SEMI GLOBAL DOMINATION

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ABSTRACT

A subset D of vertices of a graph connected graph G is called a semi global dominating set (sgd - set) iff D is a dominating set for both G and Gsc, where Gsc is the semi complementary graph of G. The semi global domination number (sgd - number) is the minimum cardinality of a semi global dominating set of G and is denoted by $\gamma_{sg}(G)$. In this paper sharp bounds for $\gamma_{sg}$ are supplied for graphs whose girth is greater than three. Exact values of this number for paths and cycles are presented as well. The characterization result for a subset of the vertex set of G to be a semi global dominating set for G is given and also characterized the graphs of order n having sgd - numbers 2, n – 1, n.

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1. INTRODUCTION & PRELIMINARIES

Domination is an active subject in graph theory, and has numerous applications to distributed computing, the web graph and adhoc networks. For a comprehensive introduction to theoretical and applied facets of domination in graphs the reader is directed to the book [2].

A set D of vertices is called a dominating set of G if each vertex not in D is joined to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set of G[2].

Many variants of the domination number have been studied. For instance a dominating set D is called a global dominating set of G if D is a dominating set of both G and its complement $G^c$. The global domination number of G, denoted by $\gamma_g(G)$ is the smallest cardinality of the global dominating set of G[5]. A dominating set D of connected graph G is called a connected dominating set of G if the induced sub graph $<D>$ is connected. The connected domination number of G, denoted by $\gamma_c(G)$ is the smallest cardinality of the connected dominating set of G[6]. A dominating set D of connected graph G is called a independent dominating set of G if the induced sub graph $<D>$ is a null graph[2].

G be a connected graph, then the Semi Complementary Graph of G is denoted by $G^c$ and it has the same vertex set as that of G and edge set being $\{uv/ u, v \in V(G), uv \notin E(G), there is w \in V(G) such that uw, vw \in E(G)\}$[4].

Recently we have introduced a new type of graph known as semi complete graph. Let G be a connected graph, then G is said to be semi complete if any pair of vertices in G have a common neighbour. The necessary and sufficient condition for a connected graph to be semi complete is any pair of vertices lie on the same triangle or lie on two different triangles having a common vertex [3].

In the present paper, we introduce a new graph parameter, the semi global domination number, for a connected graph G. We call D $\subseteq V(G)$ a semi global dominating set (sgd - set) of G if D is a dominating set for both G, $G^c$. The semi global domination number is the minimum cardinality of a semi global dominating set of G and is denoted by $\gamma_{sg}(G)$.
All graphs considered in this paper are simple, finite, undirected and connected. For all graph theoretic terminology not defined here, the reader is referred to [1].

In this paper, sharp bounds for $\gamma_{sg}$ are supplied for the graphs whose girth is greater than three. Also, we have given a characterization result for a proper subset of the vertex set of $G$ to be a sgd - set of $G$ and characterized the graphs whose sgd - numbers are 2, $n$, $n - 1$.

**Note:** Unless mentioned by $G$ we mean a connected graph.

### 2. MAIN RESULTS

Here, we obtain some bounds for the sgd - numbers of graphs whose girth is greater than three.

**Theorem 2.1:** If $G$ is a triangle free graph, then

$$2e - n(n - 3) \leq \gamma_{sg}(G) \leq n - \Delta(G) + 1.$$  

**Proof:** Suppose that $D$ be a minimum sgd - set of $G$. By our supposition each vertex in $V - D$ is non adjacent with atleast one vertex in $D$. Otherwise we get a contradiction to that $D$ is a sgd - set for $G$.

$$\Rightarrow e \leq \frac{n(n - 1) - [n - \gamma_{sg}(G)]}{2} \Rightarrow 2e - n(n - 3) \leq \gamma_{sg}(G)$$  

(1)

Suppose that $d(v) = \Delta(G)$ for some $v$ in $V(G)$.

Let $v_1, v_2, ..., v_{\Delta(G)}$ be the neighbours of $v$ in $G$. Since $G$ is triangle free, $[V - \{v_1, v_2, ..., v_{\Delta(G)}\}] \cup \{v_i : i \text{ is one of } 1, 2, ..., \Delta(G)\}$ is a sgd – set of $G$ and its cardinality is $n - \Delta(G) + 1$.

$$\Rightarrow \gamma_{sg}(G) \leq n - \Delta(G) + 1$$  

(2)

From (1) and (2)

$$\frac{2e - n(n - 3)}{2} \leq \gamma_{sg}(G) \leq n - \Delta(G) + 1$$

Furthermore the lower bound is attained in the case of $C_4$ and upper bound is attained in the case of $P_3$. Hence the bounds are sharp.

