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COEFFICIENT INEQUALITIES FOR GENERALIZED SAKAGUCHI TYPE MULTIVALENT FUNCTIONS

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ABSTRACT

In this paper we have studied two subclasses $S_p(\alpha, s, t)$and $T_p(\alpha, s, t)$concerning with generalized Sakaguchi type functions in the open unit disc U, further by using the coefficient inequalities for the classes $S_p(\alpha, s, t)$ and $T_p(\alpha, s, t)$, two subclasses $S_p^0(\alpha, s, t)$ and $T_p^0(\alpha, s, t)$are defined. Some properties of functions belonging to the class $S_p^0(\alpha, s, t)$ and $T_p^0(\alpha, s, t)$ are also discussed.

***Keywords**: Analytic functions; Starlike functions; Convex functions; Sakaguchi function; Coefficient inequalities.*

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1. INTRODUCTION

Let A_p be the class of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \tag{1.1}$$

that are analytic in the open unit disc U={z ∈ C: |z|<1}. A function $f(z) \in A_p$ is said to be in the class $S_p(\alpha, s, t)$ defined by Frasin [1] if it satisfies

$$\operatorname{Re}\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \alpha \tag{1.2}$$

for some $\alpha(0 \le \alpha < 1)$, $s, t \in \mathrm{C}, s \neq t$ and for all z ∈ U. For $p = 1$and $s = 1$, the class $S_1(\alpha, 1, t) = S(\alpha, 1, t)$was introduced and studied by Owa et al.[4], and by taking $t = -1$, the class $S_1(\alpha, 1, -1) = S_s(\alpha)$was introduced by Sakaguchi [3] and is called Sakaguchi function of order α (see [1],[5]), where $S_s(0) = S_s$ is the class of starlike functions with respect to symmetrical points in U. We also denote by $T_p(\alpha, s, t)$ the subclass of A_p consisting of all functions $f(z)$ such that $zf'(z) \in S_p(\alpha, s, t)$. Also, we note that $S_1(\alpha, 1, 0) = S^*(\alpha)$ and $T_1(\alpha, 1, 0) = T^*(\alpha)$ which are, respectively, the familiar classes of starlike functions of order α(0≤ α <1) and convex functions of order α(0≤ α <1).

2. COEFFICINT INEQUALITIES FOR SUBCLASSES $S_p^0(\alpha, s, t)$ AND $T_p^0(\alpha, s, t)$

We first prove the following two theorems which are similar to the result of Cho et al.[3] and Owa et al.[4]

Theorem 2.1: If $f(z) \in A_p$ satisfies

$$\sum_{n=1}^{\infty} \left[\left|p + n - u_{p+n}\right| + (1-\alpha)\left|u_{p+n}\right|\right] \left|a_{p+n}\right| \le p - \alpha\left|u_p\right| \tag{2.1}$$

for some α (0 ≤ α <1), then $f(z) \in S_p(\alpha, s, t)$ where

$$u_p = \sum_{j=1}^{p} s^{p-j}\, t^{j-1} \tag{2.2}$$

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Proof: To prove Theorem 2.1, we show that if $f(z)$ satisfies (2.1) then

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1\right| < 1-\alpha$$

Evidently, since

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1 = \frac{(s-t)\{pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}\}}{\{(sz)^p+\sum_{n=1}^{\infty}a_{p+n}(sz)^{p+n}\}-\{(tz)^p+\sum_{n=1}^{\infty}a_{p+n}(tz)^{p+n}\}}-1$$

$$=\frac{(p-u_p)z^p+\sum_{n=1}^{\infty}(p+n-u_{p+n})a_{p+n}z^{p+n}}{u_p z^p+\sum_{n=1}^{\infty}u_{p+n}a_{p+n}z^{p+n}}$$

We see that

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1\right| \le \frac{|p-u_p|+\sum_{n=1}^{\infty}|p+n-u_{p+n}||a_{p+n}|}{|u_p|+\sum_{n=1}^{\infty}|u_{p+n}||a_{p+n}|}$$

Therefore, if $f(z)$ satisfies (2.1), then we have

$$\frac{|p-u_p|+\sum_{n=1}^{\infty}|p+n-u_{p+n}||a_{p+n}|z^{p+n}}{|u_p|+\sum_{n=1}^{\infty}|u_{p+n}||a_{p+n}|} \le 1-\alpha$$

This completes the proof of Theorem 2.1.

Theorem 2.2: If $f(z) \in A_p$ satisfies

$$\sum_{n=1}^{\infty}(p+n)\left[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}||a_{p+n}|\right] \le p^2-\alpha p|u_p| \tag{2.3}$$

or some α(0≤ α <1), then $f(z) \in T_p(\alpha,s,t)$.

Proof: Noting that $f(z) \in T_p(\alpha,s,t)$ if and only if $zf'(z) \in T_p(\alpha,s,t)$, we can prove Theorem 2.2.

