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$g^{*S}I-closed$ Sets in Ideal Topological Spaces

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ABSTRACT

In this paper, $g^{*S}I-closed$ sets, $*s$ - additive, $*s$ - multiplicative, $g^{*S}I$ - additive and $g^{*S}I$ - multiplicative spaces are introduced and their properties are investigated. We introduce the notion of $g^{*S}I-continuous$ function and other related functions. The relationships between them are also studied.

Keywords: *Semi local function, $g^{*S}I-closed$ set, $g^{*S}I-open$ set, $g^{*S}I-continuous$ function, weakly $g^{*S}I-continuous$ function, strongly $g^{*S}I-continuous$ function, $g^{*S}I$ - additive space, $g^{*S}I$ - multiplicative space, $g^{*S}I$ - connected space, $g^{*S}I$ - compact space.*

1. Introduction

Ideals in topological spaces have been considered since 1930. In 1990, Jankovic and Hamlett [5] once again investigated applications of topological ideals. EI – Monsefetal [1] in 1992 and quite recently Khan and Noiri [7] have studied semi-local functions in ideal topological spaces. $Ig-closed$ sets were first introduced by Dontchev etal [2] in 1999. $gI-closed$ sets were introduced by M. Khan and T. Noiri [6] in 2010, $sgI-closed$ sets were introduced by M. Khan and T. Noiri [8] in 2011, and $I_{s^*g}-closed$ sets were introduced by M. Khan and M. Hamza[4] in 2011. In this paper we define $g^{*S}I-closed$ sets, $g^{*S}I-continuous$ function, and various other related properties of these closed sets and the relationship between these functions are obtained.

2. Preliminaries

An ideal I on a non empty set X is a collection of subsets of X which satisfies the following properties.

(i) $A\in I$, $B\in I \Rightarrow A\cup B\in I$
(ii) $A\in I$, $B\subset A \Rightarrow B\in I$

A topological space (X,τ) with an ideal I on X is called an ideal topological space and is denoted by (X,τ,I).

Let Y be a subset of X. $I_Y=\{I\cap Y/I\in I\}$ is an ideal on Y and by $(Y,\tau/Y,I_Y)$ we denote the ideal topological subspace. Let $P(X)$ be the power set of X, then a set operator ( )*: $P(X)\to P(X)$ called the local function [10] of A with respect to τ and I is defined as follows:

For $A\subset X$, $A^*(I,\tau)=\{x\in X/U\cap A\notin I$ for every open set U containing $x\}$.

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We simply write A^* instead of $A^*(I,\tau)$ in case there is no confusion. A Kuratowski closure operator $cl^*(\ )$ for a topology $\tau^*(I,\tau)$, called the τ^*- topology is defined by $cl^*(A) = A \cup A^*[4]$.

A subset A of a space (X,τ) is said to be semi-open[1] if $A \subset cl(\text{int}(A))$.

A set operator $(\ )^{*S} : P(X) \to P(X)$ called a semi local function and $cl^{*s}(\ )$ [1] of A with respect to τ and I are defined as follows:

For $A \subset X$, $A^{*S}(I,\tau) = \{x \in X / U \cap A \notin I$ for every semi open set U containing $x\}$. and $Cl^{*S}(A) = A \cup A^{*S}$.

Note: A^{*S} defined in [1] and A_* defined in [9] are the same. For a subset A of X, $cl(A)$ (resp $scl(A)$) denotes the closure (resp semi closure) of A in (X,τ). Similarly $cl^*(A)$ and $\text{int}^*(A)$ denote the closure of A and interior of A in (X,τ^*).

A subset A of X is called * closed [5] (resp.$*s$ - closed) if $A^* \subseteq A$ (resp $A^{*S} \subseteq A$). A is called * - dense [5] in itself (resp .$*s$ - dense). If $A \subset A^*$ (resp $A \subset A^{*S}$) A is called * - perfect [5] (resp .$*s$ - perfect). If $A = A^*$ (resp $A = A^{*S}$) A subset A of a topological space $(X,\tau)$is said to be generalized closed [5] (briefly g-closed) if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X . The complement of $g-closed$ set is said to be $g-open$.

Definition 2.1: A subset A of a space (X,τ,I) is said to be

(i) $gI-closed[6]$ if $A^{*S} \subseteq U$ wherever $A \subseteq U$ and $U-open$ in X .

(ii) $I_g-closed[2]$ if $A^* \subseteq U$ wherever $A \subseteq U$ and $U-open$ in X .

(iii) $sgI-closed[8]$ if $A^{*S} \subseteq U$ wherever $A \subseteq U$ and $U-semi\ open$ in X .

(iv) $I_{s*g}-closed[4]$ if $A^* \subset U$ wherever $A \subseteq U$ and $U-semi\ open$ in X .

The complements of $gI-closed$ (resp $I_g-closed$, $sgI-closed$, $I_{s*g}-closed$) are called $gI-open$ (resp I_g-open, $sgI-open$, $I_{s*g}-open$).

Lemma 2.2 [1]: For A, B in (X,τ,I) we have

(i) If $A \subset B$ then $A^{*S} \subset B^{*S}$

(ii) $\left(A^{*S}\right)^{*S} \subseteq A^{*S}$

(iii) $A^{*S} \cup B^{*S} \supseteq (A \cup B)^{*S}$

(iv) $(A \cap B)^{*S} \subseteq A^{*S} \cap B^{*S}$

(v) If $I = \{\phi\}$, $A^{*S} = scl(A)$ and $cl^{*S}(A) = scl(A)$

(vi) If $I = \rho(X)$ then $A^{*S} = \phi$ and $cl^{*S}(A) = A$

(vii) $A^{*S} = scl(A^{*S}) \subset scl(A)$ and A^{*S} is semi closed.

