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ON SOME COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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ABSTRACT

This paper present some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces under various conditions.

Keywords: *Fixed point, occasionally weakly compatible mappings, fuzzy metric space.*

1. INTRODUCTION

The concept of fuzzy sets introduced by Prof. Lofti Zadeh [11] in 1965 at the University of California plays an important role in topology and analysis. Since then, there were many authors who studied the fuzzy sets with applications. Espectially, Kromosil and J. Michalek [13] put forward a new concept of fuzzy metric spaces. A. George and P. Veermani [2] revised the notion of fuzzy metric spaces with the help of continuous t-norm in 1994. As a result of many fixed point theorem for various forms of mapping are obtained in fuzzy metric spaces. Recently, many researchers have proved common fixed point theorems involving fuzzy sets. Pant [15] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. Vasuki [16] and B. Singh and et al. [4] also introduced some fixed point theorems in fuzzy metric spaces for R-weakly commuting and compatible mappings respectively.

Balasubramaniam et al. [14] proved the open problem of Rhoades [3] on the existence of a contractive definition which generates a fixed point but does not force the mapping to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [17] proved an analogue of the result given by Balasubramaniam et al. [14]. Recent work on fixed point theorems in fuzzy metric space can be viewed in references [6, 18, 19, 20].

The purpose of this paper is to prove some common fixed point theorem in fuzzy metric space for more general commutative condition i.e., occasionally weakly compatible mappings.

2. PRELIMINARIES AND NOTATIONS

We recall the definitions and results that will be needed in the sequel.

Definition 2.1: A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2: A triangular norm $*$ (shortly t− norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d $\in$ [0, 1] the following conditions are satisfied:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a, \ \forall \ a \in [0,1]$;
(d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$.

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Example 2.3: Two typical examples of continuous t-norm are

(a) $a * b = ab, \quad and$
(b) $a * b = min(a, b)$

Definition 2.4 [2]**:** A 3-tuple (X, M, ∗) is said to be a fuzzy metric space, if X is an arbitrary set, ∗ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all x, y, z ∈ X and s, t > 0:

$(f1)$ $M(x, y, 0) > 0;$
$(f2)$ $M(x, y, t) = 1, \forall\, t > 0, if\ and\ only\ if\ x = y;$
$(f3)$ $M(x, y, t) = M(y, x, t);$
$(f4)$ $M(x, y, t) * M(y, z, s) \le M(x, z, t + s);$
$(f5)$ $M(x, y, *): (0, \infty) \to (0,1] is\ \ continuous,$

where $M(x, y, t)$ denote the degree of nearness between x and y with respect to t. Then M is called a fuzzy metric on X.

Example 2.5 [2]**:** Let (X, d) be a metric space. Define $a * b = ab, \forall\, a, b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t+d(x,y)}$$

Then $(X, M_d,*)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 2.6 [11]: Let $(X, M,*)$ be a fuzzy metric space. Then

(a) $a\ sequence\ \{x_n\}\ in\ X\ is\ said\ to\ converges\ to\ x\ in\ X\ if\ for\ each\ \varepsilon > 0\ and\ each\ t > 0, there\ exists\ n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon\ \forall\, n \ge n_0$.

(b) a sequence $\{x_n\}\ in\ X\ is\ said\ to\ Cauchy\ if\ for\ each\ \varepsilon > 0\ and\ each\ t > 0, there\ exists\ n_0 \in N\ such\ that$

$M(x_n, x_m, t) > 1 - \varepsilon\ \forall\, n, m \ge n_0$.

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.7: Two mappings $f\ and\ g$ of a fuzzy metric space $(X, M,*)$ into itself are said to be weakly commuting if

$$M(fgx, gfx, t) \ge M(fx, gx, t),\ \forall\, x \in X\ and\ t > 0.$$

Definition 2.8: Two mappings $f\ and\ g$ of a fuzzy metric space $(X, M,*)$ into itself are R- weakly commuting provided there exists some positive real number R such that

$$M(fgx, gfx, t) \ge M\left(fx, gx, \frac{t}{R}\right), \forall\, x \in X\ and\ t > 0.$$

Definition 2.9 [8]**:** Two self maps $f\ and\ g$ of a fuzzy metric space $(X, M,*)$ are called re-ciprocally continuous on X if $\lim_{n\to\infty} fgx_n = fx\ and\ \lim_{n\to\infty} gfx_n = gx$, whenever $\{x_n\}$ is a sequence $in\ X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x\ for\ some\ x\ in\ X.$$

Proposition 1: In a fuzzy metric space (X, M, *), if $a * a \ge a\ for\ a \in [0,1]\ then\ a * b = \min\{a, b\}\ for\ all\ a, b \in [0,1]$.

