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Numerical solution of Fuzzy Hybrid Differential Equation by Fifth order Runge Kutta Method

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ABSTRACT

In this paper the Runge kutta method of order five is considered for solving 'fuzzy hybrid differential equations' based on Seikkala derivative. We state a convergence result and give a numerical example to illustrate the theory.

Keywords: *Hybrid systems; Fuzzy differential equations; Runge–Kutta method.*

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1. INTRODUCTION

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models. The concept of fuzzy derivative was first introduced by Chang and Zadeh in [3]. It was followed up by Dubois and Prade in [4], who defined and used the extension principle. Other methods have been discussed by Puri and Ralescu in [16] and Goetshel and Voxman in [7]. The initial value problem for fuzzy differential equation (FIVP) has been studied by Kaleva in [14, 15] and by Seikkala in [21]. Pederson and Sambandham [14, 15] have investigated the numerical solution of hybrid fuzzy differential equations by using Runge Kutta method and Euler method and also they have considered the numerical solution of hybrid fuzzy differential equations by using the characterization theorem for the improved Euler's method.

The Hybrid Fuzzy Differential equations is a natural way to model control systems with embedded systems with embedded uncertainty that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics.

In this article we develop numerical methods for addressing hybrid fuzzy differential equations by an application of the Runge–Kutta method of order 5 using the Seikkala derivative which is more accurate than the one in [15]. In Section 2 we list some basic definitions for fuzzy valued functions, fuzzy differential equations and Runge kutta formula of order 5. Section 3 reviews hybrid fuzzy differential systems. Section 4 contains the Runge–Kutta method for approaching hybrid fuzzy differential equations and a convergence theorem. Section 5 contains a numerical example to illustrate the theory.

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2. PRELIMINARIES

We recall some definitions which are used throughout this paper. By $\mathcal{R}$ denote the set of all real numbers.

2.1 Definitions and Notations

A fuzzy number is a mapping $u: \mathcal{R} \to [0,1]$ with the following properties:

(a) u is upper semicontinous,

(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathcal{R}, \lambda \in [0,1]$,

(c) u is normal, i.e.,$\exists x_0 \in \mathcal{R}$ for which $u(x_0) = 1$,

(d) Supp u = $\{\mathcal{R}/u(x) > 0\}$ is the support of u, and its closure cl (supp u) is compact.

Let $\mathbb{E}$ be the set of all fuzzy number on r. The r-level set of a fuzzy number $u \in \mathbb{E}, 0 \leq r \leq 1$, denoted by$[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in \mathcal{R}/u(x) \geq r\}, & 0 < r \leq 1 \\ cl(supp\ u), & r = 0 \end{cases}$$

It is clear that the r- level set of a fuzzy number is a closed and bounded interval$[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand end point of $[u]_r$ and $\overline{u}(r)$ denotes the right- hand side end point of $[u]_r$.since each $y \in \mathcal{R}$ can be regarded as a fuzzy number $\tilde{y}$ is defined by

$$\tilde{y}(t) = \begin{cases} 1, t = y \\ 0, t \neq y \end{cases}$$

Remark 2.1: Let X be the Cartesian product of universes $X = X_1 \times \ldots \times X_n$, and $A_1 \ldots A_n$ be n fuzzy numbers in $X_1 \times \ldots \times X_n$ respectively.f is a mapping from X to a universe Y, $y = f(x_1, x_2, \ldots x_n)$.Then the extension principle allows us to define a fuzzy set B in Y by $B = \{y, u(y)/y = f(x_1 \ldots x_n). (x_1 \ldots x_n) \in X\}$, where

$$u_B(y) = \begin{cases} sup_{(x_1 \ldots x_n) \in f^{-1}(y)} min\{u_{A1}(x_1), \ldots \ldots . u_{An}(x_n)\} & if\ \ f^{-1}(y) \neq 0, \\ 0 & if\ otherwise. \end{cases}$$

where f^{-1} is the inverse of f. For n=1, the extension principle, of course, reduces to

$$B = \{y, u_B(y)/y = f(x), x \in X\}$$

where
$$u_B(y) = \begin{cases} sup_{x \in f^{-1}(y)}\, u_A(x)\ , & f^{-1}(y) \neq 0, \\ 0 & otherwise. \end{cases}$$

According to Zadeh's extension principle, operation of addition on $\mathbb{E}$ is defined by

$$(u \oplus v)(x) = \sup_{y \in \mathcal{R}} min\{u(y), v(x-y)\}, \quad x \in \mathcal{R}$$

and scalar multiplication of a fuzzy number is given by

$$(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ \hat{0}, & k = 0, \end{cases}$$

where $\hat{0} \in \mathbb{E}$. The Hausdorff distance between fuzzy numbers given by $D: \mathbb{E} \times \mathbb{E} \to \mathcal{R}_+ \cup \{0\}$,

$$D(u,v) = \sup_{r \in [0,1]} max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\},$$

It is easy to see that D is a metric in $\mathbb{E}$ and has the following properties

1) $(D(u \oplus w, v \oplus w) = D(u,v), \forall u, v, w \in \mathbb{E},)$

(i) $D(k \odot u, k \odot v) = |k|\, D(u,v), \forall\ k \in \mathcal{R}, u,v \in \mathbb{E}$,

(ii) $D(k \odot u, k \odot v) = D(u,v), \forall u, v, w \in \mathbb{E},)$

(iii) $(D, \mathbb{E})$ is a complete metric space.

Next consider the initial value problem (IVP)

$$\begin{cases} y'(t) = f(t, y(t); & a \le t \le b, \\ y(a) = \alpha \end{cases} \tag{2.1}$$

Where f is continuous mapping from $\mathcal{R}_+ \times \mathcal{R}$ into $\mathcal{R}$ and $\alpha \in \mathbb{E}$ with r level sets

$[\alpha]_r = [\underline{\alpha}(r), \overline{\alpha}(r)], r \in (0,1]$.

The extension principle of Zadeh leads to the following definition of $f(t, y)$ when $y = y(t)$ is a fuzzy number

$f(t, y)(s) = sup\{y(t)\backslash s = f(t, r)\},\ s \in \mathcal{R}$.

It follows that $[f(t, y)]_r = \left[\underline{f}(t, y, r), \overline{f}(t, y, r)\right]\ r \in (0,1]$,

where $\underline{f}(t, y, r) = min\left\{f(t, u)\backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$,

$\overline{f}(t, y, r) = max\left\{f(t, u)\backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$.

Theorem 2.1: Let f satisfy $\left|f(t, v) - f(t, \overline{v)}\right| \le g(t, |v - \overline{v}|), t \ge 0,\ v, \overline{v} \in \mathcal{R}$,

where $g{:}\mathcal{R}_+ \times \mathcal{R}_+$ is a continuous mapping such that $r \to g(t, r)$ is non decreasing and the initial value problem

$$u'(t) = g\big(t, u(t)\big),\ u(0) = u_0, \tag{2.2}$$

Has a solution on $\mathcal{R}_+$ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (2.2) for $u_0 = 0$.Then the fuzzy initial value problem (2.1) has a unique solution.

