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A new topology τ^{*b} via b-local functions in ideal topological spaces

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ABSTRACT

In this paper we introduce and study three different notions via ideals namely b-local function, the set operator Ψb and b-compatibility of τ with I. We characterize these new sorts. Several properties of them have been studied and their relationships with other types of similar operators are also investigated.

Keywords: *b-open set,*b-additive space, *b- finitely additive space,b-local function,( ) ^{*b} operator, Ψ_b operator,b-compatibility of τ with I*

1. INTRODUCTION

Ideals topological spaces have been first introduced by K. Kuratowski [4] in 1930.Vaidyanathaswamy [9]introduced local function in 1945 and defined a topology τ. M.E. Abd El Monsef , E.F. Lashien and A.a Nasef [1] introduced semi local function in 1992 and defined a topology τ^{*s} . In 2012 Sukalyan Mistry and Shyamapada Modak [8] defined Pre local function and Ψ_p operator. In this paper we introduce and study three different notions via ideals namely b-local function, the set operator Ψ_b , and b-compatibility of τ with I and investigate their relationships with other types of similar operators .

2. PRELIMINARIES

Let (X,τ) be a topological space and $A \subseteq X$. We denote closure of A and interior of A by clA and $\operatorname{int} A$ respectively.

Definition 2.1: A set A in a topological space (X,τ) is called
(a) semi open [3] if $A \subseteq cl(\operatorname{int}(A))$
(b) pre open [5] if $A \subseteq \operatorname{int}(cl(A))$
(c) b-open [2] if $A \subseteq cl(\operatorname{int}(A)) \bigcup \operatorname{int}(cl(A))$

The class of all semi open, pre open and b-open sets in X will be denoted by $SO(X,\tau)$, $PO(X,\tau)$ and $BO(X,\tau)$ respectively. The complements of these open sets are called corresponding closed sets.

Definition 2.2: [2] The intersection of all b-closed sets containing A is called b-closure of A and is denoted by $bcl(A)$. The union of all b-open sets contained in A is called b-interior of A and is denoted by $b\operatorname{int}(A)$. It is easy to prove that $b\operatorname{int}(A) = X - bcl(X-A)$, A is b-closed if and only if $A = bcl(A)$ and A is b-open if and only if $A = b\operatorname{int}(A)$. A subset $N_x \subseteq X$ is called a b-neighbourhood of x if there exists a b-open set $A \subseteq X$ such that $x \in A \subseteq N_x$. The family of all b-neighbourhoods of x will be denoted by $BN(x)$. It is seen that $bcl(A) = \{x \in X / U \bigcap A \neq \phi \text{ for every } U \in BN(x)\}$

Definition 2.3: [4] An ideal I on a non empty set X is a collection of subsets of X which satisfies the following properties:
(i) $A \in I$, $B \in I \Rightarrow A \cup B \in I$
(ii) $A \in I$, $B \subset A \Rightarrow B \in I$.

A topological space (X,τ) with an ideal I on X is called an ideal topological space and is denoted by (X,τ,I).

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Let Y be a subset of X. $I_Y = \{I \cap Y / I \in I\}$ is an ideal on Y and by $(Y, \tau / Y, I_Y)$ we denote the ideal topological subspace. Let $P(X)$ be the power set of X, then a set operator ( )*: $P(X) \to P(X)$ called the local function [7] of A with respect to τ and I is defined as follows:For $A \subset X$, $A^*(I,\tau) = \{x \in X / U \cap A \notin I$ for every open set U containing $x\}$.

We simply write A^* instead of $A^*(I,\tau)$ in case there is no confusion. A Kuratowski closure operator $cl^*(\ )$ for a topology $\tau^*(I,\tau)$, called the $\tau*$- topology is defined by $cl^*(A) = A \cup A^*$. A set operator $\psi(I,\tau) : P(X) \to P(X)$ is defined as follows: For any $A \subseteq X$, $\psi(I,\tau)(A) = \{x \in X$ such that there exists open set U such that $U - A \in I\}$. I is said to be compatible with τ, denoted by $I \sim \overleftarrow{\ }\tau$ if the following holds: for $A \subseteq X$, if for every $x \in A$ there exists open set U such that $U \cap A \in I$ then $A \in I$.

