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Uniquely Clean Idempotent 2×2 Matrices Over Integral Domains

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ABSTRACT

Let R be a ring with identity. An element of R is said to be clean if it is the sum of a unit and an idempotent and it is uniquely clean if this representation is unique. It is well known that central idempotents in any ring are uniquely clean ([2]). In this paper it has been shown that if R is an Integral Domain then the central idempotents are the only uniquely clean idempotents in $M_2(R)$.

Keywords: *Uniquely clean, central idempotents, integral domain.*

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1. INTRODUCTION

Let *R* be a ring with identity. Recall that an element *a* in *R* is called *clean* if it can be expressed as $a = e + u$, where *e* is an idempotent and *u* is a unit in *R*. It is said to be *uniquely clean* if this representation is unique. It is well known that idempotents in any ring are clean. If *e* is an idempotent in the ring, then $e = (1 - e) + (2e - 1)$ is a clean representation of *e* since $1- e$ is an idempotent and $2e - 1$ is a unit in that ring. Nicholson and Zhou [2, Example 1] have shown that central idempotents are uniquely clean in any ring.

Throughout this paper we let *R* be an Integral Domain and $M_2(R)$ be the ring of 2×2 matrices over *R*. It is well known that (see for example [1] or [3]) idempotents in $M_2(R)$ are:

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \text{ and } \begin{bmatrix}a & b\\c & 1-a\end{bmatrix} \text{ (where } bc = a - a^2\text{)}$$

One can easily verify that out of these $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ are the only central idempotents. In this paper we shall show that $\begin{bmatrix}a & b\\c & 1-a\end{bmatrix}$ (where $bc = a - a^2$) are not uniquely clean. This shows that non central idempotents are not uniquely clean. Hence uniquely clean idempotents are precisely the central idempotents in $M_2(R)$.

2. MAIN RESULTS

Theorem 2.1: Let *R* be an integral domain. In $M_2(R)$, idempotent matrix of the form $\begin{bmatrix}a & b\\c & 1-a\end{bmatrix}$ (where $bc = a - a^2$) is not uniquely clean.

Proof: For any idempotent matrix *E*, we always have $E = (I - E) + (2E - I)$ as one clean representation (since, $(I - E)$ is an idempotent and $(2E - I)^2 = I$). We shall call this the **natural clean representation** of *E*.

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Consider the idempotent matrix $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ (where $bc = a - a^2$).

If, $a = 0$, then $bc = 0$ gives

$$E = \begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ w & 1 \end{bmatrix}$$

But

$$\begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & w-p \\ 0 & 1 \end{bmatrix}$$

where $\begin{bmatrix} 1 & p \\ 0 & 0 \end{bmatrix}$ is an idempotent and $\begin{bmatrix} -1 & w-p \\ 0 & 1 \end{bmatrix}$ is a unit with determinant -1 in $\mathrm{M}_2(\mathrm{R})$,

and $\begin{bmatrix} 0 & 0 \\ w & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ w-p & 1 \end{bmatrix}$

where $\begin{bmatrix} 1 & 0 \\ p & 0 \end{bmatrix}$ is an idempotent and $\begin{bmatrix} -1 & 0 \\ w-p & 1 \end{bmatrix}$ is a unit with determinant -1 in $\mathrm{M}_2(\mathrm{R})$. Hence both $\begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ w & 1 \end{bmatrix}$ are not uniquely clean.

Similarly if $a = 1$, then for $E = \begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}$ we have the following clean representation,

$\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & w-p \\ 0 & -1 \end{bmatrix}$ and for $E = \begin{bmatrix} 1 & 0 \\ w & 0 \end{bmatrix}$

we have the following clean representation,

$$\begin{bmatrix} 1 & 0 \\ w & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ p & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ w-p & -1 \end{bmatrix}$$

Hence both $\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ w & 0 \end{bmatrix}$ are also not uniquely clean.

From now on $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ (where $bc = a - a^2 \neq 0$). In what follows we shall show that we can always express *E* as

$$E = E_1 + U$$

where $E_1 = \begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}$ (with $yw = x - x^2$) is an idempotent and $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ is a unit with determinant -1 in $\mathrm{M}_2(\mathrm{R})$. This amounts to showing that there exist an idempotent matrix $E_1 = \begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}$ (with $yw = x - x^2$) such that

$$det(E-E_1)=-1$$

or

$$det\left(\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}-\begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}\right)=-1$$

or

$$det\left(\begin{bmatrix} a-x & b-y \\ c-w & -a+x \end{bmatrix}\right)=-1$$

This problem reduces to solving the equation

$$(a-x)(x-a)-(b-y)(c-w)=-1 \qquad ....(*)$$

for *x*, *y* and *w* such that this solution $yw=x-x^2$.