3. $g^{*S}I$ - closed sets

Definition 3.1: A subset A of an ideal space (X,τ,I) is said to be $g^{*S}I$ closed, if $cl^{*S}(A) \subseteq U$ whenever $A \subseteq U$ and U is $g-open$ in X .

The complement of $g^{*S}I-closed$ set is said to be $g^{*S}I--open$. The collection of all $g^{*S}I-closed$ sets (resp $g^{*S}I-open$ sets) is denoted by $G^{*S}IC(X)$ (resp $G^{*S}IO(X)$)

Remarks: 3.1

1. If $I = p(X)$ then $cl^{*S}(A) = A \;\; \forall A \subseteq X$ and hence every subset of X is $g^{*S}I-closed$.
2. Since $A^{*S} = \{\phi\}$ for every $A \in I$, every member of I is $g^{*S}I-closed$.
3. Since every open set is $g-open$, every $g^{*S}I-closed$ set is $gI-closed$. But the converse is not true in general as seen from example (3.1).
4. Every $\tau^{*S}-closed$ set is $g^{*S}I-closed$. But the converse is not true in general as seen from example (3.3).
5. $I_{s*g}-closed$ and $g^{*S}I-closed$ are independent concepts as seen from example (3.3, 3.2).
6. $I_g-closed$ and $g^{*S}I-closed$ are independent concepts as seen from example (3.1, 3.4).
7. $sgI-closed$ and $g^{*S}I-closed$ are independent concepts as seen from example (3.1).

Example 3.1: Let (X,τ) be an indiscrete space, $x_0 \in X$ and $I = \{\phi, \{x_0\}\}$.

Then $A^{*S} = A^* = X \quad \text{if} \quad A \neq \{x_0\}$
$\qquad = \phi \quad \text{if} \quad A = \{x_0\}$.

Any subset $A \neq \{x_0\}$ is $gI-closed$, $I_g-closed$, $sgI-closed$, $I_{s*g}-closed$ but not $\tau^{*S}-closed$ and $g^{*S}I-closed$.

Example 3.2: Let (X,τ) be an indiscrete space $p \in X$ and $I = \{A \subseteq X \,/\, p \notin A\}$.

Then $A^{*S} = A^* = X \quad \text{if} \quad p \in A$
$\qquad = \phi \quad \text{if} \quad p \notin A$.

$A = \{p\}$ is $gI-closed$, $I_g-closed$, $sgI-closed$, $I_{s*g}-closed$ but not $\tau^{*S}-closed$ and $g^{*S}I-closed$.

Example 3.3: Let $X = \{a,b,c,d\}$, $\tau = \{\phi, \{a,b\}, X\}$, and $I = \{\phi\{a\}\}$, then $A = \{a,c\}$ is $g^{*S}I-closed$ but not $\tau^{*S}-closed$ and $A = \{c\}$ is $g^{*S}I-closed$ but not $I_{s*g}-closed$.

Example 3.4: Let $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$, and $I = \{\phi\}$, then $A = \{a\}$ is $g^{*S}I-closed$ but not $Ig-closed$.

Note: In an ideal space (X,τ,I), $(A \cup B)^{*S} \neq A^{*S} \cup B^{*S}$ in general as seen from example 3.5.

Example 3.5: Let $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$, $I = \{\phi\}$.

Then $\{a\}^{*S} = \{a\}$, $\{b\}^{*S} = \{b\}$ and $\{a,b\}^{*S} = X$.

This shows $(A \cup B)^{*S} \neq A^{*S} \cup B^{*S}$.

Definition 3.2: An ideal space (X,τ,I) is said to be

(i) $*S-$ finitely additive if $\left[\bigcup_{i=1}^{n} A_i\right]^{*S} = \bigcup_{i=1}^{n} (A_i)^{*S}$ for every positive integer n.

(ii) $*S-$ countably additive if $\left[\bigcup_{i=1}^{n} A_i\right]^{*S} = \bigcup_{i=1}^{\infty} (A_i)^{*S}$.

(iii) $*S-$additive if $\left[\bigcup_{\alpha\in\Omega} A_\alpha\right]^{*S} = \bigcup_{\alpha\in\Omega}(A_\alpha)^{*S}$ for all indexing sets Ω where A_i 's are subsets of X.

Similarly we define $*s$ - finitely multiplicative, countably multiplicative ideal spaces by taking intersection in the place of union.

(iv) $g^{*S}I$ - finitely additive (resp $g^{*S}I$ - countable additive, $g^{*S}I$ - additive) if finite union (resp countable union, arbitrary union) of $g^{*S}I-closed$ sets is $g^{*S}I-closed$.

Similarly we define $g^{*S}I$ - finitely multiplicative (resp $g^{*S}I$ - countably multiplicative, $g^{*S}I$ - multiplicative) if finite intersection (resp countable intersection, arbitrary intersection) of $g^{*S}I-closed$ sets is $g^{*S}I-closed$.

Remark 3.2:
1. $*s$ - finitely additive (resp $g^{*S}I$ - countable additive, additive spaces)are $g^{*S}I$ - finitely additive (resp countably additive, additive) but not conversely.
2. $*s$ - additive spaces are $*s$ countably additive and $*s$ countable additive spaces are $*s$ - finitely additive but not conversely.
3. $g^{*S}I$ additive spaces are $g^{*S}I$ - countably additive and $g^{*S}I$ - countably additive spaces are $g^{*S}I$ - additive but not conversely.

Example 3.3: Let $X = R$, $I=\{\phi\}$, τ - cofinite topology.

Then $\tau=\left\{\phi, X, A/A^c is \quad finite\right\}$. Closed sets are ϕ, X and all the finite subsets. $A^{*S}=A$ if A is finite and X if A is infinite.