Definition 2.4 [1] A set operator $(\ )^{*S} : P(X) \to P(X)$ called a semi local function with respect to τ and I is defined as follows: For $A \subset X$, $A^{*S}(I,\tau) = \{x \in X / U \cap A \notin I$ for every semi open set U containing $x\}$. A Kuratowski closure operator $cl^{*s}(\ )$ for a topology $\tau^{*s}(I,\tau)$ is defined by $Cl^{*S}(A) = A \cup A^{*S}$. A set operator $\psi_s(I,\tau) : P(X) \to P(X)$ is defined as follows: For any $A \subseteq X$, $\psi_s(I,\tau)(A) = \{x \in X$ such that there exists $U \in SN(x)$ such that $U - A \in I\}$. I is said to be s-compatible with τ, denoted by $I \overset{s}{\sim} \tau$ if the following holds: for $A \subseteq X$, if for every $x \in A$ there exists $U \in SN(x)$ such that $U \cap A \in I$ then $A \in I$.

Definition 2.5[6] A set operator $(\ )^{*p} : P(X) \to P(X)$, called the pre-local function of I with respect to τ is defined as follows. For $A \subseteq X$, $A^{*p}(I,\tau) = \{x \in X / U_x \bigcap A \notin I$ for every pre open set U containing $x\}$ when there is no ambiguity, we will simply write $A^{*p}(I)$ or $(A)^{*p}$ instead of $A^{*p}(I,\tau)$. $cl^{*p}(A)$ is defined as $A \bigcup (A)^{*p}(I)$

A set operator $\psi_p(I,\tau) : P(X) \to P(X)$ is defined as follows: For any $A \subseteq X$, $\psi_p(I,\tau)(A) = \{x \in X$ such that there exists $U \in PN(x)$ such that $U - A \in I\}$.

3. b-LOCAL FUNCTION

In this section we introduce new class of the set operator $(\ )^{*b}$ using b-neighbourhood and discuss various properties.

Definition 3.1: Given an ideal space (X,τ,I), a set operator $(\ )^{*b} : P(X) \to P(X)$, called the b-local function of I with respect to τ is defined as follows.

For $A \subseteq X$, $A^{*b}(I,\tau) = \{x \in X / U_x \bigcap A \notin I$ for every $U_x \in BN(x)\}$ when there is no ambiguity, we will simply write $A^{*b}(I)$ or $(A)^{*b}$ instead of $A^{*b}(I,\tau)$. $cl^{*b}(A)$ is defined as $A \bigcup (A)^{*b}(I)$

Remark 3.2: Since $\tau \subseteq PO(X) \subseteq BO(X)$ and $\tau \subseteq SO(X) \subseteq BO(X)$ we have the following.
(a) Every b-local function is a semi local function
(b) Every b-local function is a pre local function.

Theorem 3.3: Let (X,τ,I) be an ideal topological space and $A, B \subseteq X$. Then the following statements hold.

1. $\phi^{*b} = \phi$, $A \subseteq B \Rightarrow A^{*b} \subseteq B^{*b}$ and $E^{*b} = \phi$ if $E \in I$
2. For another ideal $J, I \subseteq J \Rightarrow A^{*b}(J) \subseteq A^{*b}(I)$
3. $A^{*b} \subseteq A^{*p} \subseteq A^* \subseteq cl(A)$

