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ANTICOMMUTATIVE ELEMENTS IN ALTERNATIVE RINGS

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ABSTRACT

*In this paper we show that if R is a semiprime 2-and 3-divisible alternative ring such that a*b = 0 implies ab = 0, then R must be commutative.*

***Keywords**: Alternative ring, Semi prime, 2 and 3-divisible ring, Nucleus.*

INTRODUCTION

Two elements a, b of any ring are said to anticommute if $a*b = ab+ba = 0$. If an element squares to zero, then it automatically anticommutes with itself. Two elements are purely anticommutative if they anticommute but do not commute. The alternative rings have no purely anticommutative elements. Kleinfeld [2] proved that a 2-and 3-divisible semiprime alternative ring R in which $a*b=0$ implies $ab = 0$ is associative. In this paper, we present Kleinfeld's conclusions in a simple way and prove that R is commutative. It is proved that if 'n' is an element in the nucleus of R such that $n^2 = 0$ then $n = 0$. By using this, we show that if R is a 2 – and 3-divisible semiprime alternative ring such that $a*b=0$ implies $ab = 0$, then R must be commutative.

PRELIMINARIES

Throughout this paper R will denote a 2-and 3-divisible semiprime alternative ring with

$$a*b = 0 \text{ implies } ab = 0. \qquad (1)$$

The associator and the commutator are defined by $(x, y, z) = (xy)z - x(yz)$ and $(x, y) = xy - yx$ for all x, y, z in R. A ring R is said to be alternative if $(x, x, y) = 0 = (y, x, x)$ for all x, y in R. The nucleus N in R is the set of elements $n \in R$ such that $(n, R, R,) = (R, n, R) = (R, R, n) = 0$. The center Z of R is the set of elements $z \in N$ such that $(z, x) = 0$ for all x in R. A ring R is semiprime if for any ideal A of R, $A^2 = 0$ implies $A=0$. R is called k-divisible if $kx = 0$ implies $x = 0$, $x \in R$ and k is a natural number.

In a 2-divisible alternative ring with generators x, y, z we know that $u = (x, y, z)$ and $v = (x, y)$ are purely anticommutative elements.

The following identity is proved in [2]:

$$(u, x, y) = vu = - uv.$$

Then this can be rewritten as

$$(u, x, y) = vu = uv = 0 \qquad (2)$$

and $$(x^2, y, z) = x* (x, y, z). \qquad (3)$$

Using (2) and (3) now yields

$$0 = ((x^2, y, z), y, z) = (x* (x, y, z), y, z).$$

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But a linearization of (3) implies that

$$\begin{aligned}(x^* (x, y, z), y, z) &= x^* ((x, y, z), y, z) + (x, y, z) * (x, y, z)\\ &= (x, y, z)^* (x, y, z).\end{aligned}$$

Hence by (1), $(x, y, z)^2 = 0$.

Linearization the last identity,

we obtain (x, y, a) * (x, y, b) = 0. So that by (1), (x, y, a) (x, y, b) = 0.

We may write the last identity as

$$(x, y, R)\ (x, y, R) = 0, \tag{4}$$

where R represents arbitrary elements of R. Linearizing (4) we get

$$(x, a', b')\ (x. c', d') = \operatorname{sgn} P\ (x, a, b)\ (x, c, d), \tag{5}$$

where P is the permentation

$$aP = a',\ bP = b',\ cP = c',\ dP = d'.$$

Clearly it follows from (5) that

$$((x, R, R), (x, R, R)) = 0. \tag{6}$$

Also beginning with (6) we have

$$0 = ((x, a, b), (x,y^2,z) = ((x, a, b), y * (x, y, z)) = y^* ((x, a, b),(x, y, z)) + (x, y, z) * ((x, a, b),y) = (x, y, z)^* ((x, a, b), y).$$

Then using (1) we obtain

$$(x, y, z)\ ((x, a, b),y) = 0 = ((x, a, b),y)\ (x ,y ,z). \tag{7}$$

We have the semi-Jacobi identity

$$(a, bc) = b\ (a, c) + (a, b)\ c - 3(a, b, c).$$

Then ((x, a, b), (x, y, z) y) = ((x, a, b), (x, y, yz)) = 0 using (6).