For every positive integer n,

let $A_n=\{-n,-n+1,.......0,.....,......n-1,n\}$ then $A_n^{*S}=A_n \forall n\,.(UA_n)^{*S}=(Z)^{*S}=R$ and $U(A_n)^{*S}=Z$
let A, B be set of all non negative and non positive integers respectively.

Then $A^{*S}=R=B^{*S}$, $A^{*S}\cap B^{*S}=R$ and $(A\cap B)^{*S}=\{0^{*S}\}=\{0\}$. Therefore this space is
1. not $*s$ - finitely multiplicative, and hence not $*s$ - countably multiplicative, and not $*s$ - multiplicative.
2. $*s$ - finitely additive but not countably $*s$ - additive, and not $*s$ - additive.
3. $g^{*S}I$ - multiplicative, $g^{*S}I$ - finitely multiplicative and $g^{*S}I$ - countably multiplicative.
4. $g^{*S}I$ - finitely additive, but not $g^{*S}I$ - countably additive and not $g^{*S}I$ - additive.

Example 3.4: Let (X,τ) be indiscrete space $x_0\in X$ and $I=\{\phi,\{x_0\}\}$. then $GO(X)=\left\{all \quad subsets\right\}$.
$A^{*S}=X$, if $A\neq\{x_0\}$ and $A^{*S}=\phi$ if $A=\{x_0\}$, $cl^{*S}(A)=X$ if $A\neq\{x_0\}$ and $cl^{*S}(A)=$ A if $A=\{x_0\}$.
$G^{*S}IC(X)=\{\phi, X,\{x_0\}\}$. If $B=\{x_0,x_1\}$ and $C=\{x_0,x_2\}$ where $x_1,x_2\neq x_0$ in X,

then $B^{*S}=X$, $C^{*S}=X$, $B^{*S}\cap C^{*S}=X$; $(B\cap C)^{*S}=\{x_0\}^{*S}=\{\phi\}$. Therefore this space is
1. not $*s$ - multiplicative, and not $*s$ - finitely multiplicative, and not $*s$ - countably multiplicative.
2. $g^{*S}I$ - additive $g^{*S}I$ - multiplicative, $g^{*S}I$ - finitely additive , $g^{*S}I$ - multiplicative and $g^{*S}I$ - countably additive and $g^{*S}I$ -countably multiplicative.
3. $*s$ - additive , $*s$ - finitely additive and $*s$ - countably additive.

Remark: 3.3: In an ideal topological space (X,τ,I) which is $*S-$ finitely additive we have following results:

1. $cl^{*S}(\phi)=\phi$
2. $cl^{*S}(X)=X$
3. $A\subseteq cl^{*S}(A)$
4. $cl^{*S}(A\cup B)=cl^{*S}(A)\cup cl^{*S}(B)$
5. $cl^{*S}\left(cl^{*S}(A)\right)=cl^{*S}(A)$

for all subsets A, B and X.

Therefore $cl^{*S}(\ )$ satisfies Kuratowski Closure axioms [10] and hence it defines a topology τ^{*S} whose closure operation is given as $cl^{*S}(A)=A\cup A^{*S}$. Note that $\tau\subseteq\tau^{*}\subseteq\tau^{*S}$. $cl^{*S}(A)$ and $\operatorname{int}^{*S}(A)$ denote the closure and interior of A in (X,τ^{*S}).

Theorem 3.1: A subset of a $*S-$ finitely additive ideal space (X,τ,I) is $g^{*S}I-open$ if and only if $F\subset Int^{*S}(A)$ wherever $F\subseteq A$ and F is a $g-closed$ subset of X.

Proof: Let A be $g^{*S}I-open$ and F be a $g-closed$ subset of X contained in A. Then $(X-F)$ is a $g-open$ set containing $X-A$ which implies $X-Int^{*S}(A)=cl^{*S}(X-A)\subset X-F$ Conversely, let $F\subset Int^{*S}(A)$ whenever $F\subseteq A$ and F is a $g-closed$ subset of X.

Let U be $g-open$ and $X-A\subset U$. Then $X-U\subset Int^{*S}(A)=X-cl^{*S}(X-A)$. Therefore $cl^{*S}(X-A)\subset U$ which proves $X-A$ is $g^{*S}I-closed$. So A is $g^{*S}I-open$.

Theorem 3.2: For each $x\in(X,\tau,I)$ either $\{x\}$ is $g-closed$ or $\{x\}^{c}$ is $g^{*S}I-closed$ in X.

Proof: Suppose $\{x\}$ is not $g-closed$, then $\{x\}^{c}$ is not $g-open$. Therefore the only $g-open$ set containing $\{x\}^{c}$ is X and $\left(\{x\}^{c}\right)^{*S}\subseteq X$ which proves that $\{x\}^{c}$ is $g^{*S}I-closed$.

Theorem 3.3: In an ideal space (X,τ,I) which is $*S$ - finitely additive, if U is semi open and A is $g^{*S}I-open$, then $U\cap A$ is $g^{*S}I-open$.

Proof: Let $X-(U\cap A)\subset G$ and G be $g-open$.

Then $(X-A)\cup(X-U)\subset G$ and this implies $X-A\subset G$ and $X-U\subset G$. $(X-A)$ is $g^{*S}I-closed$ and G is $g-open$ imply $cl^{*S}(X-A)\subset G$ and $cl^{*S}(X-U)\subset scl(X-U)=X-U\subset G$

Therefore $cl^{*S}\left[X-(A\cap U)\right]=cl^{*S}\left[(X-A)\cup(X-U)\right]$ $cl^{*S}(X-A)\cup cl^{*S}(X-U)\subset G$ (since the ideal space is $*S$ - finitely additive) This implies $A\cap U$ is $g^{*S}I-open$.

Theorem 3.4: If B is a subset of a $*S$ - finitely additive space (X,τ,I) such that $A\subset B\subset cl^{*S}(A)$ and A is $g^{*S}I-closed$, then B is also $g^{*S}I-closed$ in X.