2.1 Runge kutta method

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); & a \le t \le b, \\ y(a) = \alpha \end{cases} \tag{2.3}$$

The basis of Runge Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i, \tag{2.4}$$

where $w_i's$ are constants and for $i = 1,2,3 \ldots\ldots m$,

$$k_i = h\, f\big(t_n + c_i h,\quad y_n + h \textstyle\sum_{j=1}^{i-1} a_{ij} k_j\big). \tag{2.5}$$

For determination of c_i's & a_{ij}'s we compare (2.4) with the Taylor's series expansion about t_n.We get

$c_1 = 0, \quad c_2 = \frac{1}{4}, \; c_3 = \frac{1}{4}, \quad c_4 = \frac{1}{2}, c_5 = \frac{3}{4}, c_6 = 1, a_{21} = \frac{1}{4}, a_{31} = \frac{1}{8}, a_{32} = \frac{1}{8}, a_{41} = -\frac{1}{2},$

$a_{42} = 0, a_{43} = 1, a_{51} = \frac{3}{16}, a_{52} = 0, a_{53} = 0, a_{54} = \frac{9}{16}, a_{61} = -\frac{3}{7}, a_{62} = \frac{2}{7}, a_{63} = \frac{12}{7}, a_{64} = -\frac{12}{7}, a_{65} = \frac{8}{7}.$

Where *m* =5. Hence the fifth order Runge Kutta method is given by

$$y_{i+1} = y_i + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{4}h, y_i + \frac{1}{4}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{1}{4}h, y_i + \frac{1}{8}k_1h + \frac{1}{8}k_2h\right)$$

$$k_4 = f\left(x_i + \frac{1}{2}h, y_i - \frac{1}{2}k_2h + k_3h\right)$$

$$k_5 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{16}k_1h + \frac{9}{16}k_4h\right)$$

$$k_6 = f\left(x_i + h, y_i - \frac{3}{7}k_1h + \frac{2}{7}k_2h + \frac{12}{7}k_3h - \frac{12}{7}k_4h + \frac{8}{7}k_5h\right)$$

Lemma 2.2: If the sequence of non negative numbers $\{W_n\}$ satisfy $|W_{n+1}| \le A|W_n| + B, 0 \le n \le N-1,$ for the given positive Constants A and B, then $|W_n| \le A^n|W_0| + B\frac{A^n-1}{A-1}, 0 \le n \le N.$

Lemma 2.3: If the sequence of numbers $\{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N$ satisfy

$|W_{n+1}| \le |W_n| + A\max\{|W_n|\,|V_n|\} + B, |V_{n+1}| \le |V_n| + A\max\{|W_n|\,|V_n|\} + B$

for the given positive constants A and B, then denoting

$U_n = |W_n| + |V_n|, 0 \le n \le N,$

We have $U_n \le \overline{A^n}U_0 + \bar{B}\frac{\overline{A^n}-1}{\bar{A}-1}, 0 \le n \le N,$ where $\bar{A} = 1 + 2A, \bar{B} = 2B.$

Lemma 2.4: Let $F(t,u,v)$ & $G(t,u,v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F.Then for arbitrarily fixed $r, 0 \le r \le 1,$ $D\left(y(t_{n+1}), y^{(0)}(t_{n+1})\right) \le h^2L(1+2C),$

Where L is a bound of partial derivatives of F and G, and $C = \max\left\{\left|G[t_n, \underline{y}(t_N;r), \overline{y}(t_{N-1};r)]\right| r \in [0,1]\right\} < \infty.$

Theorem 2.5: Let $F(t,u,v)$ & $G(t,u,v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F.Then for arbitrarily fixed $r, 0 \le r \le 1,$ the numerical solutions of $\underline{y}(t_{n+1};r)$ and $\overline{y}(t_{n+1};r)$ converge to the exact solutions $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ uniformly in t.

Theorem 2.6: Let $F(t,u,v)$ & $G(t,u,v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1.$ Then for arbitrarily fixed $r, 0 \le r \le 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n;r)$ and $\overline{y}^{(j)}(t_n;r)$ converge to the numerical solution $\underline{y}(t_n;r)$ and $\overline{y}(t_n;r)$ in $t_0 \le t_n \le t_N,$ when $j \to \infty.$

3. THE HYBRID FUZZY DIFFERENTIAL SYSTEM

Consider the hybrid fuzzy differential system

$$\begin{cases} x'(t) = f\big(t, x(t), \lambda_k(x_k)\big),\ t \in [t_k, t_{k+1}], \\ x(t_k) = x_k \end{cases} \tag{3.1}$$

Where $'$ denotes the seikkala derivative,$0 \le t_0 < t_1 < \cdots < t_k < \cdots,\quad t_k \to \infty, f \in C[\mathcal{R}^+ \times \mathbb{E} \times \mathbb{E}, \mathbb{E}], \lambda_k \in C[\mathbb{E}, \mathbb{E}]$.)

To be specific the system will be as follows

$$x'(t) = \begin{cases} x_0{}'(t) = f\big(t, x_0(t), \lambda_0(x_0)\big),\ \ x(t_0) = x_0,\quad t \in [t_0, t_1], \\ x_1{}'(t) = f\big(t, x_1(t), \lambda_1(x_1)\big),\ \ x(t_1) = x_1,\quad t \in [t_1, t_2], \\ \quad . \\ \quad . \\ \quad . \\ x_k{}'(t) = f\big(t, x_k(t), \lambda_k(x_k)\big),\ \ x(t_k) = x_k,\quad t \in [t_k, t_{k+1}], \\ \quad . \\ \quad . \\ \quad . \end{cases}$$

With respect to the solution of (3.1), we determine the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), \quad t \in [t_0, t_1] \\ x_1(t), \quad t \in [t_1, t_2] \\ \quad . \\ \quad . \\ \quad . \\ x_k(t), \quad t \in [t_k, t_{k+1}] \\ \quad . \\ \quad . \\ \quad . \end{cases}$$

We note that the solutions of (3.1) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $t_k \in \mathbb{E}$ and $k = 0,1,2 \ldots$

Therefore we may replace (3.1) by an equivalent system

$$\begin{cases} \underline{x}'(t) = \underline{f}\big(t, x, \lambda_k(x_k)\big) \equiv F_k\big(t, \underline{x}, \overline{x}\big), \quad \underline{x}(t_k) = \underline{x}_k \\ \overline{x}'(t) = \overline{f}(t, x, \lambda_k(x_k) \equiv G_k\big(t, \underline{x}, \overline{x}\big), \quad \overline{x}(t_k) = \overline{x}_k \end{cases} \tag{3.2}$$

Which posseses a unique solution $\big(\underline{x}, \overline{x}\big)$ which is a fuzzy function.That is for each t, t he pair $\big[\underline{x}(t;r), \overline{x}(t;r)\big]$ is a fuzzy number, where $\underline{x}(t;r), \overline{x}(t;r)$ are respectively the solutions of the parametric form given by

$$\begin{cases} \underline{x}'(t) = F_k\big(t, \underline{x}(t;r), \overline{x}(t;r)\big), \quad \underline{x}(t_k; r) = \underline{x}_k(r) \\ \overline{x}'(t) = G_k\big(t, \underline{x}(t;r), \overline{x}(t;r)\big), \quad \overline{x}(t_k; r) = \overline{x}_k\ (r) \end{cases} \tag{3.3}$$

for $r \in [0,1]$.