4. $A^{*b} \subseteq A^{*s} \subseteq A^{*} \subseteq cl(A)$
5. $A^{*b} \subseteq bcl(A) \subseteq cl(A)$
6. $\left(A^{*b}\right)^{*b} \subseteq (A)^{*b}$
7. $(A \cap B)^{*b} \subseteq (A)^{*b} \cap (B)^{*b}$
8. $(A \cup B)^{*b} \supseteq A^{*b} \cup B^{*b}$
9. $(A)^{*b} = bcl(A)^{*b} \subseteq cl(A)$
10. If $E \in I$ then $(A \cup E)^{*b} = A^{*b} = (A \setminus E)^{*b}$
11. If $U \in \tau$ then $U \cap (A)^{*b} = U \cap (U \cap A)^{*b} \subseteq (U \cap A)^{*b}$

Proof:
(1) and (2) are obvious by definition of b-local function.
(3) Obvious since $\tau \subseteq Po(X) \subseteq Bo(X)$ and $\phi \in I$.
(4) Obvious since $\tau \subseteq So(X) \subseteq Bo(X)$ and $\phi \in I$.
(5) $x \in A^{*b}$ implies $A \cap U \notin I$ for every $U \in BN(x)$ implies $A \cap U \neq \phi$ for every $U \in BN(x)$ implies $x \in bcl(A)$.
(6) $\left(A^{*b}\right)^{*b} = \{x \in X / U_x \bigcap (A)^{*b} \notin I \text{ for every } U_x \in BN(x)\}$
$\subseteq \{x \in X / U_x \bigcap A \notin I \text{ for every } U_x \in BN(x)\}$
$= (A)^{*b}$.
(7) Follows from (1)
(8) Follows from (1)
(9) If $x \in X \setminus (A)^{*b}$ then there exists $M \in BN(x)$ such that $A \cap M \in I$. Therefore there exists $U \in Bo(X)$ such that $x \in U \subseteq M$. So $A \cap U \in I$ and this implies $U \subseteq X \setminus (A)^{*b}$. Therefore $X - (A)^{*b}$ is the union of b-open sets and hence it is b-open. So $(A)^{*b}$ is b-closed. Therefore $(A)^{*b} = bcl(A)^{*b} \subseteq bcl(bcl(A)) = bcl(A) \subseteq cl(A)$. Hence A^{*b} is b-closed sub set of $cl(A)$.
(10) $A - E \subseteq A$ implies $(A - E)^{*b} \subseteq A^{*b}$ (A)

Let $x \in A^{*b}$. Suppose $x \notin (A \setminus E)^{*b}$, then there exists $U_x \in BN(x) such that$ $U_x \cap (A \setminus E) \in I$.

Then $E \cup [U_x \cap (A \setminus E)] \in I$. This implies that $E \cup [U_x \cap A] \in I$. So, $U_x \cap A \in I$ which is a contradiction to the fact that $x \in A^{*b}$. So, $A^{*b} \subseteq (A \setminus E)^{*b}$ (B)

From (A) and (B) we get $(A \setminus E)^{*b} = A^{*b}$ when $E \in I$.
(11) Let $U \in \tau$, $x \in U \cap (A)^{*b}$ and U_x be a b-open set containing x. Then $U \cap U_x \in BO(X)$ and hence $(U_x \cap U) \cap A \notin I$ which proves $x \in (U \cap A)^{*b}$. Therefore $U \cap (A)^{*b} \subseteq (U \cap A)^{*b}$.

So $U \cap (A)^{*b} = U \cap (U \cap (A)^{*b}) \subseteq U \cap (U \cap A)^{*b}$ (A)

On the otherhand, $U \cap A \subseteq A$ implies $(U \cap A)^{*b} \subseteq A^{*b}$

Therefore $U \cap (U \cap A^{*b}) \subseteq U \cap (A)^{*b}$ (B)

From (A) and (B) it follows that $U \cap (A)^{*b} = U \cap (U \cap A)^{*b}$

Remark 3.4: In general $(A \cup B)^{*b} \neq A^{*b} \cup B^{*b}$ and $(A \cap B)^{*b} \neq A^{*b} \cap B^{*b}$ as seen from examples (3.5) and (3.6).