Definition 2.10: Two self mappings $f\ and\ g$ of a fuzzy metric space (X, M, *) are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x\ for\ some\ x\ in\ X.$$

Definition 2.11 [9]**:** An element x ∈ X is called a common fixed point of the mappings

$$F: X \times X \to X\ and\ g: X \to X\ \ if\ x = g(x) = F(x, x).$$

Definition 2.12 [5]**:** Two self maps A and B of a fuzzy metric space $(\mathrm{X}, \mathrm{M}, *)$ are called weak-compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $\mathrm{A}x = \mathrm{B}x$ for some $x \in \mathrm{X}$ then $\mathrm{AB}x = \mathrm{BA}x$.

Definition 2.13 [8]: A pair of maps $S\ and\ T$ is called weakly compatible pair if they commute at coincidence points.

The concept occasionally weakly compatible is introduced by M. Al-Thagafi and Naseer Shahzad [1]. It is stated as follows.

Definition 2.14: Two self maps $f\ and\ g\ of\ a\ set\ X$ are occasionally weakly compatible (owc) if there is a point $x\ in\ X$ which is a coincidence point of $f\ and\ g\ at\ which\ f\ and\ g\ commute.$

A. Al-Thagafi and Naseer Shahzad [1] shown that occasionally weakly is weakly compatible but converse is not true.

Example 2.15: Let R is the usual metric space. Define $f, g: R \to R$ be $f(x) = 3x\ and\ g(x) = x^2, \forall\ x \in R.$

$Then\ fx = gx, for\ x = 0{,}3\ but\ f\ g(0) = g\ f(0), and\ f\ g(3) \neq T\ g(3),\ f\ and\ g$ are occasionally weakly compatible self maps but not weakly compatible.

Example 2.16: Let R is the usual metric space. Define $f, g: R \to R$ be $f(x) = 9x\ and\ g(x) = x^3, \forall\ x \in R.$ Then $fx = gx, for\ x = 0{,}3\ but\ fg(0) = gf(0), and\ gf(3) \neq fg(3),$ because

$fg(3) = f(27) = 9 \times 27 = 243\ and\ gf(3) = g(3 \times 9) = g(27) = (27)^3 = 19683. We\ see\ that\ gf(3) \neq fg(3).$

$Thus\ f\ and\ g$ are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.17 [8]**:** Let X be a set, $f,\ g$ owc self maps of X. If $f\ and\ g$ have a unique point of coincidence,

$w = fx = gx,$ then w is the unique common fixed point of $f\ and\ g$.

The following theorem was proved by Balasubramaniam et al. [14]:

Theorem 1 [16]**:** Let $(X, M, *)$ $be\ a\ complete\ fuzzy\ metric\ space\ and\ let\ f\ and\ g\ be\ R\ weakly\ commuting\ self\ mappings\ of\ X\ satisfying\ the\ conditions$

$$M(fx, fy, t) \geq r\big(M(gx, gy, t)\big),$$

$where\ r: [0,1] \to [0,1] is\ a\ continuous\ function\ such\ that r(t) > t\ \ for\ each 0 \leq t \leq 1\ and\ \ \ r(t) = 1.$

$The\ sequences\ \{x_n\}\ and\ \{y_n\}\ in\ X\ are\ such\ that x_n \to x,\ \ y_n \to y,\ \ t > 0\ implies\ \ M(x_n, y_n, t) \to M(x, y, t).$ If the range of g contains the range of f and either f or g is continuous, then f and g have a unique common fixed point.