4. THE RUNGE–KUTTA METHOD

In this section , for a hybrid fuzzy differential equation (3.1) we develop a Runge kutta method of order five as in (2.4) and (2.5).We assume that the existence and the uniqueness of the solutions of (3.1) hold for each $[\mathrm{t_k}, \mathrm{t_{k+1}}]$.

For a fixed r, to integrate the system (3.3) in $[\mathrm{t_0}, \mathrm{t_1}], [\mathrm{t_1}, \mathrm{t_2}], \ldots, [\mathrm{t_k}, \mathrm{t_{k+1}}]$, ...,we replace each interval by a set of $\mathrm{N_k} + 1$ discrete equally spaced grid points at which the exact solution $x(t;r) = (\underline{x}(t;r), \overline{x}(t;r))$ is approximated by some

$(\underline{y_k}(t;r),\overline{y_k}(t;r))$.For the chosen grid points on $[t_k,t_{k+1}]$ $t_{k,n}=t_k+nh_k, h_k=\frac{t_{k+1}-t_k}{N_k}$,$0\le n\le N_k$,

Let $\left(\underline{Y_k}(t;r),\overline{Y_k}(t;r)\right)\equiv(\underline{x}(t;r),\overline{x}(t;r))$.$\left(\underline{Y_k}(t;r),\overline{Y_k}(t;r)\right)$ and $(\underline{y_k}(t;r),\overline{y_k}(t;r))$ may be denoted respectively by $\left(\underline{Y_{k,n}}(r),\overline{Y_{k,n}}(r)\right)$ and $\left(\underline{y_{k,n}}(r),\overline{y_{k,n}}(r)\right)$.We allow the N_k's to vary over the $[t_k,t_{k+1}]$'s so that the h_k's may be comparable.To develop the Runge kutta method of order five for (3.1),we define the above Runge kutta method of order 5

$$\underline{y_{k,n+1}}(r)-\underline{y_{k,n}}(r)=\sum_{i=1}^{6}h\,w_i\,\underline{k_i}\left(t_{k,n};y_{k,n}(r)\right)$$

$$\overline{y_{k,n+1}}(r)-\overline{y_{k,n}}(r)=\sum_{i=1}^{6}h\,w_i\,\overline{k_i}\left(t_{k,n};y_{k,n}(r)\right),$$

Where $w_{1,}w_2,w_3$ are constants and

$$\underline{k_1}\left(t_{k,n};y_{k,n}(r)\right)=min\left\{f\left(t_{k,n},u,\lambda_k(u_k)\right)\setminus u\epsilon\left[\underline{y_{k,n}}(r),\overline{y_{k,n}}(r)\right],u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\},$$

$$\overline{k_1}\left(t_{k,n};y_{k,n}(r)\right)=max\left\{f\left(t_{k,n},u,\lambda_k(u_k)\right)\setminus u\epsilon\left[\underline{y_{k,n}}(r),\overline{y_{k,n}}(r)\right],u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\},$$

$$\underline{k_2}\left(t_{k,n};y_{k,n}(r)\right)=min\left\{f\left(t_{k,n}+\frac{1}{4}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k1}}(t_{k,n},y_{k,n}(r)),\overline{z_{k1}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\overline{k_2}\left(t_{k,n};y_{k,n}(r)\right)=max\left\{f\left(t_{k,n}+\frac{1}{4}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k1}}(t_{k,n},y_{k,n}(r)),\overline{z_{k1}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\underline{k_3}\left(t_{k,n};y_{k,n}(r)\right)=min\left\{f\left(t_{k,n}+\frac{1}{4}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k2}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\overline{k_3}\left(t_{k,n};y_{k,n}(r)\right)=max\left\{f\left(t_{k,n}+\frac{1}{4}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k2}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\underline{k_4}\left(t_{k,n};y_{k,n}(r)\right)=min\left\{f\left(t_{k,n}+\frac{1}{2}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k3}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\overline{k_4}\left(t_{k,n};y_{k,n}(r)\right)=max\left\{f\left(t_{k,n}+\frac{1}{2}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k3}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\underline{k_5}\left(t_{k,n};y_{k,n}(r)\right)=min\left\{f\left(t_{k,n}+\frac{3}{4}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k4}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\overline{k_5}\left(t_{k,n};y_{k,n}(r)\right)=max\left\{f\left(t_{k,n}+\frac{3}{4}h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k4}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\underline{k_6}\left(t_{k,n};y_{k,n}(r)\right)=min\left\{f\left(t_{k,n}+h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k5}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

$$\overline{k_6}\left(t_{k,n};y_{k,n}(r)\right)=max\left\{f\left(t_{k,n}+h_k,u,\lambda_k(u_k)\right)\setminus u\epsilon[\underline{z_{k5}}(t_{k,n},y_{k,n}(r)),\overline{z_{k2}}(t_{k,n},y_{k,n}(r)),u_k\epsilon\left[\underline{y_{k,0}}(r),\overline{y_{k,0}}(r)\right]\right\}$$

where in Runge kutta method of order three

$$\underline{z_{k1}}\left(t_{k,n},y_{k,n}(r)\right)=\underline{y_{k,n}}(r)+\tfrac{1}{4}\underline{k_1}(t_{k,n},y_{k,n}(r))h_k$$

$$\overline{z_{k1}}\left(t_{k,n},y_{k,n}(r)\right)=\overline{y_{k,n}}(r)+\tfrac{1}{4}\overline{k_1}(t_{k,n},y_{k,n}(r))\,h_k$$

$$\underline{z_{k2}}\left(t_{k,n},y_{k,n}(r)\right)=\underline{y_{k,n}}(r)+\frac{1}{8}\underline{k_1}(t_{k,n},y_{k,n}(r))h_k+\frac{1}{8}\,\underline{k_2}(t_{k,n},y_{k,n}(r))h_k$$

$$\overline{z_{k2}}\left(t_{k,n}, y_{k,n}(r)\right) = \overline{y_{k,n}}(r) + \frac{1}{8}\overline{k_1}(t_{k,n}, y_{k,n}(r))h_k + \frac{1}{8}\overline{k_2}(t_{k,n}, y_{k,n}(r))h_k$$

$$\underline{z_{k3}}\left(t_{k,n}, y_{k,n}(r)\right) = \underline{y_{k,n}}(r) - \frac{1}{2}\underline{k_2}(t_{k,n}, y_{k,n}(r))h_k + \underline{k_3}(t_{k,n}, y_{k,n}(r))h_k$$

$$\overline{z_{k3}}\left(t_{k,n}, y_{k,n}(r)\right) = \overline{y_{k,n}}(r) - \frac{1}{2}\overline{k_2}(t_{k,n}, y_{k,n}(r))h_k + \overline{k_3}(t_{k,n}, y_{k,n}(r))h_k$$

$$\underline{z_{k4}}\left(t_{k,n}, y_{k,n}(r)\right) = \underline{y_{k,n}}(r) + \frac{3}{16}\underline{k_1}(t_{k,n}, y_{k,n}(r))h_k + \frac{9}{16}\underline{k_4}(t_{k,n}, y_{k,n}(r))h_k$$

