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PARETO–MINIMAL SOLUTIONS FOR ROUGH CONTINUOUS STATIC GAMES

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ABSTRACT

This paper studies a new framework to hybridize the rough set theory with the continuous static games called Rough Continuous Static Games (RCSGs). In Continuous static games, the decision possibilities need not be discrete, and the decisions and costs are related in a continuous rather than a discrete manner. The game is static in the sense that no time history is involved in relationship between cost and decisions.

In this paper, Pareto–minimal solutions are presented to solve (RCSGs). Additionally, a numerical illustrative example is given to clarify the main results developed in the paper.

Keywords: *Rough set, rough continuous static games, Pareto – minimal solutions, rough optimality, rough feasibility.*

INTRODUCTION

Since it was known by Pawlak in mid 1980's [7,8], rough set theory has become a hot topic of great interest to researchers in several fields and has been applied to many domains such as pattern recognition, Fuzzy programming, artificial intelligence, image processing, machine learning, and medical applications [2,6].

This new approach proved to be useful in many applications such as optimization theory. For the mathematical programming problems (MPPs) in the crisp seenario, the aim is to maximize or minimize an objective function over certain set of feasible solution. But in many practical situations [4,5], the decision maker may not be in a position to specify the objective and/or the feasible set exactly but rather than can specify them in a "rough sense". In such situations, it is desirable to use some rough programming type of modeling so as to provide more flexibility to the decision maker. Towards this objective, we present a new framework to hybridize the rough set theory with the continuous static games, called "Rough Continuous Static Games" (RSCGs).

Since the roughness may appear in a mathematical programming problem in many ways (e.g. the feasible set may be rough and/or the goals may be rough), the definition of rough continuous static games is not unique. This leads us to propose a new classification and characterization of the rough continuous static games (RCSGs), according to the place of roughness in the problem. We classified the (RCSGs) into the following classes [1]:

1. Problems with rough feasible set and crisp objective function,
2. Problems with crisp feasible set and rough objective function,
3. Problems with rough feasible set and rough objective function.

New definitions concerning rough optimal sets, rough optimal value, rough global optimality and rough feasibility were also proposed and discussed.

Rough set and approximation space

Rough set theory has been proven to be an excellent mathematical told dealing with vague description of objects [7, 8]. A fundamental assumption in rough set theory is that any object from a universe is perceived through available information, and such information may not be sufficient to characterize the object exactly. Pawlak has proposed rough set methodology as a new approach in handle classificatory analysis of vague concepts [9]. In this methodology any vague concept is characterized by a pair of precise concepts called the lower and the upper approximations. Rough set theory is based on equivalence relations describing partitions made of classes of indiscernible objects.

Let U be a non–empty finite set of objects, called the universe, and $E \subseteq U \times U$ be an equivalence relation on U.

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The ordered Pair $A=(U,E)$ is called an approximation space generated by *E* on *U*. The equivalence relation *E* generates a partition *U/E* = {$y_1, y_2, \ldots, y_m$} where $y_1, y_2, \ldots, y_m$ are the equivalence classes (also called elementary set generated by E, of the approximation space A. In rough set theory, any subset M $\subseteq$ U is described by the elementary sets of A and the two set

$$E_*(M)=\bigcup \ \{y_i \in U/E| \ y_i \subseteq M \}$$
$$E_*(M)=\bigcup \ \{y_i \in U/E| \ y_i \bigcap \ M \neq \varphi \}$$

are called the lower and the upper approximations of M, respectively. Therefore $E_*(M) \subseteq M \subseteq E^*(M)$. The difference between the upper and lower approximation is called the boundary of M and is denoted by $\beta N_E(M)=E^*(M)-E_*(M)$. The set M is called exact in A iff $\beta N_E(M) \neq \varphi$; otherwise the set M is inexact (rough) in A.

Classes of Rough Continuous Static Games (RCSGs).