But semi-Jacobi identity implies

$$((x, a, b), (x, y, z)\ y) = (x, y, z)\ ((x, a, b), y) + ((x, a, b), (x, y, z))\ y - 3((x, a, b), (x, y, z), y) = 0.$$

Using (7) and (6), we obtain

$$3((x, a, b), (x, y, z), y) = 0.$$

The 3-divisibility of R implies that

$$((x, a, b), (x, y, z), y) = 0.$$

Then by linearization, we have

$$((x,a',b'),c', (x,d',e')) = \operatorname{sgn} P((x,a,b),c,(x,d,e)), \tag{8}$$

for every permutation P on a, b, c, d, e. By applying (8), we see that

((x,a,b),c,(x,d,e)) = ((x,d,e),c,(x,a,b)). But the alternative identity implies that

$$((x,a,b),c,(x,d,e)) = -\ ((x,d,e),c,(x,a,b)).$$

Thus

$$((x,R,R),R,(x,R,R)) = 0. \tag{9}$$

MAIN RESULTS

First we prove the following lemmas.

Lemma 1: If n is an element of the nucleus of R such that $n^2 = 0$, then n = 0.

Proof: Since $0 = (n^2,x) = n * (n, x)$, from (1) we have $n(n, x) = 0 = (n, x)n$. So that $nRn = 0$. The ideal generated by n can be generated by all finite sums of elements of the form in, nR, Rn and RnR. Since $[xny]z = (x, ny, z) + x[nyz] = n(x, y, z) + x[nyz]$ belongs to this ideal, call it I. It is easy to show that $In = 0$. If T is the set of right annihilators of I, it follows that T is an ideal. Since T contains n, then T contains I. But then $I^2 = 0$, implies $I = 0$, since R is semiprime. Thus n = 0.

This completes the proof of the lemma.

Lemma 2: If t is an element such that $(R, R, R) t = 0 = t(R, R, R)$ and if S is the ideal of R generated by all associators, then $St = 0 = tS$.

Proof: Since $[(x, y, z)y]t =(x, y, yz)t=0$, we obtain $[(w',x',y')z']t = \text{sgn } P[(w, x, y)z]t$. However, $(wx, y, z) – (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y) z$ holds in every ring. By multiplying on the right by t, we get

$[w(x, y, z)]t = -[(w, x, y)z]t = [(x, y, z)w]t.$

Thus $((R, R, R), R) t = 0$.

Similarly $t((R, R, R), R) = 0$. But then

$(t * (w, x, y),z) = t * ((w, x, y), z) + (t, z) * (w, x, y) = (t, z) * (w, x, y).$

Then (1) implies $(w, x, y) (t, z) = 0$.

But then semi-Jacobi identity leads to

$((w, x, y) t, z) = (w, x, y) (t, z) + ((w, x, y), z) t + 3((w, x, y), t, z)$. Thus $3((w, x, y), t, z) = 0$. So that $((w, x, y), t, z) = 0$. Consequently

$$0 = (z, (w, x, y), t) = [z(w, x, y)] t.$$

Since S can be characterized as all finite sums of elements of the form (R, R, R) and R(R, R, R), we have established that $St = 0$. By going to the antiisomorphic copy of R we can establish $tS = 0$.

This completes the proof of the lemma.

The two lemmas can now be used to establish the main theorem.

Theorem 1: Let R be a 2- and 3- divisible semiprime alternative ring. If $a * b =0$ implies $ab = 0$, then R must be commutative.

Proof: Using (9) we see that $((x, y, z) * y, (x, a, b), c) = ((x, y^2, z), (x, a, b), c) = 0$, using (3) at the end.