3. MAIN RESULTS

Theorem 3.1: Let $(X, M, *)$ be a complete fuzzy metric space and let $A, B, S\ and\ T$ be self-mappings of X. Let the pairs $(A, S) and\ (B, T)$ are owc. If there exists a point $q \in (0,1),\ \ \forall\ \ \ x, y \in X\ and\ t > 0.\ Such\ that$

$$M(Ax, By, qt) \geq \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), [M(Ax, Ty, t) + M(By, Sx, t)]\} \quad (3.1)$$

then there exists a unique point of $w \in\ T, such\ that\ Aw = Sw = w\ and\ a\ unique\ point\ z \in X, such\ that\ Bz = Tz = z. Moreover, z = w, so\ that\ there\ is\ a\ unique\ common\ fixed\ \ point\ of\ A, B, S\ and\ T.$

Proof: Let the pairs $(A, S)\ and\ (B, T)$ be owc, so there are points $x, y \in X$ such that $Ax = Sx\ and\ By = Ty,$ we claim that $Ax = By.\ If\ not,$ by inequality (3.1) we have

$$M(Ax, By, qt) \geq \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), [M(Ax, Ty, t) + M(By, Sx, t)]\}$$

$$= \min\{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), [M(Ax, By, t) + M(By, Ax, t)]\}$$

$$= M(Ax, By, t).$$

Thus we have $Ax = By, i.e.\ Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3.1) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S.

Similarly there is a unique point $z \in X$ such that. $z = Bz = Tz$. Using lemma 2.17, we get w is the only common fixed point of $A,$ and S.

Assume that $w \neq z$. We $have$

$$M(w,z,qt) = M(Aw,Bw,qt) \geq \min\{M(Sw,Tz,t), M(Sw,Az,t), M(Bz,Tz,t), [M(Aw,Tz,t) + M(Bz,Sw,t)]\}$$

$$= \min\{M(w,z,t), M(w,z,t), M(z,z,t), [M(w,z,t) + M(z,w,t)]\}$$

$$= M(w,z,t).$$

Therefore, we have $z = w$ by $Lemma$ 2.17 and z is a $common$ $fixed$ $point$ of A, B, S and T.

The uniqueness of the fixed point holds from (3.1).

Theorem 3.2 Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs (A, S) and (B, T) are owc. If there exists a point $q \in (0,1)$, $\forall$ $x, y \in X$. $Such$ $that$

$$M(Ax,By,qt) \geq \delta(\min\{M(Sx,Ty,t), M(Sx,Ax,t), M(By,Ty,t), [M(Ax,Ty,t) + M(By,Sx,t)]\}) \qquad (3.2)$$

$such$ $that$ $\delta(t) > t$ for all $0 < t < 1$, and $\delta: [0,1] \to [0,1]$, $then$ $there$ $exists$ a $unique$ $common$ $fixed$ $point$ of A, B, S and T.

Proof: The proof follows from Theorem 3.1.

Theorem 3.3 Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs (A, S) and (B, T) are owc. If there exists a point $q \in (0,1)$, $such$ $that$

$$M(Ax,By,qt) \geq \delta(\, M(Sx,Ty,t), M(Sx,Ax,t), M(By,Ty,t), [M(Ax,Ty,t) + M(By,Sx,t)]\} \qquad (3.3)$$

$\forall\ x, y \in X$ and $\delta: [0,1]^5 \to [0,1]$ $such$ $that$ $\delta(t,1,1,t,t) > t$ for all $0 < t < 1$, $then$ $there$ $exists$ a $unique$ $common$ $fixed$ $point$ of A, B, S and T.

Proof: Let the pairs (A, S) and (B, T) be owc, there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$, we claim that $Ax = By$. If not, by inequality (3.3) we have

$$M(Ax,By,qt) \geq \delta\{\, M(Sx,Ty,t), M(Sx,Ax,t), M(By,Ty,t), [M(Ax,Ty,t) + M(By,Sx,t)]\}$$

$$= \delta\{M(Ax,By,t), M(Ax,Ax,t), M(By,By,t), [M(Ax,By,t) + M(By,Ax,t)]\}$$

$$= \delta\{M(Ax,By,t), 1, 1, M(Ax,By,t), M(By,Ax,t)\}$$

$$> M(Ax,By,t).$$

A contradiction, therefore $Ax = By, i.e.,$ $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3.3) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Tx$ is the unique point of coincidence of A and T. By Lemma 2.17 w is a unique common fixed point of $A,$ and S. Similarly there is a unique point $z \in X$ such that. $z = Bz = Tz$. Thus z is a common fixed point of A, B, S and T, the uniqueness of the fixed point holds from (3.3).

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