The continuous static games (CSGs) can be formulated using control notations in the following notations in the following from [3].

min $F_e(x,u), \quad e=1,\ldots,r$

S.T.

$$M=\left\{x \in R^n, u \in R^s \middle| \ g_j(x,u)=0, j=1,\ldots,n, h_k(x,u) \geq, k=1,\ldots,q\right\} \tag{1}$$

Where $x \in R^n$ is the state and. $u=(u_1,u_2,\ldots,u_r) \in R^s$, $s=s_1+s_2+\ldots+s_r$ is the composite control.

The composite control is required to be an element of a regular control constraint set $\Omega \subset R^s$ of the from

$$\Omega=\left\{ \ u \in R^s \middle| \ h \ (x,u) \geq 0\right\} \tag{2}$$

Where $x=\zeta(u)$ is the solution to g (*x*,*u*)= 0 given *u*. The functions $F_e : R^n \times R^s \to R^1, g : R^n \times R^s \to R^n$ and $h : R^n \times R^s \to R^q$ are assumed to be $C^{(1)}$ with $\left| dg(x,u)/dx \right| \neq 0$ (3)

in a boll about a solution point (*x, u*). In the above formulation, it is assumed that all entries of F_e, g and h are defined in the crisp sense. However, in many practical situation it may not be reasonable to require that the feasible set or the objective functions in continuous static games be specified in precise crisp terms. In such situations, it is desirable to use some type of rough modeling and this leads to the concept of rough continuous static games. When decision is to be made in a rough environment, many possible modifications of the above continuous static games models exist. Thus rough continuous static games models are not uniquely defined as it will very much depend upon the type of roughness and its specification as prescribed by the decision maker. Therefore, the rough continuous static games can be broadly classified as[1]:

1st Class: Continuous static games with rough feasible set and crisp objective function for all players

2nd Class: Continuous static games with crisp feasible set and rough objective function for all players.

3rd Class: Continuous static games with rough feasible set and rough objective function for all players.

In (RCSGs), wherever roughness exists, new concept like rough feasibility and rough global optimality come in the front of our interest. The "rough feasibility" arises only in the 1st and 3rd classes, where solutions have different degrees of feasibility (surely– feasible, possible – feasible and surely – not feasible).

The 1st Class of (RCSGs).

Suppose that $A=(U,E)$ is an approximation space generated by an equivalence relation E on an universe U. A rough continuous static games of the 1st class takes the following form:

$$\min F_e(x,u), \qquad e=1,2,\ldots,r \tag{4}$$

S.T.

$$M = \left\{ x \in R^n, u \in R^s \,\middle|\, g_j(x,u) = 0,\ j = 1,...,n, h_k(x,u) \geq 0, k = 1,...,q \right\}, \tag{5}$$

$$E_*(M) \subseteq M \subseteq E^*(M) \tag{6}$$

Where $M \subseteq U$ is a rough set in the approximation space $A(U,E)$ representing the feasible set of the problem. The sets $E^*(M) = M^*$ and $E_*(M) = M_*$ representing the notion of "rough – feasibility" of problem (4) – (6), where M^* is called the set of all possibly – feasible solutions, and M_* is called the set of all surely – feasible solutions. On the other hand, $U - M^*$ is called the set of all surely – not feasible solutions. The functions $F_e : M^* \to R$ is a crisp objective function that is continuous on M^*.

Pareto – Minimal Solution for (RCSGs):

In this type of game, the cooperation among all players is possible. It is assumed that each player helps the others up to the point of disadvantage to himself [3].

Definition 1: Consider the problem (4) – (6). A point $\hat{u}\left(x^*, u^*\right) \in M$ is an Pareto – minimal solution to the problem (4) – (6) if and only if there does not exist a $u\ (x,u) \in M$ such that $F_e\left(\zeta(u), u\right) \leq F_e\left(\zeta(\hat{u}, \hat{u})\right)$, for all $e \in \{1,2,...,r\}$, $F_i\left(\zeta(u), u\right) < F_i\left(\zeta(\hat{u}), \hat{u}\right)$, for some $i \in \{1,2,...,r\}$

The rough optimality arises in all classes of the (RCSGs), where solutions have different degrees of optimality (surely – Pareto optimal, possibly – Pareto optimal, and surely – not Pareto optimal). As a result of these new concepts, the Pareto optimal value of the objectives and the optimal set of the problem are defined in rough sense.