Proof: Let U be $g-open$ and $B\subset U$. Then $A\subset U$ implies $cl^{*S}(A)\subset U$

Therefore $cl^{*S}(B)\subset cl^{*S}(cl^{*S}(A))\subset cl^{*S}(A)\subset U$ which proves B is $g^{*S}I-closed$.

Note: In general intersection of g-closed sets need not be g-closed.

Definition 3.3 A topological space (X,τ) is said to be a g-multiplicative space if arbitrary intersection of g-closed sets in X is g-closed.

Remark 3.4

(i) In g-multiplicative spaces, gcl(A) which is the intersection of all g-closed sets in X containing A is also g-closed.

(ii) Any indiscrete topological space (X,τ) is g – multiplicative.

(iii) If X = {a,b,c} and τ = {X, φ, {a}} then {a,c} and {a,b} are g-closed but {a} is not g-closed and hence (X,τ) is not g-multiplicative.

Theorem 3.5: Let (X,τ,I) be a g-multiplicative ideal space and A be $g^{*S}I-closed$ subset of X. Then

(i) $scl(A^{*S})\subset U$ for all $g-open$ set U containing A.

(ii) For all $x\in scl(A^{*S})$, $gcl(\{x\})\cap A\neq\phi$

(iii) $scl(A^{*S})-A$ contains no non empty $g-closed$ set.

(iv) $(A^{*S})-A$ contains no non empty $g-closed$ set.

Proof:

(i) Since $(A^{*S})=scl\,A^{*S}$, the result follows from definition.

(ii) Let $x\in scl(A^{*S})$. Suppose $gcl(\{x\})\cap A=\phi$ then $A\subset X-gcl(\{x\})$ which is $g-open$ By (i) $scl(A^{*S})\subset X-gcl(\{x\})$ which is a contradiction to the fact $x\in scl(A^{*S})$.

(iii) Suppose that there exists a non empty $g-closed$ set F such that $F\subset scl(A^{*S})-A$.

If $x\in F$, then $gcl(\{x\})\subseteq gcl(F)=F$ and $A\cap gcl(\{x\})=\phi$.

Since $x\in scl(A^{*S})$, by (ii) $gcl(\{x\})\cap A\neq\phi$ which is a contradiction.

(iv) It follows from (iii) since $A^{*S}=scl(A^{*S})$.

Theorem 3.6: Let (X,τ,I) be a g-multiplicative ideal space and let A be $g^{*S}I-closed$. Then A is $\tau^{*S}-closed \Leftrightarrow A^{*S}-A$ is closed.

Proof:

Necessity: A is $\tau^{*S}-closed \Rightarrow A^{*S}\subset A\Rightarrow A^{*S}-A=\phi$ which is closed.

Sufficiency: Let $A^{*S}-A$ be closed. Then it is $g-closed$ By (iv) of theorem (3.5), $A^{*S}-A=\phi$ which implies $A^{*S}\subset A$.

Theorem 3.7: Let (X,τ,I) be a g-multiplicative ideal space and $A\subset X$. If A is $g^{*S}I-closed$ then $A\cup(X-A^{*S})$ is $g^{*S}I-closed$.

Proof: Let U be $g-open$ and $A\cup(X-A^{*S})\subset U$.

Then $X-U\subset X-[A\cup(X-A^{*S})]=A^{*S}-A$.Since A is $g^{*S}I-closed$, $A^{*S}-A$ contains no non empty $g-closed$ set. Therefore $X-U=\phi$ which implies $X=U$. Thus X is the only $g-open$ set containing $A\cup(X-A^{*S})$ which proves $A\cup(X-A^{*S})$ is $g^{*S}I-closed$.

Theorem 3.8: Let A be a subset of a g-multiplicative ideal space (X,τ,I). If A is $g^{*S}I-closed$ then $A^{*S}-A$ is $g^{*S}I-open$.

Proof: Since $X-(A^{*S}-A)=A\cup(X-A^{*S})$, the proof follows from theorem 3.7.

Theorem 3.9: Let (X,τ,I) be an ideal space. If every $g-open$ set is $\tau^{*S}-closed$, then every subset of X is $g^{*S}I-closed$.

Proof: Let $A\subset U$ and U be a $g-open$ set in X. Then $cl^{*S}(A)\subset cl^{*S}(U)=U$ which proves A is $g^{*S}I-closed$.

Theorem 3.10: [(6) Theorem 3.20] Let (X,τ,I) be an ideal space and $A\subset Y\subset X$ where Y is $\alpha-open$ in X. Then $A^{*S}(I_Y,\tau/Y)=A^{*S}(I,\tau)\cap Y$.

Theorem 3.11: Let (X,τ,I) be an ideal space and $A\subset Y\subset X$. If A is $g^{*S}I-closed$ in $(Y,\tau/Y,I_Y)$, Y is $\alpha-open$ and $\tau^{*S}-closed$ in X. Then A is $g^{*S}I-closed$ in X.

Proof: Let $A\subset U$ and U be $g-open$ in X. Then $A^{*S}\left(I_Y,\tau/Y\right)=A^{*S}(I,\tau)\cap Y\subset U\cap Y$.

Then $Y\subset U\cup(X-A^{*S}(I,\tau))$. Since Y is $\tau^{*S}-closed$, $Y^{*S}\subset Y$. Therefore $A^{*S}\subset Y^{*S}\subset Y\subset U\cup(X-A^{*S}(\tau,I))$. This implies $A^{*S}\subset U$ and hence $cl^{*S}(A)\subset U$. So A is $g^{*S}I-closed$ in X.