$$\overline{z_{k4}}\left(t_{k,n}, y_{k,n}(r)\right) = \overline{y_{k,n}}(r) + \frac{3}{16}\overline{k_1}(t_{k,n}, y_{k,n}(r))h_k + \frac{9}{16}\overline{k_4}(t_{k,n}, y_{k,n}(r))h_k$$

$$\underline{z_{k5}}\left(t_{k,n}, y_{k,n}(r)\right) = \underline{y_{k,n}}(r) + \frac{3}{7}\underline{k_1}(t_{k,n}, y_{k,n}(r))h_k + \frac{2}{7}\underline{k_2}(t_{k,n}, y_{k,n}(r))h_k + \frac{12}{7}\underline{k_3}(t_{k,n}, y_{k,n}(r))h_k$$
$$-\frac{12}{7}\underline{k_4}(t_{k,n}, y_{k,n}(r))h_k + \frac{8}{7}\underline{k_5}(t_{k,n}, y_{k,n}(r))h_k$$

$$\overline{z_{k5}}\left(t_{k,n}, y_{k,n}(r)\right) = \overline{y_{k,n}}(r)+) + \frac{3}{7}\overline{k_1}(t_{k,n}, y_{k,n}(r))h_k + \frac{2}{7}\overline{k_2}(t_{k,n}, y_{k,n}(r))h_k + \frac{12}{7}\overline{k_3}(t_{k,n}, y_{k,n}(r))h_k$$
$$-\frac{12}{7}\overline{k_4}(t_{k,n}, y_{k,n}(r))h_k + \frac{8}{7}\overline{k_5}(t_{k,n}, y_{k,n}(r))h_k$$

Next we define

$$S_k[t_{k,n}, \underline{y_{k,n}}(r), \overline{y_{k,n}}(r)]$$
$$= \frac{1}{90}\left(7\underline{k_1}(t_{k,n}, y_{k,n}(r)) + 32\underline{k_3}\left(t_{k,n}, y_{k,n}(r)\right) + 12\underline{k_4}\left(t_{k,n}; y_{k,n}(r)\right) + 32\underline{k_5}(t_{k,n}, y_{k,n}(r))\right.$$
$$\left. + 7\underline{k_6}\left(t_{k,n}, y_{k,n}(r)\right)\right)h_k$$

$$S_k[t_{k,n}, \underline{y_{k,n}}(r), \overline{y_{k,n}}(r)]$$
$$= \frac{1}{90}\left(7\underline{k_1}(t_{k,n}, y_{k,n}(r)) + 32\underline{k_3}\left(t_{k,n}, y_{k,n}(r)\right) + 12\underline{k_4}\left(t_{k,n}; y_{k,n}(r)\right) + 32\underline{k_5}(t_{k,n}, y_{k,n}(r))\right.$$
$$\left. + 7\underline{k_6}\left(t_{k,n}, y_{k,n}(r)\right)\right)h_k$$

The exact solution at $t_{k,n+1}$ is given by

$$\begin{cases}\underline{Y_{k,n+1}}(r) \approx \underline{Y_{k,n}}(r) + S_k[t_{k,n}, \underline{y_{k,n}}(r), \overline{y_{k,n}}(r)],\\ \overline{Y_{k,n+1}}(r) \approx \overline{Y_{k,n}}(r) + T_k[t_{k,n}, \underline{y_{k,n}}(r), \overline{y_{k,n}}(r)]\end{cases}$$

The approximate solution is given by

$$\begin{cases}\underline{y_{k,n+1}}(r) \approx \underline{y_{k,n}}(r) + S_k[t_{k,n}, \underline{y_{k,n}}(r), \overline{y_{k,n}}(r)],\\ \overline{y_{k,n+1}}(r) \approx \overline{y_{k,n}}(r) + T_k[t_{k,n}, \underline{y_{k,n}}(r), \overline{y_{k,n}}(r)]\end{cases} \quad (4.4)$$

In (4.4) we will use $\underline{y}_{0,0}(r) = \underline{x}_0(r), \overline{y}_{0,0} = \overline{x}_0(r)$ and $\underline{y}_{k,0}(r) = \underline{y}_{k-1,Nk-1}(r), \overline{y}_{k,0} = \overline{y}_{k-1,Nk-1}(r)$ if $k \geq 1$.

For a prefixed k and $r \in [0,1]$, proof of convergence of the approximations in (4.4), that is

$$\lim_{h_0 \ldots h_k \to 0} \underline{y}_{k,Nk}(r) = \underline{x}(t_{k+1}; r), \quad \lim_{h_0 \ldots h_k \to 0} \overline{y}_{k,Nk}(r) = \overline{x}(t_{k+1}; r).$$

is application of Theorem 2.5 and Lemma 4.1 below. Convergence is pointwise in *r* for a fixed *k*.

Lemma 4.1: Suppose $i \in Z^{+}, \varepsilon_i > 0, r \in [0,1]$ and $h_{\text{i}} < 1$ are fixed.Let $\{Z_{i,n}(r)\}_{n=0}^{N_i}$ be the Runge Kutta approximation with $N = N_i$ to the fuzzy IVP:

$$\begin{cases} x'(t) = f(t, x(t), \lambda_i(x_i)),\ t \in [t_i, t_{i+1}], \\ x(t_i) = x_i \end{cases} \tag{4.5}$$

If $\{y_{i,n}(r)\}_{n=0}^{N_i}$ denotes the result (4.4) from some$y_{i,0}(r)$, then there exists a $\delta_i > 0$ such that

$$\left|\underline{z}_{i,0}(r) - \underline{y}_{i,0}(r)\right| < \delta_i, \left|\overline{z}_{i,0}(r) - \overline{y}_{i,0}(r)\right| < \delta_i$$

Implies $\left|\underline{z}_{i,Ni}(r) - \underline{y}_{i,Ni}(r)\right| < \varepsilon_i, \left|\overline{z}_{i,Ni}(r) - \overline{y}_{i,Ni}(r)\right| < \varepsilon_i$

