International Journal of Mathematical Archive-3(7), 2012, 2686-2690
IJMA Available online through www.ijma.info ISSN 2229 – 5046

METRIZABILITY OF COMPLEX VALUED METRIC SPACES AND SOME REMARKS ON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES

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(Received on: 25-06-12; Revised & Accepted on: 15-07-12)

ABSTRACT

We discuss on a topological question in complex valued metric spaces, namely metrizability. We prove that every complex valued metric space is metrizable and hence complex valued metric spaces are not real generalizations of metric spaces. We also show that some of the fixed point theorems recently generalized to complex valued metric spaces are consequences of their counter parts in the setting of metric spaces and hence are redundant.

***AMS Mathematics Subject Classification (2010)**: 47H10, 54H25.*

***Keywords**: common fixed point, complex valued metric space, metrizability, contractive conditions.*

1. INTRODUCTION

A. Azam, B. Fisher and M. Khan [5] introduced the concept of a complex valued metric space and obtained a common fixed point result for a pair of mappings satisfying a certain contractive condition. Also, in [6] and [7] some fixed point results are obtained in such spaces.

In [1], [2], [3], [4], [8], [9], [11], [15], [16] and [18] the authors suggested that some generalizations of fixed point results which appeared recently are not real generalizations. However, in [19] the authors have reported some flaws on the paper [4]. Inspired by these research works, we investigated the fixed point results given in [5] – [7] and obtained that some of these results are straight forward generalizations of the corresponding results for (real valued) metric spaces and hence are redundant. Parts of our results are adopted from the results in [3].

In this paper we also investigate a topological property of complex valued metric spaces, namely metrizability. We prove that every complex valued metric space is metrizable and hence complex valued metric spaces are not a real generalization of metric spaces. But the "equivalent" metric may not satisfy the same contractive condition as the complex valued metric. In fact, two equivalent metrics on X do not satisfy the same contractive conditions, in general. Even then, most fixed point results obtained using a complex valued metric can be considered as consequences of corresponding results obtained by an appropriately defined equivalent metric.

We first review some notions and notations given in [5] and [14].

Definition 1.1: Let $\mathbb{C}$ be the set of complex numbers and let $z, w \in \mathbb{C}$.

Define a partial order $\leq$ on $\mathbb{C}$ by $z \leq w$ if and only if $\mathrm{Re}(z) \leq \mathrm{Re}(w)$ and $\mathrm{Im}(z) \leq \mathrm{Im}(w)$.

Notation: We write $z < w$ if $z \leq w$ and $z \neq w$. Similarly we write $z \ll w$ if $\mathrm{Re}(z)<\mathrm{Re}(w)$ and $\mathrm{Im}(z) < \mathrm{Im}(w)$.

Definition 1.2: Let X be a non-empty set. Suppose that the mapping D: $X \times X \to \mathbb{C}$ satisfies:

(a) $0 \leq D(x, y)$ for all $x, y \in X$ and $D(x, y)=0$ if and only if $x = y$;
(b) $D(x, y) = D(y, x)$ for all $x, y \in X$;
(c) $D(x, y) \leq D(x, z) + D(z, y)$ for all $x, y, z \in X$.

Then D is called a *complex valued metric* on X and (X, D) is called a *complex valued metric space* (briefly CVM space).

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A complex valued metric induces a Hausdorff topology on X as follows. Let $0 \ll r \in \mathbb{C}$.

Define $B(x, r) := \{y \in X: D(x, y) \ll r\}$

The family $\mathcal{F} = \{B(x, r): x \in X, 0 \ll r \in \mathbb{C}\}$ is a sub-basis for a topology τ on X and this topology is Hausdorff. We can then define other topological notions (like open set, closed set, interior point, limit point, etc) on X in the usual manner. See [5] - [7].

Definition 1.3: Let $\{x_n\}$ be a sequence in a complex valued metric space (X, *D*) and $x \in X$. We say that

a) $\{x_n\}$ *converges* to x (or equivalently x is the *limit* of x_n), written $x_n \to x$, if for every $r \in \mathbb{C}$ with $0 \ll r$ there is a positive integer N such that for all n>N, $D(x_n, x) \ll r$.

b) $\{x_n\}$ is a *Cauchy sequence* if for every $r \in \mathbb{C}$ with $0 \ll r$ there is a positive integer N such that for all m, n>N, $D(x_n, x_m) \ll r$.

c) (X, *D*) is a *complete complex valued metric space* if every cauchy sequence in (X, D) is convergent to an element in (X, *D*).

For the proofs of the following lemmas we refer to [5].

Lemma 1.1: Let (X, *D*) be a complex valued metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x if and only if and only if $|D(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.2: Let (X, *D*) be a complex valued metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|D(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

The following notions are introduced by Jungck and Rhoades [14].