Definition 3.4: $\{A_\alpha/\alpha\in\Omega\}$ is said to be a locally finite (resp. locally countable) family of sets in (X,τ,I) if for every $x\in X$, there exists an open set U in X containing x that intersects only a finite (resp. countable) number of members $A_{\alpha_1},\ldots\ldots,A_{\alpha_n}$ (resp $A_{\alpha_i}, i=1,\ldots\ldots\infty$) of $\{A_\alpha/\alpha\in\Omega\}$.

Theorem 3.13: Let (X,τ,I) be an ideal space which is $*S$ - finitely additive, and let $\{A_\alpha/\alpha\in\Omega\}$ be a locally finite family of sets in (X,τ,I). Then $\left(\bigcup_{\alpha\in\Omega}A_\alpha\right)^{*S}=\bigcup_{\alpha\in\Omega}(A_\alpha)^{*S}$.

Proof: $A_\alpha\subseteq\bigcup A_\alpha$ implies $A_\alpha^{*S}\subseteq\left(\bigcup A_\alpha\right)^{*S}$. Therefore $\bigcup_{\alpha\in\Omega}(A_\alpha)^{*S}\subseteq\left(\bigcup_{\alpha\in\Omega}A_\alpha\right)^{*S}$ (1)

On the otherhand, if $x\in\left(\bigcup_{\alpha\in\Omega}A_\alpha\right)^{*S}$ then there exists an open set U containing x, that intersects only finite number of members $A_{\alpha_1},\ldots\ldots,A_{\alpha_n}$ .Let V be a semi-open set containing x. Then $U\cap V$ is a semi-open set containing x.

which implies $(U\cap V)\cap\left(\bigcup_{\alpha\in\Omega}A_\alpha\right)\notin I$.

i.e. $\left[(U\cap V)\cap\left(\bigcup_{\alpha\neq\alpha_i}A_\alpha\right)\right]\cup\left[(U\cap V)\cap\left(\bigcup_{i=1}^{n}A_{\alpha_i}\right)\right]\notin I$

i.e. $\{\phi\}\cup\left[(U\cap V)\cap\left(\bigcup_{i=1}^{n}A_{\alpha_i}\right)\right]\notin I$ and this implies $V\cap\left(\bigcup_{i=1}^{n}A_{\alpha_i}\right)\notin I$

Therefore $x \in \left(\bigcup_{i=1}^{n} A_{\alpha_i}\right)^{*S} = \bigcup_{i=1}^{n}(A_{\alpha_i})^{*S} \subseteq \bigcup_{\alpha\in\Omega}(A_\alpha)^{*S}$

Therefore $\left(\bigcup_{\alpha\in\Omega} A_\alpha\right)^{*S} \subseteq \bigcup_{\alpha\in\Omega}(A_\alpha)^{*S}$ (2)

From 1 and 2 the result follows: $\left(\bigcup_{\alpha\in\Omega} A_\alpha\right)^{*S} = \bigcup_{\alpha\in\Omega}(A_\alpha)^{*S}$

Theorem 3.14: Let (X,τ,I) be a $*S$ - countably additive ideal space which is, and let $\{A_\alpha / \alpha \in \Omega\}$ be a locally countable family of sets in (X,τ,I). Then $\left(\bigcup_{\alpha\in\Omega} A_\alpha\right)^{*S} = \bigcup_{\alpha\in\Omega}(A_\alpha)^{*S}$.

Proof: Similar to proof of Theorem 3.13.

Theorem 3.15: Let the ideal space (X,τ,I) be $*S$ - finitely additive, and $\{A_\alpha / \alpha \in \Omega\}$ be a locally finite family of sets in (X,τ,I). If each A_α is $g^{*S}I-closed$ then $\bigcup_{\alpha\in\Omega} A_\alpha$ is $g^{*S}I-closed$ in X.

Proof: Let $\bigcup_{\alpha\in\Omega} A_\alpha \subseteq U$ and U be g - open in X. Then $A_\alpha \subseteq U \;\forall\, \alpha \in \Omega$ implies $cl^{*S}(A_\alpha) \subseteq U \;\forall\, \alpha \in \Omega$.By theorem (3.13) $cl^{*S}\left(\bigcup_{\alpha\in\Omega} A_\alpha\right) = \bigcup_{\alpha\in\Omega} cl^{*S}(A_\alpha) \subseteq U$.

Therefore $\bigcup_{\alpha\in\Omega} A_\alpha$ is $g^{*S}I-closed$.

Theorem 3.16: Let the ideal space (X,τ,I) be $*S$ - countably additive. If $\{A_\alpha / \alpha \in \Omega\}$ is a locally countable family of sets in (X,τ,I) and each A_α is $g^{*S}I-closed$ then $\bigcup_{\alpha\in\Omega} A_\alpha$ is $g^{*S}I-closed$.

Proof: Similar to proof of Theorem 3.15.

4. - $g^{*S}I-continuous$ functions

Definition 4.1: A function $f:(X,\tau,I) \to (Y,\Omega,J)$ is said to be weakly $I^{*S}-continuous$ if for each $x \in X$ and for every V in Ω containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq cl^{*S}(V)$.

Definition 4.2: A function $f:(X,\tau,I) \to (Y,\sigma)$ is said to be $g^{*S}I-continuous$ if for every $U \in \Omega$, $f^{-1}(U)$ is $g^{*S}I-open$ in X. Equivalently for every closed set V in Y, $f^{-1}(V)$ is $g^{*S}I-closed$ in X.

Definition 4.3: A function $f:(X,\tau,I) \to (Y,\Omega,J)$ is said to be strongly $g^{*S}I-continuous$ if for $g^{*S}I-open$ (resp $g^{*S}I-closed$) set V in Y, $f^{-1}(V)$ is open (resp closed) in X.

Definition 4.4: A function $f:(X,\tau,I) \to (Y,\Omega,J)$ is said to be weakly $g^{*S}I-continuous$ if for every $x \in X$ and for every $V \in \Omega$ containing $f(x)$, there exists $g^{*S}I-open$ set U in X such that $x \in U$ and $f(U) \subseteq cl^{*S}(V)$.