**Note:** The upper bound holds good for any graph $G$.

**Proposition 2.2:**

1. $\gamma_{sg}(K_n) = n$, $n \geq 3$
2. $\gamma_{sg}(S_n) = 2$, $n \geq 3$
3. $\gamma_{sg}(K_{m,n}) = 2$, $m + n \geq 3$
4. $\gamma_{sg}(P_n) = \lfloor n/3 \rfloor$, $n = 3m + 1$  
   $= \lfloor n/3 \rfloor + 2$, $n = 3m$, $3m + 2$

Here $n \geq 4$.

5. $\gamma_{sg}(C_n) = \lfloor n/3 \rfloor$, $n = 3m$
   $= \lfloor n/3 \rfloor + 1$, $n = 3m + 1$, $3m + 2$
6. $\gamma_{sg}(C_nO_K_2) = n$.

**Proposition 2.3:** $G = P_n(n \geq 4)$. Then there is an independent sgd – set for $G$ iff $n = 3m + 1$.

**Proposition 2.4:** $G = C_n(n \geq 4)$. Then there is an independent sgd – set for $G$ iff $n = 3m$.

**Proposition 2.5:** $G = P_n(n \geq 3)$. Then $\gamma_{sg}(G) = n - 2$ iff $n = 4, 5$.

**Proposition 2.6:** $G = C_n(n \geq 4)$. Then $\gamma_{sg}(G) = n - 2$ iff $n = 4, 5$. 

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Proposition 2.7: If \( T \) is a tree of order \( n \geq 3 \), then \( \gamma_{sg}(T) = 2 \) if and only if \( T \) is obtained from \( P_3 \) or \( P_4 \) by adding zero or more leaves to the stems of the path.

Note: \( 2 \leq \gamma_{sg}(G) \leq n \).

Theorem 2.8: \( \gamma_{sg}(G) = n \) if and only if \( G \cong K_n \).

Theorem 2.9: \( \gamma_{sg}(G) = n - 1 \) if and only if \( G \cong K_n - \{e\} \), where \( e \) is any edge in \( K_n \).

Proof: Assume that \( \gamma_{sg}(G) = n - 1 \). Suppose \( \text{diam}(G) = l, l \geq 3 \). W.l.g. assume that \( d_G(u, v) = l \) for some \( u, v \) in \( G \). Clearly \( u \) or \( v \) is not a cut vertex in \( G \). Hence \( D = \{u, v\} \) is a connected dominating set in \( G \). Follows that \( D = \{u, v\} \) is a sgd - set in \( G \) of cardinality \( n - 2 \), which is a contradiction to our assumption. So \( \text{diam}(G) \leq 2 \). If \( \text{diam}(G) = 1 \), then \( G = K_1 \).

This implies \( \gamma_{sg}(G) = n \), a contrary to our assumption. Hence \( \text{diam}(G) = 2 \). This implies \( G \) has at least one pair of non adjacent vertices. If \( G \) has a pendant vertex, then \( \gamma_{sg}(G) = 2 \).

Clearly \( n \geq 4 \). Hence \( \gamma_{sg}(G) < n - 1 \), a contrary to our assumption. Let \( u_1, v_1, u_2, v_2, \ldots, u_s, v_s \) be distinct pairs of non adjacent vertices in \( G \). Since \( \text{diam}(G) = 2 \), \( <u_1, v_1>, <u_2, v_2>, \ldots, <u_s, v_s> \) are paths in \( G \) for some \( w_1, w_2, \ldots, w_s \) in \( G \). Clearly \( V - \{u_1, u_2, \ldots, u_s\} \) or \( V - \{v_1, v_2, \ldots, v_s\} \) is a sgd - set in \( G \). If \( |s| \geq 2 \), then we get a contradiction to our assumption. So \( |s| = 1 \). This implies there is exactly one pair of non adjacent vertices in \( G \).