Definition 2: In (RCSGs), the Pareto optimal value of the objective function for each player is a rough real number $\overline{F}_e, e = 1,...r,$ that is determined roughly by lower and upper bounds denoted by $\overline{F}_{e*}, \overline{F}_e^*, e = 1,...,r$ respectively [1].

Remark 1: If $\overline{F}_{e*} = \overline{F}_e^*$ for any player e, then the optimal value $\overline{F}_e$ is exact, otherwise $\overline{F}_e$ is rough [1].

Also, the single optimal set of the crisp continuous static games is replaced by four optimal sets covering all possible degrees of feasibility and optimality. See table 1.

		Optimality	
		Possibly	Surly
Feasibility	Possibly	$FO_{P(PP)}$	$FO_{P(SP)}$
	surely	$FO_{S(PP)}$	$FO_{S(SP)}$

Table 1

Remark 2: The set of all surely– feasible, surely – Pareto optimal solutions is denoted by $FO_{S(SP)}$.

Remark 3: The set of all surely–feasible, possibly–Pareto optimal solutions is denoted by $FO_{S(PP)}$.

Remark 4: The set of all possible– feasible, surely – Pareto optimal solutions is denoted by $FO_{P(SP)}$.

Remark 5: The set of all possible – feasible, possibly – Pareto optimal solutions is denoted by $FO_{P(PP)}$.

Proposition 1: In problem (4) – (6), the lower and the upper bounds of the optimal objective value $\overline{F}_e$ for each players is given by

$$\overline{F}_{e*} = \inf \left\{ a_e, b_e \right\}$$

$$\overline{F}_e^{\,*} = \inf \left\{ a_e, c_e \right\}$$

Where

$$a_e = \min_{(x,u)\in M_*} F_e(x,u), \quad e = 1,\dots,r$$

$$b_e = \inf \bigcup_{\substack{y\in U/E \\ y\subseteq M_{BN}}} \left\{ \max_{(x,u)\in y} F_e(x,u) \right\}, \quad e = 1,\dots,r$$

$$C_e = \min_{(x,u)\in M_{BN}} F_e(x,u), e = 1,\dots,r$$

Definition 3: A solution $(x,u) \in M^*$ is surely– Pareto optimal solution if and only if $F_e(x,u) = \overline{F}_e^{\,*}$.

Definition 4: A solution $(x,u) \in M^*$ is possible–Pareto optimal solution if and only if $F_e(x,u) \le \overline{F}_{e*}$.

Definition 5: A solution $(x,u) \in M^*$ is surely–not Pareto optimal solution if and only if $F_e(x,u) > \overline{F}_{e*}$.

Definition 6: The Pareto optimal set of the 1st class (RCSGs) are defined as:

$$F0_{S(SP)} = \left\{ (x,u) \in M_* \middle|\ F_e(x,u) = \overline{F}^{\,*}{}_e, e = 1,\dots,r \right\}$$
$$F0_{S(PP)} = \left\{ (x,u) \in M_* \middle|\ F_e(x,u) \le \overline{F}_{e*}, e = 1,\dots,r \right\}$$
$$F0_{P(SP)} = \left\{ (x,u) \in M^* \middle|\ F_e(x,u) = \overline{F}_e^{\,*}, e = 1,\dots,r \right\}$$
$$F0_{P(PP)} = \left\{ (x,u) \in M^* \middle|\ F_e(x,u) \le \overline{F}_{e*}, e = 1,\dots,r \right\}$$

Necessary conditions for determining Pareto – minimal solutions are stated in the following theorem [3].