Definition 1.4: Let f and g be self-maps on a set X.

i) If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g; and w is called a *point of coincidence* of f and g.

ii) The function f and g are said to be *weakly compatible* if they commute at all of their coincidence points.

The following fixed point result is proved in [5]. In [13] the same theorem is proved in the setting of (real valued) metric spaces.

Theorem 1.1: *Let (X, D) be a complex valued metric space and let the mappings S, T: X→X satisfy:*

$$D(Sx, Ty) \leq \lambda D(x,y) + \frac{\mu D(x,Sx)D(y,Ty)}{1+D(x,y)}$$

for all x, y ϵX *where* λ, μ *are nonnegative reals with* $\lambda + \mu < 1$. *Then S and T have a unique common fixed point.*

In [6], the following result is proved. In [12] similar result is proved in the setting of metric spaces.

Theorem 1.2: *Let (X, D) be a complex valued metric space and let f, g, S and T are four self-maps of X such that* $T(X) \subseteq f(X)$ *and* $S(X) \subseteq g(X)$. *Suppose there exist nonnegative real numbers a, b, and c with* $a + 2b + 2c < 1$ *such that*

$$D(Sx, Ty) \leq aD(fx, gy) + b\,[D(fx, Sx) + D(gy, Ty)] + c[D(fx, Ty) + D(gy, Sx)]$$

Suppose that the pairs {f, S} and {g, T} are weakly compatible. Then f, g, S and T have a unique common fixed point.

In [7], the following result is proved. In [10] and [17], its real valued counterpart is proved.

Theorem 1.3: *Let (X, D) be a complex valued metric space and let mappings S, T: X→X satisfy*

$$D(Sx, Ty) \leq \frac{a[D(x,Sx)D(x,Ty)+D(y,Ty)D(y,Sx)]}{D(x,Ty)+D(y,Sx)}$$

for all x, y ϵX *where* $0 \leq a < 1$. *Then S and T have a unique common fixed point.*

2. MAIN RESULTS

We begin by showing that every complex valued metric space (X, *D*) is metrizable.

Theorem 2.1: *For every complex valued metric* $D: X\times X\rightarrow \mathbb{C}$*, there is a real valued metric* $d: X\times X\rightarrow \mathbb{R}$ *such that D and d induce the same topology on X.*

Proof: Define $d(x, y) := \text{Max}\,\{Re\,(D(x, y), Im(D(x, y))\}$. First we show that *d* is a metric. Clearly $d(x, y) \geq 0$ for all x, y $\in$X and $d(x, y) = 0$ if and only if x=y. Also, $d(x, y) = d(y, x)$ for all x, y $\in$X.

Now, let x, y, z $\in$X. Let $D(x, y) := (a, b)$, $D(y, z) := (c, d)$ and $D(x, z) := (e, f)$. Since *D* is a complex valued metric, so we have $(e, f) \leq (a + c, b + d)$. This implies

$d\,(x, z) = max\,\{e, f\} \leq max\,\{a+ c, b + d\} \leq max\,\{a, b\} + max\,\{c, d\} = d(x, y) + d(y, z).$

Next we show that *d* and *D* induce the same topology on X. To this end it suffices to show that every open ball in (X, d) is an open set in (X, *D*) and every open ball in (X, *D*) is an open set in (X, *d*). Let B(x, c) be an open ball in (X, *D*) where $0 \ll c= (c_1, c_2) \in \mathbb{C}$. Let z $\in$B(x, c). Then $D(x, z) := (a_1, a_2) \ll (c_1, c_2)$. If we choose $r \in \mathbb{R}$ such that $r \leq min\,\{c_1- a_1, c_2 - a_2\}$, then y $\in$ B (z, r) implies $D(x, y) \ll c$. Therefore, we get $B\,(z, r) \subseteq B(x, c)$. Thus B(x, c) is an open set in (X, d). By same argument it can be shown that any open ball B(x, r) in (X, d) is open in (X, D).

Thus D and d induce the same topology on X.

Remark 2.1: In fact the real valued metric defined by $d(x, y) := |\,D(x, y)|$ also induces the same topology on X as *D*.

In line with the definition of metric equivalence, we define equivalence between a complex valued metric and a metric as follows.

Definition 2.1: A complex valued metric D and a metric d on X are said to be equivalent if they give rise to the same topology on X.

Theorem 2.1 states that for every complex valued metric D on X there exists a metric d on X which is equivalent to D.

In metric fixed point theory, it is a well known fact that the classes of maps contractive with respect to two metrics d and d' are not the same, in general, even for equivalent metrics d and d'. Thus a mapping T that satisfies a certain contractive type condition with respect to D may not satisfy the same contractive type condition with respect to d and vice versa. However, for d and D, this happens rarely as the following theorem depicts.