Hence \( G \cong K_2 - \{e\} \).

The converse part is clear.

Corollary 2.10: If \( G \) is a tree, then \( \gamma_{sg}(G) = n - 1 \) if and only if \( G \cong P_3 \).

Note: By Theorem 2.9
(i) \( \gamma_{sg}(C_n) \neq n - 1 \) for any \( n \).
(ii) \( \gamma_{sg}(P_n) = n - 1 \) if and only if \( n = 3 \).

Theorem 2.11: \( \gamma_{sg}(G) = 2 \) if and only if
(i) There is an edge \( u \) in \( G \) such that each vertex in \( V - \{u, v\} \) is adjacent to \( u \) or \( v \) but not both.

or

(ii) There is a path \( P_4 \) in \( G \) with end vertices \( u \) and \( v \) such that all vertices in \( V - \{u, v\} \) are adjacent to either \( u \) or \( v \) but not both.

Proof: Suppose that \( \gamma_{sg}(G) = 2 \). W.l.g. assume that \( D = \{u, v\} \) be \( \gamma_{sg} \)-set in \( G \).

Case: 1 \( <D> \) is connected in \( G \).

Clearly \( uv \) is an edge in \( G \). If any vertex \( w \) in \( V - \{u, v\} \) is adjacent to both \( u \) and \( v \), then \( D \) is not a dominating set for \( G \).

Case: 2 \( <D> \) is not connected in \( G \).

Clearly any vertex in \( V - D \) cannot be adjacent to both \( u \) and \( v \). Hence there is a path \( P_4 \) from \( u \) to \( v \) in \( G \), say \( <uv, v_2, v> \). Let \( v_3 \in V - V(P_4) \). Since \( D \) is an sgd - set in \( G \), \( v_3 \) is adjacent to \( u \) or \( v \) (in \( G \)) but not both. W.l.g. assume that \( v_3 v_4 \) is in \( G \). For \( v_4 \) to be dominated by a vertex in \( D \), \( v_4 \) and \( v \) are to be connected by a path of length two in \( G \), say \( <v_4, v_5, v> \).

Hence \( v_4 \) lies on an edge \( v_3 v_4 \) and \( v_4, v_5 \) are totally dominated by \( u, v \) (end vertices in \( P_4 \) ) respectively. Hence (ii) holds.

The converse part is clear.

Result 2.12: A sgd – set for \( G \) is a global dominating set for \( G \).

Note: \( \gamma_{g}(G) \leq \gamma_{sg}(G) \).

Result 2.13: If \( \text{diam}(G) = 2 \), then \( D \) is a sgd – set in \( G \) if and only if \( D \) is a global dominating set in \( G \).

Corollary 2.14: \( G \) be a semi complete graph \( D \subset V \). Then \( D \) is a sgd-set in \( G \) if and only if \( D \) is a global dominating set in \( G \).
Proof: By hypothesis, $diam(G) = 2$. Hence proof follows from the above result.

Now, we give the characterization result for a non empty subset of $V$ to be sgd – set in G

Theorem 2.15: $D \subseteq V$ is a sgd – set in G if and only if each vertex in $V-D$ lies on an edge whose end points are totally dominated by distinct vertices in D.

Proof: Assume that D is a sgd–set in G. Let $v_1 \notin V-D$. By our assumption, there exists $v_2, v_3$ in $D$ ($v_2 \neq v_3$) such that $v_1 v_2$ is in $E(G)$ and $v_1 v_3$ is in $E(G^c)$. Since $v_1 v_3$ is in $E(G^c)$, there is $v_4$ in V such that $<v_1 v_2 v_4>$ is a path in G.

Now, we have the following cases:

Case: 1 $v_4 = v_2$.

Then $<v_1 v_2 v_3>$ is a path in G, which implies $v_1$ lies on the edge $v_1 v_2$ and $v_1, v_2$ are dominated by $v_2, v_3$ respectively from $D - \{v_1\}$, $D - \{v_1, v_2\}$.

Case: 2 $v_4 \neq v_2$.

Then $<v_1 v_2 v_3 v_4>$ is a path in G which implies $v_1$ lies on the edge $v_1 v_2$ and $v_1, v_4$ are dominated by $v_2, v_3$ respectively from $D - \{v_1, v_4\}$.

Hence in either case the claimant holds.

