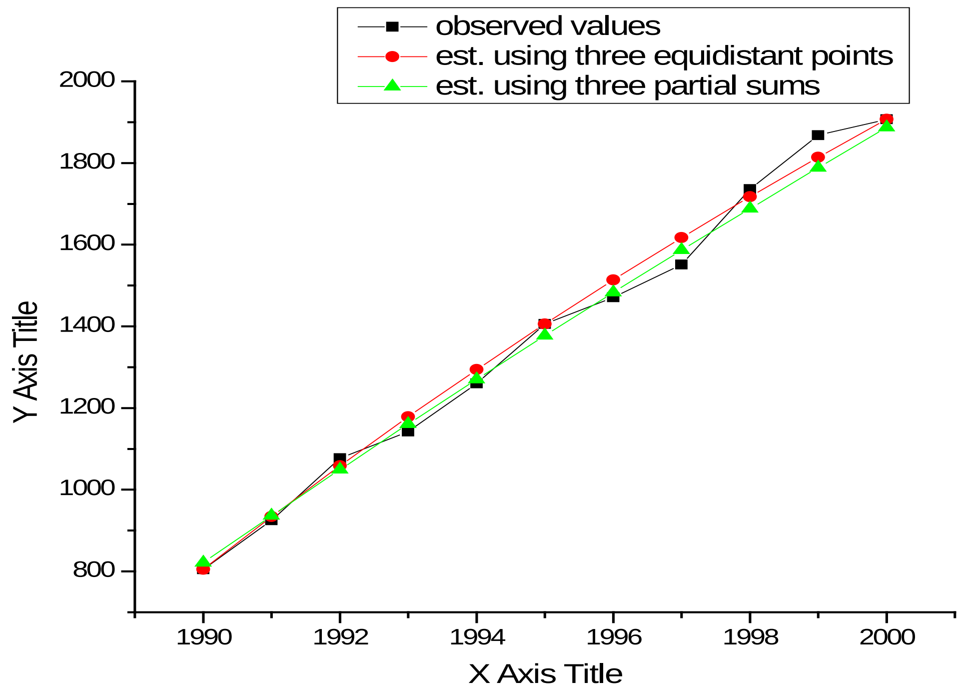

Table 3

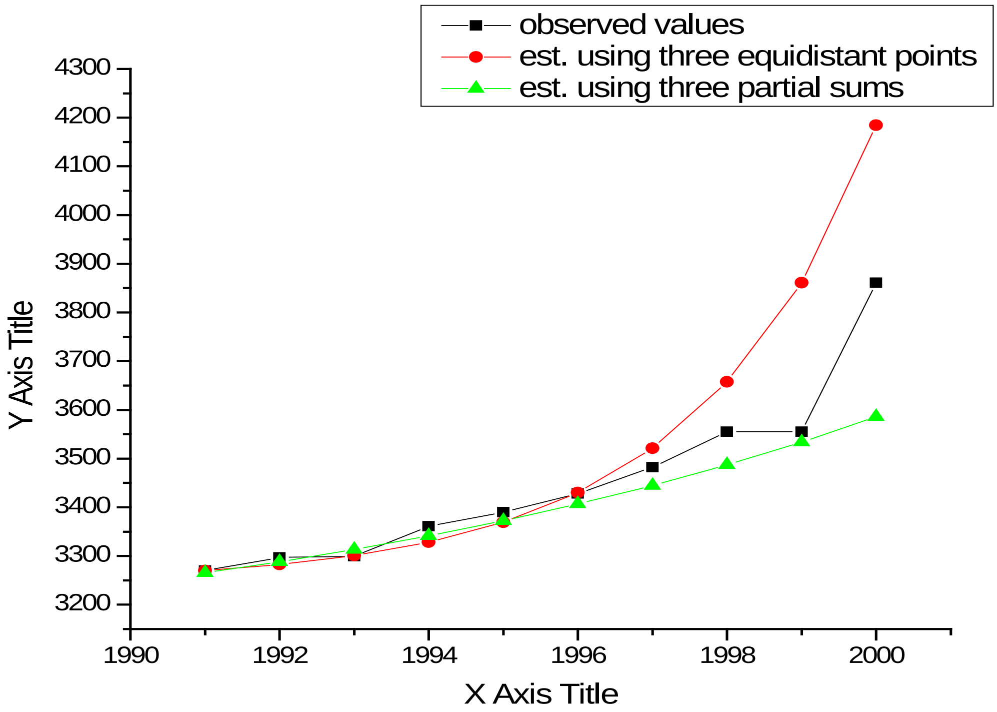

Table 4

3.3 CONCLUSION

From Chi square test for the set of data given in TABLE 1, TABLE 2, TABLE 3, TABLE 4 we see that the method of three equidistant points gives us the best fit than the method of three partial sums which is also clear from the graphs drawn.