Definition 4.5: A function $f:(X,\tau,I)\to(Y,\Omega,J)$ is said to be $g^{*S}I-irresolute$ if for every $g^{*S}I-open$ (resp $g^{*S}I-closed$ set) set V in Y, $f^{-1}(V)$ is $g^{*S}I-open$ (resp $g^{*S}I-closed$) in X.

Remarks: 4.1:

1. Every continuous function is $g^{*S}I-continuous$ (since every open set is $g^{*S}I-open$) but the converse is not true as seen from example 4.1.
2. Every strongly $g^{*S}I-continuous$ function is continuous and hence it is $g^{*S}I-continuous$ but the converse is not true as seen from example (4.2).
3. Every $g^{*S}I-continuous$ function is weakly $g^{*S}I-continuous$ but the converse is not true as seen from example (4.2).
4. Every weakly $I^{*S}-continuous$ function is weakly $g^{*S}I-continuous$
5. Every strongly $g^{*S}I-continuous$ function is $g^{*S}I-irresolute$ and $g^{*S}I-irresolute$ function is $g^{*S}I-continuous$

Example 4.1: Let $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a\}\}$, $I=\{\phi,\{a\}\}$, $Y=X$, $\Omega=\tau$.

Let $f:(X,\tau,I)\to(Y,\Omega,J)$ be defined by $f(a)=c$, $f(b)=f(c)=a$.

Then f is $g^{*S}I-continuous$ but not continuous.

Example 4.2: Let $X=Y$ be indiscrete space and $I=\{\phi,x_0\}=J$ where $x_0\in X$.

Let $f:(X,\tau,I)\to(Y,\Omega,J)$ be identity map. Y, ϕ, $X-\{x_0\}$ are the only $g^{*S}I-open$ sets in Y. $f^{-1}(X-\{x_0\})=X-\{x_0\}$ is not open in X. Therefore f is not strongly $g^{*S}I-continuous$ but f is both continuous and $g^{*S}I-continuous$.

Example 4.3: Let (X,τ) be an indiscrete space, $x_0\in X$ and $I=\{\phi,x_0\}=J$.

Let $Y=X$, $\Omega=\tau$, $J=g$ and $f:(X,\tau,I)\to(Y,\Omega,J)$ be identity function. Then f is an irresolute function. $A=\{x_0\}$ is $g^{*S}I-closed$ in Y, but $f^{-1}(A)=\{x_0\}$ is not closed in X.

Therefore f is not strongly $g^{*S}I-continuous$.

Let $x_1\neq x_0$ be a point of X and $f:(X,\tau,I)\to(Y,\Omega,J)$ be defined by $f(x_0)=x_1$, $f(x_1)=x_0$ and $f(x)=x\,\forall x\neq x_0,x_1$. Then f is $g^{*S}I-continuous$. $A=\{x_0\}$ is $g^{*S}I-closed$ in Y and $f^{-1}(A)=\{x_1\}$ is not $g^{*S}I-closed$ in X. Therefore f is not $g^{*S}I-irresolute$.

Definition 4.6: Let N be a subset of (X,τ,I) and $x\in X$. The subset N of X called a $g^{*S}I-open$ neighbourhood of x if there exists $g^{*S}I-open$ set U containing x such that $U\subset N$.

Theorem 4.1: Let (X,τ,I) be an ideal space which is $g^{*S}I-multiplicative$. Then the following are equivalent.

1. f is $g^{*S}I-continuous$
2. For each $x\in X$ and each open set V in Y with $f(x)\in V$, there exists an $g^{*S}I-open$ set U containing x such that $f(U)\subset V$.

3. For each $x \in X$ and each open set V in Y with $f(x) \in V$, $f^{-1}(V)$ is a $g^{*S}I-open$ neighbourhood of x.

Proof: Since X is $g^{*S}I-multiplicative$, arbitrary union of $g^{*S}I-open$ sets is $g^{*S}I-open$.
1 $\Rightarrow$ 2: Let $x \in X$ and V be open in Y containing $f(x)$. Then $U = f^{-1}(V)$ is $g^{*S}I-open$, $x \in U$ and $f(U) \subseteq V$.

2 $\Rightarrow$ 3: Let $x \in X$, V open in Y containing $f(x)$. By (2), there exists an $g^{*S}I-open$ set U containing x such that $f(U) \subseteq V$. So $x \in U \subseteq f^{-1}(V)$ which proves $f^{-1}(V)$ is an $g^{*S}I-open$ neighbourhood of x.

3 $\Rightarrow$ 1: Let V be open in Y and $x \in f^{-1}(V)$. Then $f^{-1}(V)$ is a $g^{*S}I-open$ neighbourhood of x.

Thus for each $x \in f^{-1}(V)$, there exists an $g^{*S}I-open$ set U_x containing x such that $x \in U_x \subset f^{-1}(V)$.

Therefore $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \bigcup_x$ is a $g^{*S}I-open$ which proves that f is $g^{*S}I-continuous$.

Theorem 4.2: Let (X, τ, I) be a $g^{*S}I-multiplicative$ ideal space in which every open set is $*S-closed$. Then a function $f:(X,\tau,I) \to (Y,\Omega)$ is weakly $g^{*S}I-continuous$ if and only if it is $g^{*S}I-continuous$.

Proof: Obviously $g^{*S}I-continuity \Rightarrow$ weakly $g^{*S}I-continuity$. Conversely, let f be weakly $g^{*S}I-continuous$. Then for each $x \in X$ and open set V containing $f(x)$, there exists $g^{*S}I-open$ set U such that $x \in U$ and $f(U) \subset cl^*(V) = V$. Therefore by theorem 4.1, f is $g^{*S}I-continuous$.