Example 3.5: Let Z be the set of integers, τ be the cofinite topology in X and $I = \{\phi\}$.
Then $BO(X) = \{\varphi,$ all infinite subsets of X $\}$ and $BC(X) = \{X, A / A^C$ is infinite $\}$.

Let $A = Z^+$ and $B = Z^-$. Then $A \cup B = Z - \{o\}$.

In this space, for a subset $K \subseteq Z$, $K^{*b} = bcl(K) = K$ if K^C is infinite
$= Z$ if K^C is finite.

So $A^{*b} = A$, $B^{*b} = B$, $A^{*b} \cup B^{*b} = Z - \{o\}$ whereas $(A \cup B)^{*b} = Z$.

Therefore $(A \cup B)^{*b} \neq A^{*b} \cup B^{*b}$ and $cl^{*b}(A \cup B) \neq cl^{*b}(A) \cup cl^{*b}(B)$.

Example 3.6: In the ideal space given in example (3.5), let $A = Z \setminus \{-n, -n+1, \ldots\ldots\ldots, n-1, n\}$ and $B = \{-n, -n+1, \ldots\ldots\ldots, n-1, n\}$. Then $(A \cap B) = \phi$, $(A \cap B)^{*b} = \phi$, $A^{*b} = Z$, $B^{*b} = B$ and $A^{*b} \cap B^{*b} = B \neq (A \cap B)^{*b}$. Therefore $cl^{*b}(A \cap B) \neq cl^{*b}(A) \cap cl^{*b}(B)$.

Remark 3.7: In the ideal space (X, τ, I) because $cl^{*b}(A \cup B) \neq cl^{*b}(A) \cup cl^{*b}(B)$ in general, we are not able to define a topology using the operator $cl^{*b}(\ )$. To define a topology we need the following definitions.

Definition 3.8: An ideal space (X, τ, I) said to be

1. $*b-$ finitely additive if $\left[\bigcup_{i=1}^{n} A_i\right]^{*b} = \bigcup_{i=1}^{n} (A_i)^{*b}$ for every finite positive integer n.
2. $*b-$ additive if $\left[\bigcup_{\alpha\in\Omega} A_\alpha\right]^{*b} = \bigcup_{\alpha\in\Omega} (A_\alpha)^{*b}$ for every indexing set Ω.
3. $*b-$ finitely multiplicative if $\left[\bigcap_{i=1}^{n} A_i\right]^{*b} = \bigcap_{i=1}^{n} [A_i]^{*b}$ for every finite positive integer n.
4. $*b-$ multiplicative if $\left[\bigcap_{\alpha\in\Omega} A_\alpha\right]^{*b} = \bigcap_{\alpha\in\Omega} [A_\alpha]^{*b}$ for every indexing set Ω.

Remark 3.9:
1. Every $*b-$ additive (resp $*b-$ multiplicative) space is $*b-$ finitely additive (resp $*b-$ finitely multiplicative).
2. $A \subseteq cl^{*b}(A)$
3. If (X, τ, I) is $*b-$ finitely additive then
(a) $cl^{*b}(A \cup B) = cl^{*b}(A) \cup cl^{*b}(B)$ and
(b) $cl^{*b}(cl^{*b}(A)) = cl^{*b}(A)$.

Therefore in a $*b-$ finitely additive space, $cl^{*b}(\ )$ satisfies Kuratowski closure axioms.

Definition 3.10: Let (X, τ, I) be a $*b-$ finitely additive space. If a $*b-$closed set A is defined to be one for which $cl^{*b}(A) = A$, then the class of all complements of such sets is a topology on X denoted by τ^{*b}, whose closure operation is given as $cl^{*b}(A) = A \cup (A)^{*b}$.

Example 3.11: The ideal space given in example (3.5) is not $*b-$ finitely additive, not $*b-$ finitely multiplicative and hence not $*b-$ additive and $*b-$ multiplicative.