We now define

$$S_p^0(\alpha,s,t) = \{f(z) \in A_p \text{ such that } f(z) \text{ satisfies } (\mathbf{2.1})\}$$

$$T_p^0(\alpha,s,t) = \{f(z) \in A_p \text{ such that } f(z) \text{ satisfies } (\mathbf{2.3})\}$$

3. COEFFICIENT INEQUALITIES FOR SUBCLASSES $S_p(\alpha,s,t)$ AND $T_p(\alpha,s,t)$

Next applying Caratheodry function $\varphi(z)$ defined by

$$\varphi(z) = 1+\sum_{n=1}^{\infty}\varphi_n z^n \tag{3.1}$$

We discuss the coefficient inequalities for functions $f(z)$ in $S_p(\alpha,s,t)$ and $T_p(\alpha,s,t)$.

Theorem 3.1: If $f(z) \in S_p(\alpha,s,t)$, then

$$|a_{p+n}| \le \frac{\beta|u_p|}{|\vartheta_{p+n}|}\left\{1+\beta\sum_{j=p+1}^{p+n-1}\frac{|u_j|}{|\vartheta_j|}+\beta^2\sum_{j_2>j_1}^{p+n-1}\sum_{j_1=p+1}^{p+n-2}\frac{|u_{j_1}u_{j_2}|}{|\vartheta_{j_1}\vartheta_{j_2}|}+\cdots+\beta^{n-1}\prod_{j=p+1}^{p+n-1}\frac{|u_j|}{|\vartheta_j|}\right\} \tag{3.2}$$

where

$$\beta = 2(p-\alpha u_p), \qquad \vartheta_n = nu_p - pu_n \tag{3.3}$$

for some α(0 ≤ α <1), $s,t \in \mathbb{C}, s \neq t$.

Proof: We define the function $\varphi(z)$ by

$$\varphi(z) = \frac{u_p}{(p-\alpha u_p)}\left(\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-\alpha\right) = 1+\sum_{n=1}^{\infty}\varphi_n z^n \tag{3.4}$$

for $f(z) \in S_p(\alpha, s, t)$. Then $\varphi(z)$ is caratheodry function and satisfies

$$|\varphi_n| \le 2 \qquad (n \ge 1) \tag{3.5}$$

Since

$$(s-t)zf'(z) = \{f(sz) - f(tz)\}\left\{\alpha + \frac{(p-\alpha u_p)}{u_p}\varphi(z)\right\}$$

we have

$$(s-t)z^p\left[p + \sum_{n=1}^{\infty}(p+n)a_{p+n}z^n\right] = z^p\left[(s^p - t^p) + \sum_{n=1}^{\infty}(s^{p+n} - t^{p+n})a_{p+n}z^n\right]\left[\frac{p}{u_p} + \frac{(p-\alpha u_p)}{u_p}\sum_{n=1}^{\infty}\varphi_n z^n\right]$$

$$u_p\left[p + \sum_{n=1}^{\infty}(p+n)a_{p+n}z^n\right] = \left[u_p + \sum_{n=1}^{\infty}u_{p+n}a_{p+n}z^n\right]\left[p + (p-\alpha u_p)\sum_{n=1}^{\infty}\varphi_n z^n\right]$$

So we get

$$a_{p+n} = \frac{(p-\alpha u_p)}{((p+n)u_p - pu_{p+n})}\left[u_{p+n-1}a_{p+n-1}\varphi_1 + u_{p+n-2}a_{p+n-2}\varphi_2 + \cdots + u_{p+1}a_{p+1}\varphi_{n-1} + u_p\varphi_n\right] \tag{3.6}$$

From equation (3.6), we easily have that

$$|a_{p+1}| = \left|\frac{(p-\alpha u_p)}{((p+1)u_p - pu_{p+1})}\varphi_1 u_p\right| \le 2|(p-\alpha u_p)|\left[\frac{|u_p|}{|(p+1)u_p - pu_{p+1}|}\right]$$

$$|a_{p+2}| = \left|\frac{(p-\alpha u_p)}{((p+2)u_p - pu_{p+2})}\left[\varphi_2 u_p + u_{p+1}a_{p+1}\varphi_1\right]\right|$$

$$\le \frac{2(p-\alpha u_p)|u_p|}{|(p+2)u_p - pu_{p+2}|}\left[1 + 2(p-\alpha u_p)\frac{|u_{p+1}|}{|((p+1)u_p - pu_{p+1})|}\right]$$

$$|a_{p+3}| \le \frac{2(p-\alpha u_p)|u_p|}{|(p+3)u_p - pu_{p+3}|}\left[1 + \frac{2(p-\alpha u_p)|u_{p+1}|}{|((p+1)u_p - pu_{p+1})|} + \frac{2(p-\alpha u_p)|u_{p+2}|}{|((p+2)u_p - pu_{p+2})|} + \frac{2^2(p-\alpha u_p)^2|u_{p+1}||u_{p+2}|}{|((p+1)u_p - pu_{p+1})||((p+2)u_p - pu_{p+2})|}\right]$$

Thus, using mathematical induction, we obtain the inequality (3.2).