Theorem 4.3: Let $f:(X,\tau,I) \to (Y,\Omega,J)$ and $g:(Y,\sigma,\tau) \to (Z,\Omega,K)$ be functions between ideal spaces.
1. If f is strongly $g^{*S}I-continuous$ and g is $g^{*S}I-continuous$ then gof is continuous.
2. If f is $g^{*S}I-continuous$ and g is continuous then gof is $g^{*S}I-continuous$.
3. If f is strongly $g^{*S}I-continuous$ and g is $g^{*S}I-irresolute$ then gof is strongly $g^{*S}I-continuous$ and $g^{*S}I-irresolute$.
4. If f and g are $g^{*S}I-irresolute$ then gof is $g^{*S}I-irresolute$.

Proof: Obvious from definition.

Theorem 4.4: Let (X,τ,I) be $*s-finitely$ additive. Let $f:(X,\tau,I) \to (Y,\Omega)$ be $g^{*S}I-continuous$ and U be $g^{*S}I-open$ in X. Then $f/U:(U,\tau_U,I_U) \to (Y,\Omega)$ is $g^{*S}I-continuous$.

Proof: Since (X,τ,I) is $*s-finitely$ additive, finite intersection $g^{*S}I-open$ sets is $g^{*S}I-open$. Let V be open in (Y,Ω). Then $f^{-1}(V)$ is $g^{*S}I-open$ in X.

Therefore $(f/U)^{-1}(V) = f^{-1}(V) \cap U$ is $g^{*S}I-open$.Therefore (f/U) is $g^{*S}I-continuous$.

Note: The result is true if $g^{*S}I-continuous$ is replaced by $g^{*S}I-irresolute$.

Theorem 4.5: Let (X,τ,I) be an ideal space which is $*S-multiplicative$ finitely additive and $g^{*S}I-multiplicative$.

Then $f:(X,\tau,I)\to(Y,\Omega)$ is $g^{*S}I-continuous$ if and only if the graph function $g:X\to X\times Y$ defined by $g(x)=(x,f(x))$ for each $x\in X$ is $g^{*S}I-continuous$.

Proof:

Necessity: Let $x\in X$ and W be any open set in $X\times Y$ containing $g(x)=(x,f(x))$. Then there exists basic open set $U\times V$ such that $g(x)\in U\times V\subseteq W$. Therefore $f(x)\in V$.

Since f is $g^{*S}I-continuous$, there exists $g^{*S}I-open$ set U_1 containing X such that $x\in U_1$ and $f(U_1)\subseteq V$ (by theorem 4.1) and By theorem 3.3 $U_1\cap V$ is $g^{*S}I-open$ in X. Then $x\in U_1\cap U$ and $g(U_1\cap U)\subset U\cap V\subset W$. Therefore g is $g^{*S}I-continuous$.

Sufficiency: Let $g:X\to X\times Y$ be $g^{*S}I-continuous$. Let $x\in X$ and V be an open set in Y containing $f(x)$. Then $X\times V$ is an open set in $X\times Y$. Since g is $g^{*S}I-continuous$, there exists $g^{*S}I-open$ set U in X such that $x\in U$ and $g(U)\subset X\times V$. Hence $x\in U$ and $f(U)\subseteq V$ which proves that f is $g^{*S}I-continuous$.

Theorem 4.6: Let $\{X_\alpha/\alpha\in\nabla\}$ be any family of topological spaces. If $f:(X,\tau,I)\to\prod_{\alpha\in\nabla}X_\alpha$ is a $g^{*S}I-continuous$, function, then $P_\alpha\, o\, f:X\to X_\alpha$ is $g^{*S}I-continuous$ for each $\alpha\in\nabla$ where P_α is projection of $\prod X_\alpha$ onto X_α.

Proof: Consider a fixed $\alpha_o\in\nabla$ .Let G_{α_o} be an open set in X_{α_o} .Then $P_{\alpha_o}^{-1}(G_{\alpha_o})$ is open in X_{α_o}. (P_{α_o} is continuous).Therefore $f^{-1}\left[(P_{\alpha_o})^{-1}(G_{\alpha_o})\right]=(P_{\alpha_o}of)^{-1}(G_{\alpha_o})$ is $g^{*S}I-open$ in X.

Therefore $P_{\alpha_o}\, o\, f$ is $g^{*S}I-continuous$.

Theorem 4.7: For any bijection $f:(X,\tau,I)\to(Y,\Omega,J)$ the following are equivalent.

(i) $f^{-1}:(X,\tau,I)\to(Y,\Omega,J)$ is $g^{*S}I-continuous$.

(ii) $f(U)$ is $g^{*S}I-open$ in Y for every open set U in X.

(iii) (ii) $f(U)$ is $g^{*S}I-closed$ in Y for every closed set U in X.

Proof: Obvious.

5. $g^{*S}I$ - compact spaces and $g^{*S}I$ - connected spaces

Definition 5.1: A collection $\{A_\alpha/\alpha\in\Omega\}$ of $g^{*S}I-open$ set in an ideal topological space X is called $g^{*S}I-open$ cover of a subset B of X if $B\subseteq U\{A_\alpha/\alpha\in\Omega\}$.

Definition 5.2: An ideal topological space (X,τ,I) is called $g^{*S}I$ - compact modules I, if for every $g^{*S}I-open$ cover $\{A_\alpha/\alpha\in\Omega\}$ of (X,τ,I), there exists a finite subset Ω_0 and Ω such that $X-U\{A_\alpha/\alpha\in\Delta_0\}\in I$.

Theorem 5.1: The image of $g^{*S}I$ - compact modulo I space (X,τ,I) under a $g^{*S}I-continuous$ subjective function f is $f(I)$ - compact.