Theorem 1: If $\hat{u} = (\hat{u}_1, \hat{u}_2) \in M$ is a regular local surely Pareto – minimal solution and possibly Pareto minimal solution respectively for the problem (4) – (6) and if $\hat{x} = \zeta(\hat{u})$ is the corresponding solution to $g(x,\hat{u}) = 0$, then there exists the vectors $\eta \in R^r, \lambda \in R^n, \mu \in R^q$ such that

$$\partial L(\hat{x}, \hat{u}, \eta, \lambda, \mu) / \partial x = 0$$
$$\partial L(\hat{x}, \hat{u}, \eta, \lambda, \mu) / \partial u = 0$$
$$g_i(\hat{x},\hat{u}) = 0, j = 1,\dots,n$$
$$h_k(\hat{x},\hat{u}) \ge 0, k = 1,\dots,q$$

$$\mu_k^t h_k(\hat{x},\hat{u}) = 0,\ k = 1,2,\dots,q$$
$$\eta_e \ge 0,\ e = 1,\dots,r$$
$$\sum_{e=1}^{r} \eta_e = 1, \mu_k \ge 0, k = 1,2,\dots,q$$

Where

$$L\ (x, u, \eta, \lambda, \mu) = \eta^t\ F(x,u) - \lambda^t g\ (x,u) - \mu^t h\ (x,u)$$

Example: Let U be a universal set defined as $U = \{u = (u_1, u_2) \in R^2 | u_1^2 + u_2^2 \le 9\}$ and let k be a polytope generated by the following closed half planes

$$h_1 = 2 - u_1 - u_2 \ge 0 \quad , \quad h_2 = 2 + u_1 - u_2 \ge 0$$
$$h_3 = u_2 - u_1 + 2 \ge 0 \quad , \quad h_4 = u_1 + u_2 + 2 \ge 0$$

Suppose that E is an equivalence relation on U such that: $U/E = \{E_1, E_2, E_3\}$,

$$E_1 = \{(u_1,u_2) \in U \mid (u_1,u_2) \text{ is an interior point of polytope k}\}$$
$$E_2 = \{(u_1,u_2) \in U \mid (u_1,u_2) \text{ is a boundary point of polytope k}\}$$
$$E_3 = \{(u_1,u_2) \in U \mid (u_1,u_2) \text{ is an exterior point of polytope k}\}$$

Consider the following 1st class RCSG:

$$\min F_1(u_1,u_2) = (u_1 - 2.5)^2 + u_2^2$$
$$\min F_2(u_1,u_2) = -u_1 - u_2$$
$$s.t.$$
$$M_* = E_1 \cup E_2, \quad M^* = E_1 \cup E_2 \cup E_3$$

Where player (1) selects $u_1 \in R^1$ to minimize $F_1(u_1,u_2)$ and player (2) selects $u_2 \in R^1$ to minimize $F_2(u_1,u_2)$. Also, M is a rough feasible region in the approximation space A (U, E) and M_*, M^* are the lower and upper approximations of M; respectively, and the boundary region of M is given by. $M_{BN} = E_3$.

Solution:

1st step: finding the rough minimal value $\overline{F} = F_{\min}$, where $F = (F_1, F_2)$

$$\overline{F_*} = \inf\{a,b\}$$
$$a = \min_{(u_1,u_2)\in M_*} F(u_1,u_2)$$

Using theorem (1) to solve a, b and c.

When $\mu = 0$, the Pareto-minimal control set is

$$P = \{(u_1,u_2) \in R^2 \mid u_1 = u_2 + 2.5,\ u_2 = \eta_2 / 2\eta_1,\ 0 < \eta_1 \le 1,\ 0 \le \eta_2 \le 1\}$$

$$\therefore\ a_1 = \min F_1(u_1,u_2) = F_1(2,0) = 0.25$$
$$a_2 = \min F_2(u_1,u_2) = F_2(2,0) = -2$$

$$\because \sup_{(u_1,u_2)\in E_3} F_1(u_1,u_2) = F_1(-3,0) = -30.25$$
$$\sup_{(u_1,u_2)\in E_3} F_2(u_1,u_2) = F_2(-3,0) = -3$$

$$\therefore b = \inf \bigcup_{\substack{y\in U/E \\ y\subseteq M_{BN}}} \left\{\max_{(u_1,u_2)\in y} F(u_1,u_2)\right\} = \max_{(u_1,u_2)\in E_3} F(u_1,u_2)$$
$$\therefore b_1 = F_1(-3,0) = -30.25$$
$$b_2 = F_2(-3,0) = -3$$