Conversely assume that $v_1 \notin V-D$. By our assumption there is an edge $v_1 v_2$ in G such that $v_1 v_2, v_2 v_3$ are in G and $v_2, v_3$ are in $D (v_3 \neq v_4)$.

If $v_3 = v_2$, then $<v_1 v_2 v_4>$ is a path in G and $v_1 v_2$ is in G, $v_1 v_4$ is in $G^c$.

If $v_2 \neq v_3$, then $<v_4 v_1 v_2 v_4>$ is a path in G, which implies $v_1 v_3$ is in G and $v_1 v_4$ is in $G^c$.

Hence, in either case for $v_1$ in D, there are $v_2, v_4$ in D such that $v_1 v_3$ is in G and $v_1 v_4$ is in $G^c$. Hence D is a sgd – set in G.

Theorem 2.16: G be a connected graph and D be a $\gamma_{SG}$ – set in G. Then $d_{D \cup \{v\}}(v) < n$ for each $v$ in $V-D$ if and only if D is a sgd – set in G.

Proof: Assume that $d_{D \cup \{v\}}(v) < n$ for each $v$ in $V-D$. Let $v \in V-D$.

Then by our assumption $d_{D \cup \{v\}}(v) < n$. This implies there is $v_1$ in D such that $d(v, v_1) \neq 1$. Since $<D \cup \{v_1\}>$ is connected, this implies there is a $v-v_1$ path in $D \cup \{v_1\}$ (say) $P = <v v_2 v_3 ... v_n>$, where $v_2, v_3, ..., v_n \in D$. Since $d_{D \cup \{v\}}(v) < n$, there is a $v_1 \in D$ such that $d_{D}(v, v_1) = 2$. This implies $vv_2 \in E(G^c)$. D is a sgd – set in G.

Conversely assume that $v \in V-D$. By our assumption, there is $v_1$ in D such that $d_{D}(v, v_1) = 2$. This implies $vv_1 \notin G$.

Hence $d_{D \cup \{v\}}(v) < n$.

Theorem 2.17: G be a connected graph such that $\delta(G) \geq 2$ and D is an independent sgd – set for G. If $D^c$ is independent, then $D^c$ is a sgd – set in G.

Proof: Assume that $D^c$ is independent. Let $v \in V-D^c = D$. This implies there is $v_1$ in $D^c$ such that $vv_1$ is in G (since $\delta(G) \geq 2$). Since $v_1$ is in $D^c$ and D is independent sgd – set in G, there is $v_2$ in $D, v_3$ in V such that $<v_1 v_2 v_3>$ is a path in G. Clearly $v_2 \in D^c$. Since $D^c$ is independent, $<vv_1 v_2 v_3>$ is a path in G and $v_1 v_3$ is not an edge in G. For $v \in V-D^c$, there is $v_1 \in D^c$ such that $vv_1$ is in G and $v_2 v_3$ is in $G^c$. Since v is arbitrary, $D^c$ is a sgd – set in G.

Note: The converse is not true in view of P7.

Result 2.18: For a semi complete graph G, $\gamma_{SG}(G) \geq 3$.

Proof: Suppose claimant does not hold. Since $\gamma_{SG}(G) \neq 1, \gamma_{SG}(G) = 2$. Let $D = \{v_1, v_2\}$ be a sgd – set in G.

Case: 1 $<D>$ is connected in G.
Then \(v_1, v_2\) is an edge in \(G\). By the nature of semi complete graph there is a \(v_3\) in \(G\) such that \(<v_1, v_2, v_3>\) is a triangle in \(G\). This implies \(D\) is not a dominating set in \(G^c\), which is a contradiction to \(D\) is a \(sgd\) - set in \(G\).

**Case: 2 \(<D>\) is disconnected in \(G\).**

Since \(G\) is semi complete there is \(v_3\) in \(G\) such that \(<v_1, v_3, v_2>\) is a path in \(G\). Then in \(G^c\), \(v_j\) is not dominated by vertex in \(D\), a contradiction to \(D\) is a \(sgd\) - set in \(G\).

Hence in either case, we get a contradiction to \(D\) is a \(sgd\) - set in \(G\).

So, Our supposition is false. This implies \(\gamma_{sg}(G) \geq 3\).

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**REFERENCES**


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