Proof: Fix $i \in Z^{+}, \varepsilon_i > 0, r \in [0,1]$ and $h_{\text{i}} < 1$.Let $\{Z_{i,n}(r)\}_{n=0}^{N_i}$ be the Runge Kutta approximation with $N = N_i$ to the fuzzy IVP (4.5). Suppose $\{y_{i,n}(r)\}_{n=0}^{N_i}$ denotes the result of (4.4) from some $y_{i,0}(r)$, then by (4.4), for each $l = 0, \dots \dots N_{i-1}$,

$$\left|\underline{z}_{i,i+1}(r) - \underline{y}_{i,l+1}(r)\right| = \left|\underline{z}_{i,l}(r) + h_i S_i\left[t_{i,l}, \underline{z}_{i,l}(r), \overline{z}_{i,l}(r)\right] - \underline{y}_{i,l}(r) - h_i S_i[t_{i,l}, \underline{y}_{i,l}(r), \overline{y}_{i,l}(r)]\right|$$

$$\leq \left|\underline{z}_{i,l}(r) - \underline{y}_{i,l}(r)\right| + h_i \left|S_i\left[t_{i,l}, \underline{z}_{i,l}(r), \overline{z}_{i,l}(r)\right] - S_i[t_{i,l}, \underline{y}_{i,l}(r), \overline{y}_{i,l}(r)]\right| \tag{4.6}$$

$$\left|\overline{z}_{i,i+1}(r) - \overline{y}_{i,l+1}(r)\right| = \left|\overline{z}_{i,l}(r) + h_i T_i\left[t_{i,l}, \underline{z}_{i,l}(r), \overline{z}_{i,l}(r)\right] - \overline{y}_{i,l}(r) - h_i T_i[t_{i,l}, \underline{y}_{i,l}(r), \overline{y}_{i,l}(r)]\right|$$

$$\leq \left|\overline{z}_{i,l}(r) - \overline{y}_{i,l}(r)\right| + h_i \left|T_i\left[t_{i,l}, \underline{z}_{i,l}(r), \overline{z}_{i,l}(r)\right] - T_i[t_{i,l}, \underline{y}_{i,l}(r), \overline{y}_{i,l}(r)]\right| \tag{4.7}$$

Let $\alpha_{Ni} = \varepsilon_{i.}$ Since S_i and T_i are continuous there exists a $\eta_{Ni} > 0$ such that

$\left|\underline{z}_{i,Ni-1}(r) - \underline{y}_{i,Ni-1}(r)\right| < \eta_{Ni}$ and $\left|\overline{z}_{i,Ni-1}(r) - \overline{y}_{i,Ni-1}(r)\right| < \eta_{Ni}$ imply

$$\left|S_i\left[t_{i,Ni-1}, \underline{z}_{i,Ni-1}(r), \overline{z}_{i,Ni-1}(r)\right] - S_i\left[t_{i,Ni-1}, \underline{y}_{i,Ni-1}(r), \overline{y}_{i,Ni-1}(r)\right]\right| < \frac{\varepsilon_i}{2} = \frac{\alpha N_i}{2} \tag{4.8}$$

$$\left|T_i\left[t_{i,Ni-1}, \underline{z}_{i,Ni-1}(r), \overline{z}_{i,Ni-1}(r)\right] - T_i\left[t_{i,Ni-1}, \underline{y}_{i,Ni-1}(r), \overline{y}_{i,Ni-1}(r)\right]\right| < \frac{\varepsilon_i}{2} = \frac{\alpha N_i}{2} \tag{4.9}$$

Let $\alpha N_{i-1} = \min\left\{\frac{\varepsilon_i}{2}, \frac{\eta N_i}{2}\right\}$.

If $\left|\underline{z}_{i,Ni-1}(r) - \underline{y}_{i,Ni-1}(r)\right| < \alpha N_{i-1}$ and $\left|\overline{z}_{i,Ni-1}(r) - \overline{y}_{i,Ni-1}(r)\right| < \alpha N_{i-1}$ then by (4.6) and (4.7) with $l = N_i - 1$ and (4.8) and (4.9) we have

$$\left|\underline{z}_{i,Ni}(r) - \underline{y}_{i,Ni}(r)\right|$$

$$\leq \left|\underline{z}_{i,Ni-1}(r) - \underline{y}_{i,Ni-1}(r)\right| + h_i \left|S_i\left[t_{i,Ni-1}, \underline{z}_{i,Ni-1}(r), \overline{z}_{i,Ni-1}(r)\right] - S_i\left[t_{i,Ni-1}, \underline{y}_{i,Ni-1}(r), \overline{y}_{i,Ni-1}(r)\right]\right|$$

$$< \alpha N_{i-1} + h_i \frac{\varepsilon_i}{2} \leq \frac{\varepsilon_i}{2} + h_i \frac{\varepsilon_i}{2} < \varepsilon_i, \tag{4.10}$$

$$\left|\overline{z}_{i,Ni}(r)-\overline{y}_{i,Ni}(r)\right|$$

$$\leq \left|\overline{z}_{i,Ni-1}(r)-\overline{y}_{i,Ni-1}(r)\right| + h_i\left|T_i\left[t_{i,Ni-1},\underline{z}_{i,Ni-1}(r),\overline{z}_{i,Ni-1}(r)\right]-T_i\left[t_{i,Ni-1},\underline{y}_{i,Ni-1}(r),\overline{y}_{i,Ni-1}(r)\right]\right|$$

$$< \alpha N_{i-1} + h_i\frac{\varepsilon_i}{2} \leq \frac{\varepsilon_i}{2} + h_i\frac{\varepsilon_i}{2} < \varepsilon_i \qquad (4.11)$$

Continue inductively for each $j = 2,3 \ldots\ldots\ldots.. N_i$ as follows. Since S_i and T_i are continuous there exists a $\eta_{Ni-(j-1)} > 0$ such that

$\left|\underline{z}_{i,Ni-j}(r)-\underline{y}_{i,Ni-j}(r)\right| < \eta_{Ni-(j-1)}$ and $\left|\overline{z}_{i,Ni-j}(r)-\overline{y}_{i,Ni-j}(r)\right| < \eta_{Ni-(j-1)}$ imply

$$\left|S_i\left[t_{i,Ni-j},\underline{z}_{i,Ni-j}(r),\overline{z}_{i,Ni-j}(r)\right]-S_i\left[t_{i,Ni-j},\underline{y}_{i,Ni-j}(r),\overline{y}_{i,Ni-j}(r)\right]\right| = \frac{\alpha N_{i-(j-1)}}{2} \qquad (4.12)$$

$$\left|T_i\left[t_{i,Ni-j},\underline{z}_{i,Ni-j}(r),\overline{z}_{i,Ni-j}(r)\right]-T_i\left[t_{i,Ni-j},\underline{y}_{i,Ni-j}(r),\overline{y}_{i,Ni-j}(r)\right]\right| = \frac{\alpha N_{i-(j-1)}}{2} \qquad (4.13)$$

where $\frac{\alpha N_{i-(j-1)}}{2}$ is defined in the previous step. Let $\alpha_{Ni-j} = \min\left\{\frac{\alpha N_{i-(j-1)}}{2},\frac{\eta N_{i-(j-1)}}{2}\right\}$.