Proof: Let (X,τ,I) be a $g^{*S}I$- compact modulo *I* space and $f:(X,\tau,I)\to(Y,\eta)$ be a subjective $g^{*S}I-continuous$ function. Then $f(I)$ is an ideal in (Y,η).

Let $\{A_\alpha / \alpha\in\Omega\}$ be an open cover for $(Y,\eta,f(I))$

Then $f^{-1}(A_\alpha)$ is $g^{*S}I-open$ in X for every $\alpha\in\Omega$ so $\{f^{-1}(A_\alpha)/\alpha\in\Omega\}$ is a $g^{*S}I-open$ cover for X.

Since (X,τ,I) is $g^{*S}I$- compact modulo *I*, there exists a finite subset Ω_0 of Ω such that $X-\bigcup_{\alpha\in\Omega_0}\{f^{-1}(A_\alpha)\}\in I$. Therefore $Y-\bigcup_{\alpha\in\Omega_0}\{A_\alpha\}\in f(I)$ which proves that $(Y,\eta,f(I))$ is compact modulo $f(I)$.

The following examples prove that there exist spaces which are $g^{*S}I$- compact and spaces which are not $g^{*S}I$- compact.

Example 5.1: Consider the space in example (3.1) Here the only $g^{*S}I-open$ covers are $\{X\}$ and $\{X, X-\{x_0\}\}$. Hence the space is $g^{*S}I$- compact modulo I.

Example 5.2: Let $X=Z$, τ - cofinite topology and $I=\{\phi\}$.

Let for every positive integer n, $A_n=\{-n,-n+1,.......0,1,.........n-1,n\}$. Then $\{A_n^c\}_{n=1}^{\infty}$ is a $g^{*S}I-open$ cover for Z. Suppose there exists a finite subset $\{\alpha_1,.....\alpha_k\}$ of positive integers such that $X-\bigcup_{i=1}^{k}A_{\alpha_i}^c=\Phi$ then $X=\bigcup_{i=1}^{k}A_{\alpha_i}^c$ and hence $\Phi=\bigcap_{i=1}^{n}A_{\alpha_i}=A_\alpha$ where $\alpha=\min\{\alpha_1,\alpha_2,.....\alpha_k\}$ which is not true, since $A_\alpha=\{-\alpha,-\alpha+1,.......0,.........\alpha\}=\Phi$.

Therefore this space is not $g^{*S}I$- compact modulo *I*.

Definition 5.3: An ideal topological space (X,τ,I) is said to be $g^{*S}I-connected$ if X cannot be written as the disjoint union of two non-empty $g^{*S}I-open$ sets. A subset of X is said to be $g^{*S}I$ connected if it is $g^{*S}I-connected$ as a subspace A space which is not $g^{*S}I-connected$ is said to be $g^{*S}I-disconnected$.

Remark 5.1: An ideal space (X,τ,I) is $g^{*S}I-disconnected$ if and only if there exists a proper subset which is both $g^{*S}I-open$ and $g^{*S}I-closed$.

Theorem 5.2: Let $f:(X,\tau,I)\to(Y,\sigma,J)$ be an onto function.
1. If f is continuous and X is $g^{*S}I-connected$ then Y is connected.
2. If f is $g^{*S}I-irresolute$ and X is $g^{*S}I-connected$ then Y is also $g^{*S}I-connected$.
3. If f is strongly $g^{*S}I-continuous$ and X is connected then Y is $g^{*S}I-connected$

Proof:
1. Suppose Y is disconnected the Y can be written as disjoint union of open sets A and B.

Then $X=f^{-1}(Y)=f^{-1}(A)Uf^{-1}(B)$ which is a disjoint union of $g^{*S}I-open$ sets. This is a contradiction to the fact that X is $g^{*S}I-connected$. Therefore Y is connected.

2. Similar to the proof of 1.
3. Similar to the proof of 1.

Definition 5.4: An ideal topological space (X,τ,I) is called $g^{*S}I-normal$ if for every pair of disjoint closed sets A and B of subset of (X,τ,I) there exists disjoint $g^{*S}I-open$ sets U, $V\subseteq X$ such that $A\subseteq U$ and $B\subseteq V$.

We now give examples of spaces which are $g^{*S}I-normal$ and not $g^{*S}I-normal$.

Example 5.3: Let X be an finite set τ - cofinite topology and $I=\{\phi\}$. Here $G^{*}SIO(X)=\{\phi,X,A/A^{c}$ is finite. Suppose U and V are two disjoint $g^{*S}I-open$ sets then $U\cap V=\phi$. Therefore $U^{c}\bigcup V^{c}=X$ which is a contradiction since U^{c} and V^{c} are finite.

Hence (X,τ,I) is not $g^{*S}I-normal$.

In the above example, if $I = P$ (X) then (X,τ,I) is $g^{*S}I-normal$.

Theorem5.3: Let $f:(X,\tau,I)\to(Y,\sigma,J)$ be a closed, injective function.
1. If f is $g^{*S}I-continuous$ then Y is normal $\Rightarrow$ X is $g^{*S}I-normal$.
2. If f is $g^{*S}I-irresolute$ then Y is $g^{*S}I-normal$ $\Rightarrow$ X is $g^{*S}I-normal$.
3. If f is strongly $g^{*S}I-continuous$ then Y is $g^{*S}I-normal$ $\Rightarrow$ X is normal.

Proof:
(i) Let F_1 and F_2 be two disjoint closed sets in X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed sets in Y. Then there exists open sets V_1 and V_2 in Y such that $f(F_1)\subseteq V_1$ and $f(F_2)\subseteq V_2$. Then $F_1\subseteq f^{-1}(V_1)$, $F_2\subseteq f^{-1}(V_2)$ where $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are two disjoint $g^{*S}I-open$ sets in X. Hence X is $g^{*S}I-normal$.
(ii) Similar to the proof of (i).
(iii) Similar to the proof of (i).

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