$$\therefore \overline{F_{1^*}} = \inf\{a_1,b_1\} = -30.25$$
$$\overline{F_{2^*}} = \inf\{a_2,b_2\} = -3$$
$$\overline{F}^* = \inf\{a,c\}$$
$$\because \inf_{(u_1,u_2)\in E_3} F_1(u_1,u_2) = F_1(2.5,0) = 0$$

$$\inf_{(u_1,u_2)\in E_3} F_2(u_1,u_2)=F_2(2.5,0)=-2.5$$
$$c=\min_{(u_1,u_2)\in E_3} F(u_1,u_2)=\min_{(u_1,u_2)\in E_3} F(u_1,u_2)=F(2.5,0)$$
$$c_1=F_1(2.5,0)=0$$
$$c_2=F_2(2.5,0)=-2.5$$
$$\therefore \overline{F}_1^{*}=\inf\{a_1,c_1\}=0$$
$$\overline{F}_2^{*}=\inf\{a_2,c_2\}=-2.5$$
$$\therefore \overline{F}_1\in[-30.25,0]$$
$$\overline{F}_2\in[-3,-2.5]$$

2nd step: finding the rough minimal sets:

$$FO_{S(SP)}=\{(u_1^*,u_2^*)\in E_1\cup E_2 : F_1(u_1^*,u_2^*)=0, F_2(u_1^*,u_2^*)=-2.5\}=\varphi$$
$$FO_{S(PP)}=\{(u_1^*,u_2^*)\in E_1\cup E_2 : F_1(u_1^*,u_2^*)\le -30.25, F_2(u_1^*,u_2^*)=-3\}=\{(-3.0)\}$$
$$FO_{P(SP)}=\{(u_1^*,u_2^*)\in E_1\cup E_2\cup E_3 : F_1(u_1^*,u_2^*)=0, F_2(u_1^*,u_2^*)=-2.5\}=\{(2.5,0)\}$$
$$FO_{P(PP)}=\{(u_1^*,u_2^*)\in E_1\cup E_2\cup E_3 : F_1(u_1^*,u_2^*)\le -30.25, F_2(u_1^*,u_2^*)\le -3\}=$$
$$\{(u_1,u_2):(u_1-2.5)^2+u_2^2\le -30.25,-u_1-u_2\le -3\}$$

CONCLUSION

This paper proposes a new formulation, classification and definition of the rough continuous static games by using Pareto-minimal solution. Only the 1st class of RCSGs is defined and its optimal sets are characterized in this paper.

REFERENCES

[1] E. A. Youness. Characterizing Solution of Rough Programming problems,European Journal of Operational Research, 168(3), 1019 – 1029, 2006.

[2] H. Zhang, H. Liang, D. Liu. Two New Operators in Rough set Theory with Applications to Fuzzy sets, International Journal of Information Sciences, 166, 147 – 165, 2004

[3] J. Walter and Thomas, Optimality in Parametric System, Johnwiley and Sons, New York, London, 1981.

[4] M. S. Osman, E. F. Lashein, E.A. Youness and T. E. M. Atteya. Rough Mathematical Programming, Optimization A Journal of Mathematical Programming and Operations Research, 58, 1 – 8, 2009

[5] M. S. Osman, W. F. Abd El.Wahab, M. M. K. El-Shafei and H. B. Abd El-Wahab. A proposed Approach for solving Rough Bi – Level Programming Problems by Genetic Algorithm, International Journal of Contemporary Mathematical science, 6, 29 – 32, 2011.

[6] Y. Y. Yao. A comparative Study of Fuzzy Sets and Rough Sets, International Journal of Information Sciences, 109, 227 – 242, 2004.

[7] Z. Pawlak. Rough sets, International Journal of Computer and Information Science, 11, 341 – 356, 1982.

[8] Z. Pawlak. Rough sets, Rough Relations and Rough Functions, Fundamental Inform, 27(213), 103 – 108, 1996.

[9] Z. Pawlak. Vagueness, A rough set view, In : Structures in Logic and Computer Science, Springer, 106 – 117, 1997.

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