Now from (*) we have

$$(2ax - a^2-x^2)-(bc-bw-yc+wy)=-1$$

using $bc=a-a^2$ and $yw=x-x^2$ we get

$$(2a-1)x-a+bw+yc=-1$$

Multiplying both sides by *y* and rearranging the terms we get

$$(2a-1)xy+y^2(c)+bwy+(-a+1)y=0$$

again using $yw=x-x^2$, we get

$$(-b)x^2+(2a-1)xy+(c)y^2+(b)x+(-a+1)y=0$$

Multiplying both sides by (-4*b*), we get

$$4b^2x^2+4b(1-2a)xy-4bcy^2-4b^2x+4b(a-1)y=0$$
or
$$(2bx+(1-2a)y-b)^2-((1-2a)y-b)^2-4bcy^2+4b(a-1)y=0$$
or
$$(2bx+(1-2a)y-b)^2-y^2-2by-b^2=0$$
or
$$(2bx+(1-2a)y-b)^2-(y+b)^2=0$$

Letting $x'=2bx+(1-2a)y-b$ and $y'=y+b$, we get

$$(x')^2-(y')^2=0$$
or
$$-(x')^2+(y')^2=0$$
or
$$(y'-x')(y'+x')=0 \qquad ....(**)$$

One of the parenthesis must be zero, say

$$(y'+x')=0$$

or

$$y+b+2bx+(1-2a)y-b=0$$

or

$$2bx+(2-2a)y=0$$

or

$$bx+(1-a)y=0$$

Clearly its general solution is

$x=(1-a)t$, $y=-bt$

taking $t = 1 - a$, we get

$x=(1-a)^2$, $y=-b(1-a)$ such that

$$\begin{aligned} x(1-x) &= (1-a)^2(1-(1-a)^2) \\ &= (1-a)^2(-a)(a-2) \\ &= -bc(1-a)(a-2) \quad (\text{as } bc = a - a^2) \\ &= -b(1-a)[c(a-2)] \\ &= y(c(a-2)) \\ &= yw \qquad\qquad (\text{letting } w = c(a-2)) \end{aligned}$$

Thus we get a solution $x=(1-a)^2$, $y=-b(1-a)$ and $w=c(a-2)$ of eq. (*) satisfying $yw=x-x^2$ and hence an idempotent matrix

$$E_1=\begin{bmatrix}(1-a)^2 & -b(1-a)\\ c(a-2) & 1-(1-a)^2\end{bmatrix}=\begin{bmatrix}(1-a)^2 & -b(1-a)\\ c(a-2) & -a^2+2a\end{bmatrix}$$

which gives the following clean representation for E

$$\begin{bmatrix}a & b\\ c & 1-a\end{bmatrix}=\begin{bmatrix}(1-a)^2 & -b(1-a)\\ c(a-2) & -a^2+2a\end{bmatrix}+\begin{bmatrix}a-(1-a)^2 & 2b-2a\\ c-c(a-2) & (a-1)^2-a\end{bmatrix}$$

where one can check that $det\left(\begin{bmatrix}a-(1-a)^2 & 2b-2a\\ c-c(a-2) & (a-1)^2-a\end{bmatrix}\right)=-1$

From (**), we can also have

$$(y'-x')=0$$

or

$$y+b-2bx-(1-2a)y+b=0$$

or

$$-2bx+2ay+2b=0$$

or

$bx+(-a)y=b$

Its general solution is

$x = 1 + at,\ y = bt$

taking $t = -a$, we get

$x=1-a^2\ ,\ y=-ba$

such that

$$\begin{aligned}
x(1-x) &= (1-a^2)(1-(1-a^2))\\
&= (1-a^2)(a^2)\\
&= (1-a)a[a(1+a)]\\
&= bc[a(1+a)]\\
&= (-ba)[-c(1+a)]\\
&= y[-c(1+a)]\\
&= yw \qquad\qquad (\text{letting } w = -c(1+a))
\end{aligned}$$

Hence we get another solution $x=1-a^2$, $y=-ba$ and $w=-c(1+a)$ for eq. (*) satisfying $yw=x-x^2$ and hence another idempotent matrix

$$E_1=\begin{bmatrix}1-a^2 & -ba\\ -c(1+a) & 1-1+a^2\end{bmatrix}=\begin{bmatrix}1-a^2 & -ba\\ -c(1+a) & a^2\end{bmatrix}$$

which gives the following clean representation for E

$$\begin{bmatrix}a & b\\ c & 1-a\end{bmatrix}=\begin{bmatrix}1-a^2 & -ba\\ -c(1+a) & a^2\end{bmatrix}+\begin{bmatrix}a-1+a^2 & b+ba\\ c+c(1+a) & 1-a-a^2\end{bmatrix}$$

where one can check that $det\left(\begin{bmatrix}a-1+a^2 & b+ba\\ c+c(1+a) & 1-a-a^2\end{bmatrix}\right)=-1$

Thus we get two different clean representations for E. Note that these clean representations are entirely different from the natural clean representation for E which is

$E = (I - E) + (2E - I)$

Or

$$\begin{bmatrix}a & b\\ c & 1-a\end{bmatrix}=\begin{bmatrix}1-a & -b\\ -c & a\end{bmatrix}+\begin{bmatrix}2a-1 & 2b\\ 2c & 1-2a\end{bmatrix}$$

Theorem 2.2: Let R be an integral domain. In $M_2(R)$, an idempotent is uniquely clean if and only if it is central. In other words, central idempotents are the only uniquely clean idempotents in $M_2(R)$.

Proof: The proof is clear from the above theorem and from the fact that central idempotents are uniquely clean [2].

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