3.5 The Fortran programming used for the estimations is given below:

```
C          FITTING OF GOMPERTZ-MAKEHAM MODEL WITH THREE PARAMETERS
C          Y=A+B*E**(C*T)
C           1. METHOD OF THREE EQUIDISTANT POINTS
C           2. MEHTOD OF THREE PARTIAL SUMS
            INTEGER T1,T2,T3
            DIMENSION T(40), P(40), X(40),Y(40),YCAL(40),XX(40,6)
            DIMENSION  Y1(40), Y2(40), Y3(40),Z(40), PP(40)
            OPEN(UNIT=5, FILE='TEA.DAT')
            OPEN(UNIT=6, FILE='GOMPART3.DOC')
            READ(5,*) N,M
            P(I)=XX(I,1)
C           WRITE(*,31) T(I),P(I)
    10      CONTINUE
            CALL THREEP(N,P,A,B,C)
            WRITE(6,55)
            WRITE(6,*) '(a):  ESTIMATION USING THREE EQUIDISTANT POINTS'
            AN=N
            AA1=A
            BB1=B
            CC1=C
            DO 26 I=1,N
            Y1(I)=AA1+BB1*(EXP(CC1*I))
C           WRITE(6,31) T(I),PP(I),Y1(I)
   26       CONTINUE
            CH1=0.0
            DO 301 I=1,N
            CH1=CH1+(P(I)-Y1(I))**2/ABS(Y1(I))
  301       CONTINUE
```



```
C              WRITE(6,57) AA1,BB1,CC1,CH1
               CALL PARTSUM(N,P,A,B,C)
               WRITE(6,*) '(b): ESTIMATION USING THREE PARTIAL SUMS'
               AA2=A
               BB2=B
               CC2=C
               DO 24 I=1.N
               Y2(I)=AA2+BB2*(EXP(CC2*I))
C              WRITE(6,31) T(),P(),Y2()
   24          CONTINUE
               CH2=0.0
               DO 302 I=1.N
               CH2=CH2+(P(I)-Y2(I))**2/ABS(Y2(I))
  302          CONTINUE
C              WRITE(6,57) AA2,BB2,CC2,CH2
               WRITE(6,*)
               WRITE(6,55)
               WRITE(6,56)
               WRITE(6,51) AA1,AA2
               WRITE(6,52) BB1,BB2
               WRITE(6,53)CC1,CC2
               WRITE(6,54)
               DO 45 I=1,N
               WRITE(6,31) T(I),P(I),Y1(I),Y2(I)
  45           CONTIN UE
               WRITE(6,56)
               WRITE(6,54) CH1,CH2
  31           FORMAT(5X, 4F10.3)
  51           FORMAT(20X, 'A= ',2F10.3)

  52           FORMAT(20X, 'B= ',2F10.3)
  53           FORMAT(20X, 'C= ',2F10.3)
  54           FORMAT(25X, 'Chi= ',2F10.3)
  55           FORMAT(5X, 'FITTING OF GOMPERTZ MODEL WITH THREE PARAMETERS'/)
  56           FORMAT(15X, '................................................................................')
  57           FORMAT(6X, 'A=',F10.3, 'B=',F10.3, 'C=',F10.3,
        1                'Chi-Sq=',F10.3 /)
                  STOP
                  END
               SUBROUTINE THREEP(N,P,A,B,C)
C              FITTING OF MODEL USING THREE EQUIDISTANT POINTS
C
C              WRITE(6,*) '(a): ESTIMATION USING THREE EQUIDISTANT POINTS'
               INTEGER T1, T2, T3
               DIMENSION P(40)
               T1=1
               T3=N
               T2=(T3+T1)/2
               U1=P(T1)
               U2=P(T2)
               U3=P(T3)
               AM=T2-T1
               AM1=1/AM
               AM2=T1/AM
               DU=U3-(2*U2)+U
               C=AM1*LOG((U3-U2)/(U2-U1))
               B=(((U2-U1)**2/DU)*((U2-U1)/(U3-U2))**AM2
               A=((U3*U1)-(U2*U2))/DU
               RETURN
               END
               SUBROUTINE PARTSUM(N,P,A,B,C)
C              FITTING OF MODEL USING THREE PARTIAL SUMS
               DIMENSION  T(40),P(40),X(40),Y(40)
               AN=N
```

```
      N1=AN/3
      N2=N1+N1
      N3=N2+N1
      AAN1=N1
      AN11=1/AAN1
      S1=0.0
      S2=0.0
      S3=0.0
      DO 21 I=1,N
      S1=S1+P(I)
21    CONTINUE
      DO 22 I=N1+1,N2
      S2=S2+P(I)
23    CONTINUE
      C=AN11*LOG((S3-S2)/(S2-S1))
      U1=((S2-S1)**3*((S3-S2)**AN11-(S2-S1)**AN11)
      U2=((S3-(2*S2)+S1)**2*((S3-S2)**AN11)
      B=U1/U2
C     B=(((S2-S1)**3)/(S3-(2*S2)+S1)**2)*(1-(((S2-S1)/(S3-S2))**AN11)))
      A=(((S1*S3)-(S2*S2))/(S3-(2*S2)+S1)**AN11
      RETURN
      END
```

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