Example 3.12: Let (X,τ) be an indiscrete space, $x_0 \in X$ and $I = \{\phi, \{x_0\}\}$. In this space all subsets are b-open and b-closed. $A^{*b} = A - \{x_0\}$ if $x_0 \in A$

$= A$ if $x_0 \notin A$

This space is both $*b-$ additive and $*b-$ multiplicative.

Example 3.13: Let (X,τ) be an indiscrete space, $p \in X$ and $I = \{A \subseteq X \,/\, p \notin A\}$.

In this space $A^{*b} = \{p\}$ if $p \in A$

$= \phi$ if $p \notin A$

This space is both $*b-$ additive and $*b-$ multiplicative.

These examples show that spaces which are $*b-$ additive, $*b-$ multiplicative and spaces which are not $*b-$ additive, not $*b-$ multiplicative do exist.

Remark 3.14: In a $*b-$ finitely additive space, (X,τ,I),
(1) $\tau^{*b} = \{A \subseteq X \,/\, cl^{*b}(X-A) = X - A\}$
(2) $cl^{*b}(A) \subseteq cl^{*s}(A) \subseteq cl^{*}(A) \subseteq cl(A)$ and hence $\tau \subseteq \tau^{*} \subseteq \tau^{*s} \subseteq \tau^{*b}$.

Thus a new topology τ^{*b} is defined in a $*b-$ finitely additive ideal space (X,τ,I), with the help of b-local function and this topology is finer than τ^{*}- topology.

Theorem 3.15: Let (X,τ,I) be an ideal space. For $A \subseteq X$, we have the following results.
1.If $I = \{\phi\}$ then $A^{*b} = bcl(A)$ and $cl^{*b}(A) = bcl(A)$.
2.If $I = P(X)$ then $A^{*b} = \phi$ and $cl^{*b}(A) = A$.

Proof: Obvious from the definition of $(A)^{*b}$

Remark 3.16: In a $*b-$ finitely additive space (X,τ,I) with $I = P(X)$, τ^{*b} is the discrete topology since every subset is $*b-$open and $*b-$closed.

Theorem 3.17: If I and J are two ideals in a $*b-$ finitely additive space such that $I \subseteq J$. Then $\tau^{*b}(I) \subseteq \tau^{*b}(J)$.

Proof: Let A be closed in $\tau^{*b}(I)$ topology

$$I \subseteq J \Rightarrow A^{*b}(J) \subseteq A^{*b}(I).$$

Therefore $cl_J^{*b}(A) \subseteq cl_I^{*b}(A)$

Then $A \subseteq cl_J^{*b}(A) \subseteq cl_I^{*b}(A) = A$. which proves $A \subseteq cl_J^{*b}(A)$ and so A is closed in $\tau^{*b}(J)$.

Definition 3.18: A subset A in an ideal space (X,τ,I) is said to be
1. $*b-$dense subset in X if $cl^{*b}(A) = X$
2. $*b-$perfect if $A^{*b} = A$.
3. $*b-$closed in X if $cl^{*b}(A) = A$

Theorem 3.19: In a $*b-$ finitely additive ideal space (X,τ,I) the following are equivalent

1. $W \in \tau^{*b}$
2. $X - W$ is τ^{*b} - closed
3. $(X-W)^{*b} \subseteq (X-W)$
4. $W \subseteq X-(X\setminus W)^{*b}$

Proof: Obvious.

4. THE SET OPERATOR $\psi_b(I,\tau)$.

Definition 4.1: Let (X,τ,I) be an ideal space. A set operator $\psi^b(I,\tau): P(X) \to P(X)$ is defined as follows.

For any $A \subseteq X$, $\psi_b(I,\tau)(A) = \{x \in X$ / there exists $U \in BN(x)$ such that $U - A \in I\}$.

Remark 4.2:

1. Obviously $x \in \psi_b(I,\tau)(A)$ if and only if $x \notin (A^c)^{*b}$. Therefore $\psi_b(I,\tau)(A) = X \setminus (X \setminus A)^{*b}$.
2. We denote $\psi_b(I,\tau)$ simply by ψ_b when no ambiguity is present.
3. $\psi(A) \subseteq \psi^s(A) \subseteq \psi_b(B)$
4. $\psi(A) \subseteq \psi_p\ (A) \subseteq \psi_b(B)$

Theorem 4.3: For a subset A in an ideal space (X,τ,I) the following results are true.