Remark (1): Equality in Theorem (3.1) are attended for the function f(z) given by

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} = 1 + \frac{(1-2\alpha)z}{1-z} \tag{3.7}$$

Remark (2): If we put $s = 1, p = 1, \alpha = 0, t = 0$ in Theorem (3.1), then we have well known result

$f(z) \in S^* \to |a_n| \le n$ where $S^*$is a usual starlike class

And if we put $s = 1, p = 1, \alpha = 0, t = -1$, then we have the results due to Sakaguchi [2].

$f(z) \in STS \to |a_n| \le 1$ where STS is Sakaguchi function class.

For the function class $T_p(\alpha, s, t)$similarly we have,

Theorem 3.2: If $f(z) \in T_p(\alpha, s, t)$, then

$$|a_{p+n}| \le \frac{\beta p|u_p|}{(p+n)|\vartheta_{p+n}|}\left\{1 + \beta\sum_{j=p+1}^{p+n-1}\frac{|u_j|}{|\vartheta_j|} + \beta^2\sum_{j_2>j_1}^{p+n-1}\sum_{j_1=p+1}^{p+n-2}\frac{|u_{j_1}u_{j_2}|}{|\vartheta_{j_1}\vartheta_{j_2}|} + \cdots + \beta^{n-2}\prod_{j=p+1}^{p+n-1}\frac{|u_j|}{|\vartheta_j|}\right\} \tag{3.8}$$

4. DISTORTION INEQUALITIES FOR SUBCLASSES $S_p^0(\alpha,s,t)$ AND $T_p^0(\alpha,s,t)$

For functions f(z) in the classes $S_p^0(\alpha,s,t)$ and $T_p^0(\alpha,s,t)$we derive

Theorem 4.1: If $f(z) \in S_p^0(\alpha,s,t)$, then

$$|z|^p - \sum_{n=1}^{j}|a_{p+n}|\,|z|^{p+n} - A_j\,|z|^{p+j+1} \le |z|^p + \sum_{n=1}^{j}|a_{p+n}|\,|z|^{p+n} + A_j\,|z|^{p+j+1} \tag{4.1}$$

Where

$$A_j = \frac{p-\alpha|u_p|-\sum_{n=1}^{j}[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|}{p+j+1-\alpha|u_{p+n}|}(j \ge 1) \tag{4.2}$$

Proof: From the inequality 2.1, we know that

$$\sum_{n=j+1}^{\infty}[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}| \le p-\alpha|u_p| - \sum_{n=1}^{j}[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|$$

On the other hand we know that,

$$[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}| \ge p+n-\alpha|u_{p+n}|$$

and hence $p+n-\alpha|u_{p+n}|$ is monotonically increasing with respect to n. Thus we deduce

$$p+j+1-\alpha|u_{p+n}|\sum_{n=j+1}^{\infty}|a_{p+n}| \le p-\alpha|u_p| - \sum_{n=1}^{j}[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|$$

which implies that therefore

$$\sum_{n=j+1}^{\infty}|a_{p+n}| \le A_j$$

Therefore we have that

$$|f(z)| \le |z|^p + \sum_{n=1}^{j}|a_{p+n}|\,|z|^{p+n} + A_j\,|z|^{p+j+1}$$

And

$$|f(z)| \ge |z|^p - \sum_{n=1}^{j}|a_{p+n}|\,|z|^{p+n} - A_j\,|z|^{p+j+1}$$

This completes the proof.

For function $T_p(\alpha,s,t)$, similarly we have,

Theorem 4.2: If $f(z) \in T_p^0(\alpha,s,t)$, then

$$|z|^p - \sum_{n=1}^{j}|a_{p+n}|\,|z|^{p+n} - B_j\,|z|^{p+j+1} \le |f(z)| \le |z|^p + \sum_{n=1}^{j}|a_{p+n}|\,|z|^{p+n} + B_j\,|z|^{p+j+1} \tag{4.4}$$

And

$$p|z|^{p-1} - \sum_{n=1}^{j}(p+n)|a_{p+n}||z|^{p+n-1} - C_j\,|z|^{p+j} \le |f'(z)| \le p|z|^{p-1} + \sum_{n=1}^{j}(p+n)|a_{p+n}||z|^{p+n-1} + C_j\,|z|^{p+j} \tag{4.5}$$

where

$$B_j = \frac{p^2-\alpha p\,|u_p|-\sum_{n=1}^{j}(p+n)\left[[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|\right]}{(p+j+1)(p+j+1-\alpha|u_{p+n}|)} \tag{4.6}$$

and

$$C_j = \frac{p^2-\alpha p\,|u_p|-\sum_{n=1}^{j}(p+n)\left[[|p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|\right]}{(p+j+1)(p+j+1-\alpha|u_{p+n}|)} \tag{4.7}$$

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