Theorem 2.2*: Let (X, D) be a complex valued metric space and d be the metric defined by:*

$d(x, y) = max\,\{Re\,(D(x, y), Im(D(x,y)))\}.$

Let S, T, f, g: $X\rightarrow X$ *be mappings and* $\alpha, \beta, \mu, \lambda, a, b, c, e$ *are nonnegative reals with* $\mu+\lambda < 1$, $\alpha, \beta \in [0, 1)$, $e \in [0, ½)$ *and* $a + 2b + 2c < 1$.

Then

i) If $D\,(Tx, Ty) \leq \alpha D(x, y)$, then $d\,(Tx, Ty) \leq \alpha d(x, y)$
ii) If $D\,(Tx, Ty) \leq e[D(Tx, x) + D(Ty, y)]$, then $d\,(Tx, Ty) \leq e[d(Tx, x) + d(Ty, y)]$,
iii) If $D\,(Tx, Ty) \leq e[D(Tx, y) + D(Ty, x)]$, then $d\,(Tx, Ty) \leq e[d(Tx, y) + d(Ty, x)]$,
iv) If $D\,(Tx, Ty) \leq \alpha D(Tx, y) + \beta D(Ty, x)]$, then $d\,(Tx, Ty) \leq \alpha d(Tx, y) + \beta d(Ty, x)$,
v) If $D(Sx,Ty) \leq aD(fx,gy) + b\,[D(fx,Sx) + D(gy,Ty] + c[D(fx,Ty) + D(gy,Sx]$, then $d(Sx,Ty) \leq ad(fx,gy) + b\,[d(fx,Sx) + d(gy,Ty] + c[d(fx,Ty) + d(gy,Sx]$

Proof: We prove only (v). The same argument may be used to prove the remaining conclusions of the theorem.

Let $D\,(Sx, Ty) = (p,q)$, $D(fx, gy) =(r, s)$, $D(fx, Sx)= (t, u)$, $D(gy, Ty)=(v, w)$, $D(fx, Ty) = (i, j)$, and $D(gy, Sx) = (k, l)$.

By assumption, $(p,q) \leq a(r, s) + b[(t, u) +(v, w)] + c[(i, j) + (k, l)]$. This implies

$\max\,\{p, q\} \leq a\,max\{r, s\} + b\,[max\,\{t, u\} + max\{v, w\}] + c[max\{i, j\} + max\{k, l\}].$

Hence $d(Sx,Ty) \leq ad(fx,gy) + b\,[d(fx,Sx) + d(gy,Ty] + c[d(fx,Ty) + d(gy,Sx]$.

In view of Theorem 2.2(v) we can conclude that Theorem1.2 is a consequence of the corresponding result in [12]. We indicate in the form of open questions whether or not Theorems 1.1 and 1.3 can also be viewed as corollaries of the corresponding results in metric spaces in the same way.

The following example is given in [5] to illustrate the motivation behind the introduction of complex valued metric spaces. We remark that this example cannot justify the need to introduce complex valued metric spaces because this problem can still be solved in the setting of metric spaces (i.e., by using a real valued metric d on X, different from d_u, under which T becomes a contraction).

Example: Let $X_1 := \{(x, 0): 0 \le x \le 1\}$, $X_2 := \{(0, x): 0 \le x \le 1\}$, $X := X_1 \cup X_2$

Define $D: X\times X \to \mathbb{C}$ by

$$D(x, y) := \begin{cases} D((x,0),(y,0)) = (\frac{2}{3}|x-y|, \frac{1}{2}|x-y|) \\ D((0,x),(0,y)) = (\frac{1}{2}|x-y|, \frac{1}{3}|x-y|) \\ D((x,0),(0,y)) = (\frac{2}{3}x + \frac{1}{2}y, \frac{1}{2}x + \frac{1}{3}y) \end{cases}$$

The space (X, D) is a complete complex valued metric space.

Let $T: X \to X$ defined by

$$T(x, y) := \begin{cases} (0, x) \, if \ (x,y) \in X_1 \\ \left(\frac{1}{2}y, 0\right) if \ (x,y) \in X_2 \end{cases}$$

If d_u is the usual metric on X, then T is not contractive as $d_u(T(x,0), T(y,0)) = |x-y| = d(x, y)$.

In [5] the authors claim that Banach Contraction Principle is not valid for this mapping as T is not a contraction with respect to the usual (Euclidean) metric d_u and they have shown that it is a contraction with respect to the complex valued metric D. They used this example as an illustration of the necessity to introduce complex valued metric spaces. The fact that T is not a contraction mapping with respect to the usual metric does not mean that it is not a contraction mapping with respect to all real valued metrics on X. One can find an appropriate real valued metric d on X such that T is a contraction mapping in (X, d). One such d is d(x, y): = max (Re (D(x, y), Im D(x, y))).

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Source of support: Nil, Conflict of interest: None Declared