1. If $I = \{\phi\}$ then $\psi_b(A) = b\,\mathrm{int}(A)$.
2. If $I = P(X)$ then $\psi_b(A) = X$.

Proof:

1. $\psi_b(A) = X \setminus (X \setminus A)^{*b} = X \setminus bcl(X \setminus A) = b\,\mathrm{int}(A)$
2. $\psi_b(A) = X \setminus (X \setminus A)^{*b} = X - \phi = X$

The following theorem gives many basic and useful facts for the operator ψ_b .

Theorem 4.4: Let A and B subsets in an ideal space (X,τ,I) .

1. If $A \subseteq B$ then $\psi_b(A) \subseteq \psi_b(B)$.
2. $\psi_b(A \cap B) \subseteq \psi_b(A) \cap \psi_b(B)$.

Proof:

1. $A \subseteq B \Rightarrow X \setminus B \subseteq X \setminus A \Rightarrow (X \setminus B)^{*b} \subseteq (X \setminus A)^{*b} \Rightarrow X \setminus (X \setminus A)^{*b} \subseteq X \setminus (X \setminus B)^{*b} \Rightarrow \psi_b(A) \subseteq \psi_b(B)$
2. Follows from 1.

Theorem 4.5: Let (X,τ,I) be a $*b-$ finitely additive space. Then

1. If $U \in \tau^{*b}$ then $U \subseteq \psi_b(U)$.
2. For every $A \subseteq X$, then $\psi_b(A) \in \tau$.
3. For every $A \subseteq X$, then $\psi_b(A) \subseteq \psi_b(\psi_b(A))$.
4. For every $A \subseteq X$ and $E \in I$ then $\psi_b(A \setminus E) = \psi_b(A) = \psi_b(A \cup E)$.
5. If $A \in Bo(X)$ then $A \subseteq \psi_b(A)$.
6. If $A \in \tau$ then $A \subseteq \psi_b(A)$.

7. If A is semi open then $A \subseteq \psi_b(A)$.
8. If A is pre open, then $A \subseteq \psi_b(A)$.
9. If $A \in \tau^\alpha$ then $A \subseteq \psi_b(A)$.
10. If $(A \setminus B) \cup (B \setminus A) \in I$ then $\psi_b(A) = \psi_b(B)$.

Proof:
1. $U \in \tau^{*b} \Rightarrow (X \setminus U)^{*b} \subseteq X \setminus U$. Then $\psi_b(U) = X \setminus (X \setminus U)^{*b} \supseteq U$.
2. By theorem 3.3, $(X - A)^{*b}$ is b- closed. Therefore $cl^{*b}[(X \setminus A)^{*b}] \subseteq bcl(X \setminus A)^{*b} = (X \setminus A)^{*b}$.

Therefore $(X \setminus A)^{*b}$ is $*b-$ closed and hence $\psi_b(A) = X \setminus (X \setminus A)^{*b} \in \tau^{*b}$.
3. By (2) $\psi_b(A) \in \tau^{*b}$ and hence $\psi_b(A) \subseteq \psi_b(\psi_b A))$ by (1).
4. $\psi_b(A \setminus E) = X \setminus [(X \setminus (A \setminus E)]^{*b} = X \setminus [(X \setminus A) \cup E]^{*b} = X \setminus [X \setminus A]^{*b} = \psi_b(A)$.(by thm (4.3))
$\psi_b(A \cup E) = X \setminus [X \setminus (A \cup E)]^{*b} = X \setminus [(X \setminus A) \setminus E]^{*b} = X \setminus [X \setminus A]^{*b} = \psi_b(A)$.
5. If $A \in Bo(X)$ then $X \setminus A$ is b-closed. Therefore $(X \setminus A)^{*b} \subseteq bcl(X \setminus A) = (X \setminus A)$ and this implies A is $*b-$ open. So by (1) $A \subseteq \psi_b(A)$.
6. Follows from 5 since $\tau \subseteq Bo(X)$.
7. Follows from 5 since $So(X) \subseteq Bo(X)$.
8. Follows from 5 since $Po(X) \subseteq Bo(X)$.
9. Follows from 5 since $\tau^\alpha = So(X, \tau) \bigcap Po(X, \tau)$.
10. Let $A \setminus B = E$ and $B \setminus A = H$. Then $E \cup H \in I$ implies E and H are in *I*.