If $\left|\underline{z}_{i,Ni-j}(r)-\underline{y}_{i,Ni-j}(r)\right| < \alpha N_{i-j}$ and $\left|\overline{z}_{i,Ni-j}(r)-\overline{y}_{i,Ni-j}(r)\right| < \alpha N_{i-j}$ then by (4.6) and (4.7) with $l = N_{i-j}$ and by (4.12), (4.13) we have

$$\left|\underline{z}_{i,Ni-(j-1)}(r)-\underline{y}_{i,Ni-j}(r)\right|$$

$$\leq \left|\underline{z}_{i,Ni-j}(r)-\underline{y}_{i,Ni-j}(r)\right| + h_i\left|S_i\left[t_{i,Ni-j},\underline{z}_{i,Ni-j}(r),\overline{z}_{i,Ni-1}(r)\right]-S_i\left[t_{i,Ni-j},\underline{y}_{i,Ni-1}(r),\overline{y}_{i,Ni-j}(r)\right]\right|$$

$$\leq \frac{\alpha N_{i-(j-1)}}{2} + h_i\frac{\alpha N_{i-(j-1)}}{2} < \alpha N_{i-(j-1)}\ , \qquad (4.14)$$

$$\left|\overline{z}_{i,Ni-(j-1)}(r)-\overline{y}_{i,Ni-j}(r)\right|$$

$$\leq \left|\overline{z}_{i,Ni-1}(r)-\overline{y}_{i,Ni-1}(r)\right| + h_i\left|T_i\left[t_{i,Ni-j},\underline{z}_{i,Ni-1}(r),\overline{z}_{i,Ni-j}(r)\right]-T_i\left[t_{i,Ni-j},\underline{y}_{i,Ni-j}(r),\overline{y}_{i,Ni-j}(r)\right]\right|$$

$$\leq \frac{\alpha N_{i-(j-1)}}{2} + h_i\frac{\alpha N_{i-(j-1)}}{2} < \alpha N_{i-(j-1)} \qquad (4.15)$$

Then for $j = N_i$ we see $\left|\underline{z}_{i,0}(r)-\underline{y}_{i,0}(r)\right| < \alpha_0$ and $\left|\overline{z}_{i,0}(r)-\overline{y}_{i,0}(r)\right| < \alpha_0$ imply

$\left|\underline{z}_{i,1}(r)-\underline{y}_{i,1}(r)\right| < \alpha_1$ and $\left|\overline{z}_{i,1}(r)-\overline{y}_{i,1}(r)\right| < \alpha_1$.

For $j = N_{i-j}$ we see $\left|\underline{z}_{i,1}(r)-\underline{y}_{i,1}(r)\right| < \alpha_1$ and $\left|\overline{z}_{i,1}(r)-\overline{y}_{i,1}(r)\right| < \alpha_1$ imply

$\left|\underline{z}_{i,2}(r)-\underline{y}_{i,2}(r)\right| < \alpha_2$ and $\left|\overline{z}_{i,2}(r)-\overline{y}_{i,2}(r)\right| < \alpha_2$.

Continue decreasing to $j = 2$ to see

$\left|\underline{z}_{i,Ni-2}(r)-\underline{y}_{i,Ni-2}(r)\right| < \alpha_{Ni-2}$ and $\left|\overline{z}_{i,Ni-2}(r)-\overline{y}_{i,Ni-2}(r)\right| < \alpha_{Ni-2}$. imply

$\left|\underline{z}_{i,Ni-1}(r)-\underline{y}_{i,Ni-1}(r)\right| < \alpha_{Ni-1}$ and $\left|\overline{z}_{i,Ni-1}(r)-\overline{y}_{i,Ni-1}(r)\right| < \alpha_{Ni-1}$

But it was already shown in (4.10) and (4.11) that

$\left|\underline{z}_{i,Ni-1}(r) - \underline{y}_{i,Ni-1}(r)\right| < \alpha_{Ni-1}$ and $\left|\overline{z}_{i,Ni-1}(r) - \overline{y}_{i,Ni-1}(r)\right| < \alpha_{Ni-1}$ imply

$\left|\underline{z}_{i,Ni}(r) - \underline{y}_{i,Ni}(r)\right| < \varepsilon_i$ and $\left|\overline{z}_{i,Ni}(r) - \overline{y}_{i,Ni}(r)\right| < \varepsilon_i$

This proves the lemma with $\delta_i = \alpha_0$.

Theorem 4.1: Consider the systems (3.2) and (4.4).For a fixed $k \in Z^+$ and $r \in [0,1]$,

$$\lim_{h_0,\ldots,h_k \to 0} \underline{y}_{k,Nk}(r) = \underline{x}(t_{k+1}; r), \tag{4.16}$$

$$\lim_{h_0,\ldots,h_k \to 0} \overline{y}_{k,Nk}(r) = \overline{x}(t_{k+1}; r). \tag{4.17}$$

Proof: Fix $k \in Z^+$ and $r \in [0,1]$,.Choose $\varepsilon > 0$. For each $i = 0, \ldots \ldots \ldots k$ we will find a $\delta_i^* > 0$ such that $h_i < \delta_i^*$ implies

$\left|\underline{x}(t_{k+1}; r) - \underline{y}_{k,Nk}(r)\right| < \varepsilon$ and $\left|\overline{x}(t_{k+1}; r) - \overline{y}_{k,Nk}(r)\right| < \varepsilon$ where the h_i values are allowable by regular partition of the $[t_i, t_{i+1}]'s$.By theorem 2.5, there exists a $\delta_k^* > 0$ such that if $h_k < \delta_k^*$ then

$\left|\underline{z}_{k,Nk}(r) - \underline{x}(t_{k+1}; r)\right| < \frac{\varepsilon}{2}, \left|\overline{z}_{k,Nk}(r) - \overline{x}(t_{k+1}; r)\right| < \frac{\varepsilon}{2}.$

We may assume $\delta_k^* < 1$, then $h_k < 1$. By lemma 4.1 there exists a $\delta_k > 0$ such that

$$\left|\underline{z}_{k,o}(r) - \underline{y}_{k,0}(r)\right| < \delta_k, \left|\overline{z}_{k,0}(r) - \overline{y}_{k,0}(r)\right| < \delta_k \tag{4.18}$$

Implies $\left|\underline{z}_{k,Nk}(r) - \underline{y}_{k,Nk}(r)\right| < \frac{\varepsilon}{2}, \left|\overline{z}_{k,Nk}(r) - \overline{y}_{k,Nk}(r)\right| < \frac{\varepsilon}{2}$.

Therefore if $h_k < \delta_k^*$ and (4.18) holds then

$$\left|\underline{x}(t_{k+1}; r) - \underline{y}_{k,Nk}(r)\right| \le \left|\underline{x}(t_{k+1}; r) - \underline{z}_{k,Nk}(r)\right| + \left|\underline{z}_{k,Nk}(r) - \underline{y}_{k,Nk}(r)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{4.19}$$

$$\left|\overline{x}(t_{k+1}; r) - \overline{y}_{k,Nk}(r)\right| \le \left|\overline{x}(t_{k+1}; r) - \overline{z}_{k,Nk}(r)\right| + \left|\overline{z}_{k,Nk}(r) - \overline{y}_{k,Nk}(r)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{4.20}$$

By theorem 2.5, there exists a $\delta_{k-1}^* > 0$ such that if $h_{k-1} < \delta_{k-1}^*$ then

$\left|\underline{z}_{k-1,Nk-1}(r) - \underline{x}(t_k; r)\right| < \frac{\delta_k}{2}$ and $\left|\overline{z}_{k-1,Nk-1}(r) - \overline{x}(t_k; r)\right| < \frac{\delta_k}{2}$.