But also

$((x, y, z)* y, (x, a, b), c) = (x, y, z) * (y, (x, a, b), c) + y * ((x, y, z), (x, a, b), c) = (x, y, z) * (y, (x, a, b), c)$. Then by (1)

$(x, y, z) (y,(x, a, b),c) = 0$. By the linearization of this identity, we obtain

$$(x, y', z') (d',(x, a, b),c') = \text{sgn } P(x, y, z) (d,(x, a, b),c), \tag{10}$$

for every permutation P on c, d, y, z. A linearization of (2) shows that

$(r,(x,r,s),t) = – (r,(t,r,s),x) = – (r,(r,s,t),x).$

Thus (x,y,z) (r,(x,r,s),t) = – (x,y,z) (r,(r,s,t),x) = (x,y,r) (z,(r,s,t),x), using (5). However, (10), with x replaced by r, now makes the last expression zero. Hence

$$(x,y,z)\,(q', (x,r',s'),t') = \operatorname{sgn} P(x,y,z)\,(q,(x,r,s),t) \tag{11}$$

By combining (10) and (11) we obtain

$$(x,y',z')\,(q',(x,r',s'),t) = \operatorname{sgn} P(x,y,z)\,(q,(x,r,s),t) \tag{12}$$

Now (y,(x,y,z) * (x,r,s),t) = (x,y,z) * (y,(x,r,s),t) + (x,r,s) * (y,(x,y,z),t) = 0

using (12). However, (6) implies that (x, y, z) * (x, r, s) = 2(x, r, s) (x, y, z).

Thus (y, (x, r, s) (x, y, z),t) = 0. Since (a, b, R) = 0 always implies that (a, b) is in the nucleus of R, we can establish that (y,(x, r, s) (x, y, z)) is in the nucleus. Because of (4) every associator u squares to zero. As before (1) implies that

$$uRu = 0.$$

But then $[(x, r, s)\,(x, y, (y, z))]^2 = 0$, where u is taken to be either (x, r, s) or (x, y,(y, z)). Now lemma 1 yields the identity

$$(x,r,s)\,(x,y,(y,z)) = 0. \tag{13}$$

From (9), we have ((x,R,R),R, (x,R,R,)) = 0.

By using the alternative property,

$$((x, R, R,), (x, R, R), R) = 0.$$

Linearization of this yields that

$$((x,R,R), (w,R,R,),R) + ((w,R,R), (x,R,R),R) = 0.$$

Then $2((R,R,R), (R,R,R),R) = 0.$

Since R is 2-divisible,

$$((R,R,R), (R,R,R),R) = 0 \tag{14}$$

and $$((R,R,R), (R,R,R)) = 0. \tag{15}$$

Then from Lemma 2, (14), (15) and hypothesis it follows that

$$(R,R,R)\,(x,y,(y,z)) = 0 = (x,y,(y,z))\,(R,R,R). \tag{16}$$

By linearizaing (16), we obtain

$$(R,R,R)\,(x,y,(r,z)) = -(R,R,R)\,(x,r,(y,z)).$$

Then $(R,R,R)\,(R,R,(R,R)) = 0$, since R is 2-divisible.

Now by the hypothesis, we have

$$(R,R,(R,R)) = 0. \tag{17}$$

So the commutator is in the nucleus and hence in the center.

We have $((x, y)x, y) = (xy\cdot x - yx\cdot x, y) = (x\cdot yx - yx\cdot x, y) = ((x, yx),y) = 0.$

If v = (x, y) in the identity $(v, x, y) = v(x, y) + (v, y)x + 3(v, x, y)$, we obtain $v(x, y) = 0$.

Thus $(x, y)^2 = 0$. From lemma 1, we get that (x, y) = 0.

Hence R is commutative.

REFERENCES

[1] Kleinfeld, E, “A characterization of the Cayley numbers”, studies in Modern Algebra, Vol.2.

[2] Kleinfeld, E, “Anticommutative Elements in Alternative Rings”, Journal of Algebra 83, 1983, 65-71.

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