By (4) $B = (A \setminus E) \cup H$ implies $\psi_b(A) = \psi_b(A \setminus E) . = \psi_b[(A \setminus E) \cup H]$ (since $H \in I$) $= \psi_b(B)$.

Definition 4.6: In an ideal space (X, τ, I), we say two subsets A and B are congruent modulo I (in notation $A \equiv B \bmod I$) if $(A \setminus B) \cup (B \setminus A) \in I$. Obviously " $\equiv \bmod I$ " is an equivalence relation.

Theorem 4.7 Let A and B are two subsets in $*b-$ finitely additive ideal space (X, τ, I) .If $A \equiv B \bmod I$ then $\psi_b(A) = \psi_b(B)$.

Proof: It follows from definition of $A \equiv B (\bmod I)$ and by (10) of theorem (4.5).

5. B-COMPATABILITY OF τ WITH I

Definition 5.1 Given a space $(X, \tau, I), I$ is said to be b-compatible with τ, denoted by $I \overset{b}{\sim} \tau$ if the following holds: for $A \subseteq X$, if for every $x \in A$ there exists $U \in BN(x)$ such that $U \cap A \in I$ then $A \in I$.

Remark 5.2

Since $\tau \subseteq SO(X) \subseteq BO(x),\ I \overset{b}{\sim} \tau \Rightarrow I \overset{s}{\sim} \tau \Rightarrow I \sim \tau$

.
The following example shows the existence of this compatibility.

Example 5.3 Let (X, τ) be an indiscrete space, $p \in X$ and $I = \{A \subseteq X \,/\, p \notin A\}$. In this space $I \overset{b}{\sim} \tau$.

Theorem 5.4 If (X, τ, I) is a *b-finitely additive ideal space then the following are equivalent.

(1) $I \overset{b}{\sim} \tau$
(2) If A has a cover of b-open set each of whose intersections with A is in I then A is in I.

(3) For every $A \subseteq X, A \cap A^{*b} = \phi \Rightarrow A \in I$

(4) For every $A \subseteq X, A \setminus A^{*b} \in I$

(5) For every τ^{*b}-closed subset $A, A - A^{*b} \in I$

(6) For every $A \subseteq X$, if A contains no non-empty subset $B \subset B^{*b}$ then $A \in I.$

Proof: $(1) \Rightarrow (2)$ Let $I \overset{b}{\sim} \tau$ and $A = \cup A_\alpha$ where each A_α is b-open and $A \cap A_\alpha \in I$. Then by definition $A \in I.$

$(2) \Rightarrow (3)$ for $A \subseteq X$ .Let $A \cap A^{*b} = \phi$ .So if $x \in A$ then $x \notin A^{*b}$

Therefore there exists $U_x \in BN(x)$ such hat $U_x \cap A \in I$. Then $\{U_x / x \in A\}$ is an open cover for A and $U_x \cap A \in I$ hence $A \in I$.

$(3) \Rightarrow (4)$ Let $x \in A = A^{*b}$. Suppose $x \in (A - A^{*b})^{*b}$ then for every $U \in BN(x), U \cap (A - A^{*b}) \notin I.$

This implies $U \cap A \notin I$ which implies $x \in A^{*b}$ which is a contradiction.