We may assume $\delta_{k-1}^* < 1$ .Then $h_{k-1} < 1$. By lemma 4.1 there exists a $\delta_{k-1} > 0$ such that

$$\left|\underline{z}_{k-1,o}(r) - \underline{y}_{k-1,0}(r)\right| < \delta_{k-1}, \left|\overline{z}_{k-1,0}(r) - \overline{y}_{k-1,0}(r)\right| < \delta_{k-1} \tag{4.21}$$

Implies $\left|\underline{z}_{k-1,Nk-1}(r) - \underline{y}_{k-1,Nk-1}(r)\right| < \frac{\delta_k}{2}$ and $\left|\overline{z}_{k-1,Nk-1}(r) - \overline{y}_{k-1,Nk-1}(r)\right| < \frac{\delta_k}{2}$.

Therefore if $h_{k-1} < \delta_{k-1}^*$ and (4.21) holds then

$$\left|\underline{x}(t_k; r) - \underline{y}_{k-1,Nk-1}(r)\right| \le \left|\underline{x}(t_k; r) - \underline{z}_{k-1,Nk-1}(r)\right| + \left|\underline{z}_{k-1,Nk-1}(r) - \underline{y}_{k-1,Nk-1}(r)\right| < \frac{\delta_k}{2} + \frac{\delta_k}{2} = \delta_k, \tag{4.22}$$

$$\left|\overline{x}(t_k;r)-\overline{y}_{k-1,Nk-1}(r)\right| \le \left|\overline{x}(t_k;r)-\overline{z}_{k-1,Nk-1}(r)\right|+\left|\overline{z}_{k-1,Nk-1}(r)-\overline{y}_{k-1,Nk-1}(r)\right| < \frac{\delta_k}{2}+\frac{\delta_k}{2}=\delta_k \qquad (4.23)$$

Continue inductively for each $i=k-2,\dots\dots\dots\dots 2,1$ to find a $\delta_i^*>0$ such that if $h_i<\delta_i^*$ then

$\left|\underline{z}_{i,Ni}(r)-\underline{x}(t_{i+1};r)\right| < \frac{\delta_{i+1}}{2}$ and $\left|\overline{z}_{i,Ni}(r)-\overline{x}(t_{i+1};r)\right| < \frac{\delta_{i+1}}{2}$.

We may assume each $\delta_i^*<1$.Then for each $h_i<1$, by lemma 4.1 there exists a $\delta_i>0$ such that

$$\left|\underline{z}_{i,o}(r)-\underline{y}_{i,0}(r)\right| < \delta_i, \left|\overline{z}_{i,0}(r)-\overline{y}_{i,0}(r)\right| < \delta_i \qquad (4.24)$$

Implies $\left|\underline{z}_{i,Ni}(r)-\underline{y}_{i,Ni}(r)\right| < \frac{\delta_{i+1}}{2}, \left|\overline{z}_{i,Ni}(r)-\overline{y}_{i,Ni}(r)\right| < \frac{\delta_{i+1}}{2}$. (4.25)

Therefore if $h_i<\delta_i^*$ and (4.24) holds then

$$\left|\underline{x}(t_{i+1};r)-\underline{y}_{i,Ni}(r)\right| \le \left|\underline{x}(t_{i+1};r)-\underline{z}_{i,Ni}(r)\right|+\left|\underline{z}_{i,Ni}(r)-\underline{y}_{i,Ni}(r)\right| < \frac{\delta_{i+1}}{2}+\frac{\delta_{i+1}}{2}=\delta_{i+1}$$

$$\left|\overline{x}(t_{i+1};r)-\overline{y}_{i,Ni}(r)\right| \le \left|\overline{x}(t_{i+1};r)-\overline{z}_{i,Ni}(r)\right|+\left|\overline{z}_{i,Ni}(r)-\overline{y}_{i,Ni}(r)\right| < \frac{\delta_{i+1}}{2}+\frac{\delta_{i+1}}{2}=\delta_{i+1}$$

In particular there exists a $\delta_1^*>0$ such that if $h_1<\delta_1^*$ and (4.24) holds with $i=1$ then

$\left|\underline{x}(t_2;r)-\underline{y}_{1,N1}(r)\right| < \delta_2$ and $\left|\overline{x}(t_2;r)-\overline{y}_{1,N1}(r)\right| < \delta_2$.

By theorem 2.5 we may choose $\delta_0^*>0$ such that $h_0<\delta_0^*$ implies

$\left|\underline{x}(t_1;r)-\underline{y}_{0,N0}(r)\right| < \delta_1$ and $\left|\overline{x}(t_1;r)-\overline{y}_{0,N0}(r)\right| < \delta_1$. (4.26)

Suppose for each $i=0,\dots..k$ that $h_i<\delta_i^*$ since (4.26) is the same as (4.24) with $i=1$, we obtain (4.25) with $i=1$. Since (4.25) with $i=1$ implies (4.24) with $i=2$, we obtain (4.25) with $i=2$.Continue inductively to obtain {(4.18)-(4.20)},proving (4.16) and (4.17).

5. NUMERICAL EXAMPLE

Example 1: Consider the following hybrid fuzzy IVP

$$\begin{cases} y'(t)=y(t)+m(t)\lambda_k\big(y(t_k)\big), & t\in[t_k,t_{k+1}],\\ y(0,r)=[0.75+0.25r,1.125-0.125r], & 0\le r\le 1. \end{cases} \qquad (5.1)$$

$$m(t)=\begin{cases} 2\big(t(mod\ 1)\big) & if \quad t(mod\ 1)\le 0.5\\ 2\big(1-t(mod\ 1)\big), & if \quad t(mod\ 1)>0.5 \end{cases}$$

$$\lambda_k(\mu)=\begin{cases} \hat{0}, & \text{if } k=0\\ \mu, & \text{if } k\{1,2,\dots\} \end{cases}$$

In (5.1), $y(t)+m(t)\lambda_k\big(x(t_k)\big)$is a continuous function of $t, y,$ and $\lambda_k\big(x(t_k)\big)$. Therefore by Example 6.1 of Kaleva [8] and Theorem 4.2 of Buckley and Feuring [2] for each k=0, 1, 2,…, the fuzzy IVP

$$\begin{cases} y'(t) = y(t) + m(t)\lambda_k\big(y(t_k)\big), & t \in [t_k, t_{k+1}],\ \ t_k = k \\ y(t_k) = y_{tk}\ , \end{cases}$$

has a unique solution on $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP (5.1) we will apply the Runge–Kutta method for hybrid fuzzy differential equations from Section 4 with N=2 to obtain $y_{1,2}(r)$ approximating $y(2.0; r)$.