$\therefore (A - A^{*b}) \cap (A - A^{*b})^{*b} = \phi$ and hence $A - A^{*b} \in I$ by (3).

$(4) \Rightarrow (5)$ Proof is obvious.

$(5) \Rightarrow (1)$ Let $A \subseteq X$ and for every $x \in A$ there exists $U \in BN(x)$ such that $U \cap A \in I$.

Then $A \cap A^{*b} = \phi$ .Since (X, τ, I) is *b-finitely additive

$(A \cup A^{*b})^{*b} = A^{*b} \cup (A^{*b})^{*b} \subseteq A^{*b} \cup A^{*b} = A^{*b} \subseteq A \cup A^{*b}$

$\therefore A \cap A^{*b}$ is *b-closed. By (5), $(A \cup A^{*b}) - (A \cup A^{*b})^{*b} \in I.$

But $(A \cup A^{*b}) - (A \cup A^{*b})^{*b} = (A \cup A^{*b}) - A^{*b} = A. \therefore A \in I$

$(4) \Rightarrow (6)$. Let $A \subseteq X$. By (4) $A - A^{*b} \in I$. Let $x \in A \cap A^{*b}$.

Suppose $x \notin (A \cap A^{*b})^{*b}$ then there exists $U \in BN(x)$ such that $U \cap (A \cap A^{*b}) \in I$. This implies that $U \cap A \in I$ which is a contradiction. Therefore $A \cap A^{*b} \subseteq (A \cap A^{*b})^{*b}$.

By (6) $A \cap A^{*b} = \phi$ and this implies $A = A - A^{*b} \in I$.(Since $(4) \Rightarrow (5)$.)

$(6) \Rightarrow (4).$ Let $A \subseteq X$ .since $(A - A^{*b}) \cap (A - A^{*b})^{*b} = \phi$,we have $(A - A^{*b}) \in I$ by (6).

Theorem 5.5 Let (X, τ, I) be an ideal space. Then $I \overset{b}{\sim} \tau$ if and only if $\psi_b(A) - A \in I$ for all $A \subseteq X.$

Proof: Necessity: Assume that $I \overset{b}{\sim} \tau$ .Let $A \subseteq X$, $x \in \psi_b(A) - A$ .Then $x \notin A.$ and there exists $U_x \in BN(x)$ such that $U_x - A \in I$ .Therefore for each $x \in \psi_b(A) - A$ there exists $U_x \in BN(x)$ such that

$U_x \cap (\psi_b(A) - A) \in I.$ This implies $\psi_b(A) - A \in I$.

Sufficiency: Let $A \subseteq X$ and for each $x \in A$ there exists $U_x \in BN(x) \ni U_x \cap A \in I$ .By definition of $\psi_b(A), \psi_B(X - A) = \{x \in A / \exists U_x \in BN(x) \ni U_x \cap A \in I\}$

$\therefore A \subseteq \psi_b(X - A) - (X - A) \in I$

Theorem 5.6 Let (X, τ, I) be *b-finitely additive ideal space with $I \overset{b}{\sim} \tau$. Then $\psi_b(\psi_b(A)) = \psi_b(A) \forall A \subseteq X$.

Proof: From theorem (4.1) $\psi_b(A) \subseteq \psi_b(\psi_b(A))$ .By theorem (5.5) $\psi_b(A) - A = E$ for some $E \in I$.

Therefore $\psi_b(A) = A \cup E$.

So, $\psi_b(\psi_b(A)) = \psi_b(A \cup E) = \psi_b(A)$, by theorem (4.5).

Theorem 5.7 let (X, τ, I) be a*b-finitely additive ideal space with $I \overset{b}{\sim} \tau$. If $U, V \in BO(x)$, and $\psi_b(U) = \psi_b(V)$ then $U \equiv V \bmod I$.

Proof: By theorem (4.5) $U \subseteq \psi_b(U)$

$\therefore U \setminus V \subseteq \psi_b(U) - V = \psi_b(V) - V \in I$

Therefore $U \equiv V \bmod I$

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