Let $f: [0, \infty) \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be given by $\mathrm{f(t, y, \lambda_k(x(t_k))) = y(t) + m(t)\lambda_k\big(y(t_k)\big)}$ $\mathrm{t_k = k, k = 1,2,3, \dots}$..

where $\lambda_k: \mathcal{R} \to \mathcal{R}$ is given by $\lambda_k(y) = \begin{cases} 0, & \text{if } k = 0 \\ y, & \text{if } k\epsilon\{1,2, \dots\} \end{cases}$

Since the exact solution of (5.1) for t $\in [1,1.5]$ is $\mathrm{Y(t; r) = Y(1; r)(3e^{t-1} - 2t), 0 \le r \le 1}$,

$\mathrm{Y(1.5; r) = Y(1; r)\big(3\sqrt{e} - 3\big), 0 \le r \le 1}$.Then$Y(1.5; 1)$is approximately5.29 and $y_{1,1}(1)$ is approximately 5.29.Since the exact solution of (5.1) for

$$t \in [1.5,2] \text{ is } Y(t; r) = Y(1; r)\big(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)\big), 0 \le r \le 1,$$

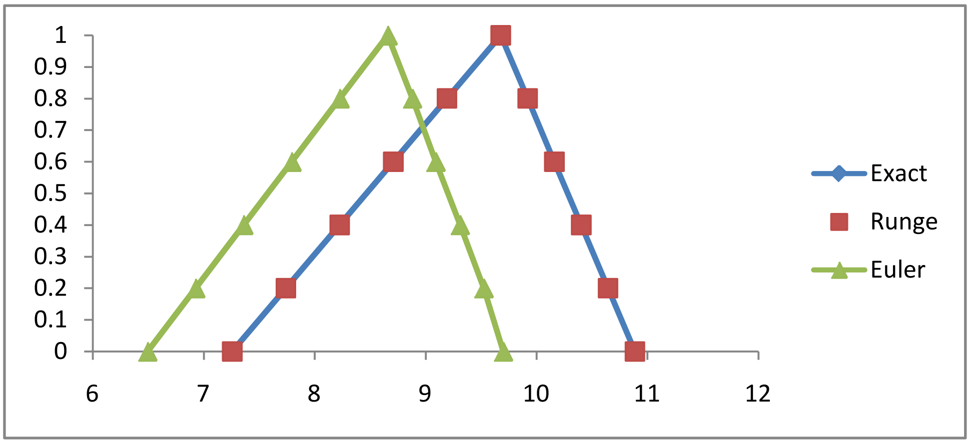
$\mathrm{(Y(2.0; r) = Y(1; r)\big(2 + 3e - 4\sqrt{e}\big), 0 \le r \le 1}$, Then $Y(2.0; 1)$ is approximately 9.68 and $y_{1,2}(1)$ is approximately 9.677. These observations are summarized in Table 5.1 For additional comparison, Fig 5.1 shows the graphs of $Y(2.0), y_{1,2}$, and the corresponding Euler approximation.)

Table 5.1

At t=2

r	Exact solution		Approximate solution	
	$\underline{y}$	$\overline{y}$	$\underline{y}$	$\overline{y}$
1	9.676975672	9.676975672	9.67700504	9.67700504
0.8	9.193126888	9.918900064	9.19315479	9.91893017
0.6	8.709278105	10.16082446	8.70930454	10.1608553
0.4	8.225429321	10.40274885	8.22545428	10.4027804
0.2	7.741580538	10.64467324	7.74160403	10.6447055
0	7.257731754	10.88659763	7.25775378	10.8866307

Fig 5.1



Example 2: Next consider the following hybrid fuzzy IVP

$$\begin{cases} y'(t) = y(t) + m(t)\lambda_k\big(y(t_k)\big), & t \in [t_k, t_{k+1}], \\ y(0,r) = [0.75 + 0.25r, 1.125 - 0.125r], & 0 \le r \le 1. \end{cases} \tag{5.2}$$

$m(t) = |\sin(\pi t)|$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0 \\ \mu, & \text{if } k \epsilon \{1,2,\dots\} \end{cases}$$

In (5.2), $y(t) + m(t)\lambda_k\big(x(t_k)\big)$ is a continuous function of $t, y,$ and $\lambda_k\big(x(t_k)\big)$. Therefore by Example 6.1 of Kaleva [8] and Theorem 4.2 of Buckley and Feuring [2] for each, $k = 0,1,2,\dots$ the fuzzy IVP)

$$\begin{cases} y'(t) = y(t) + m(t)\lambda_k\big(y(t_k)\big), & t \in [t_k, t_{k+1}],\ \ t_k = k \\ y(t_k) = y_{tk}\ , \end{cases}$$

has a unique solution on $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP (5.1) we will apply the Runge–Kutta method for hybrid fuzzy differential equations from Section 4 with N=2 to obtain $y_{1,2}(r)$ approximating $y(2.0; r)$.

Let $f: [0, \infty) \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be given by

$f(t, y, \lambda_k(x(t_k))) = y(t) + m(t)\lambda_k\big(y(t_k)\big) \qquad t_k = k, k = 1,2,3, \dots$ ..where $\lambda_k: \mathcal{R} \to \mathcal{R}$ is given by

$$\lambda_k(y) = \begin{cases} 0, & \text{if } k = 0 \\ y, & \text{if } k \epsilon \{1,2,\dots\} \end{cases}$$

Since the exact solution of (5.1) for

$(t \in [2, r]$ is $Y(t; r) = Y(1; r)\left(\frac{\pi}{\pi^2+1} + e\left(1 + \frac{\pi}{\pi^2+1}\right)\right), 0 \le r \le 1$,

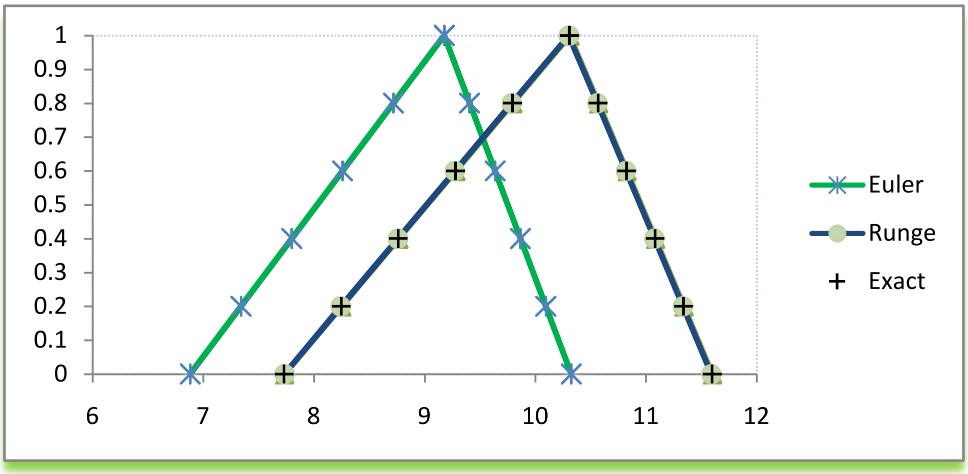
Then $Y(2.0; 1)$ is approximately 10.31 and $y_{1,2}(1)$ is approximately 10.30 .These observations are summarized in Table 5.2 For additional comparison, Fig 5.2 shows the graphs of $x(2.0), y_{1,2}$, and the corresponding Euler approximation.

Table 5.2

At t=2

r	Exact solution		Approximate solution	
	$\underline{y}$	$\overline{y}$	$\underline{y}$	$\overline{y}$
1	10.31154322	10.31154322	10.30288017	10.30288017
0.8	9.795966054	10.5693318	9.787736162	10.56045217
0.6	9.280388898	10.82712038	9.272592153	10.81802418
0.4	8.764811737	11.08490896	8.757448145	11.07559618
0.2	8.249234576	11.34269754	8.242304136	11.33316819
0	7.733657415	11.60048612	7.733657415	11.59074019

Fig 5.2



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