UNIQUENESS OF MEROMORPHIC FUNCTIONS

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ABSTRACT

In this paper, we investigate the uniqueness of meromorphic functions concerning differential polynomials with weighted sharing method. Also study the uniqueness of meromorphic functions sharing a small function and a positive answer is given to the open problem posed by Dyavanal[11].

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1. INTRODUCTION AND MAIN RESULTS

In this paper, meromorphic function means meromorphic in the complex plane. We adopt the standard notations in Nevanlinna theory of meromorphic functions as explained in [1,2]. Let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function \( f \), we denote \( T(r, f) \) the Nevanlinna characteristic of \( f \) and \( S(r, f) \) any quantity satisfying

\[
S(r, f) = o(T(r, f)) \quad (r \to \infty, r \notin E).
\]

Let \( f \) and \( g \) be two nonconstant meromorphic functions, and let \( a \) be a finite value. We say that \( f \) and \( g \) share the value \( a \) CM, provided that \( f - a \) and \( g - a \) have the same zeros with same multiplicities. Similarly, we say that \( f \) and \( g \) share the value \( a \) IM, provided that \( f - a \) and \( g - a \) have the same zeros with ignoring multiplicities. For convenience, we give following notations and definitions.

For any constant \( a \), we denote by \( N_k(r, \frac{1}{f-a}) \) the counting function for zeros of \( f(z) - a \) with multiplicity no more than \( k \) and \( \overline{N}_k(r, \frac{1}{f-a}) \) the corresponding for which multiplicity is not counted. Let \( N_k(r, \frac{1}{f-a}) \) be the counting function for zeros of \( f(z) - a \) with multiplicity at least \( k \) and \( \overline{N}(r, \frac{1}{f-a}) \) the corresponding for which the multiplicity is not counted.

Set

\[
N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}(2 \frac{1}{f-a}) + \cdots + \overline{N}(k \frac{1}{f-a})
\]

We define,

\[
\delta_k(a, f) = 1 - \limsup_{r \to \infty} \frac{N_k(r, \frac{1}{f-a})}{T(r, f)}
\]

\[
\Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)}
\]

Let \( l \) be non-negative integer or \( \infty \). For any \( a \in C \cup \infty \), we denote by \( E_l(a, f) \) the set of all \( a \)-points of \( f(z) \) where an \( a \)-points of multiplicity \( m \) is counted \( m \) times if \( m \leq l \) and \( l + 1 \) times if \( m > l \). If \( E_l(a, f) = E_l(a, g) \), we say that \( f \) and \( g \) share the value \( a \) with weight \( l \). When \( l = 0, f \) and \( g \) share \( 1 \) IM.[8]

In 2007, Bhoosnurmath and Dyavanal[3] proved the following theorem.

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Theorem A. Let $f$ and $g$ be two nonconstant meromorphic functions, and $n, k$ be two positive integers with $n > 3k + 8$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM then either $f = tg$ for some $n^{th}$ root of unity or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c, c_1$ and $c_2$ are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

Theorem B. Let $f$ and $g$ be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{3}{n+1}$ and let $n, k$ be two positive integers with $n \geq 3k + 13$. If $[f^n(f - 1)]^{(k)}$ and $[g^n(g - 1)]^{(k)}$ share 1 CM then $f(z) \equiv g(z)$


Theorem C. Let $f$ and $g$ be two trancendental meromorphic functions and let $n, k$ be two positive integers with $n > 9k + 14$. Suppose $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a nonzero constant b IM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c, c_1$ and $c_2$ are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$ or $f = tg$ for some $n^{th}$ root of unity.

In 2010, Pulak Sahoo[5] obtained the following result.

Theorem D. Let $f$ and $g$ be two trancendental meromorphic functions and let $n(\geq 1), k(\geq 1)$ and $m(\geq 0)$ be three integers. Let $[f^n(f - 1)^m]^{(k)}$ and $[g^n(g - 1)^m]^{(k)}$ share 1 IM. Then one of the following holds:

i) when $m = 0$, if $f(z) \neq \infty$, $g(z) \neq \infty$ and $n > 9k + 14$, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c, c_1$ and $c_2$ are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant $t$ such that $t^n = 1$.

ii) when $m = 1$, $n > 9k + 20$ and $\Theta(\infty, f) > \frac{2}{n}$, the either $[f^n(f - 1)^m]^{(k)}[g^n(g - 1)^m]^{(k)} \equiv 1$ or $f = g$.

iii) when $m \geq 2, n > 9k + 4m + 16$ then either $[f^n(f - 1)^m]^{(k)}[g^n(g - 1)^m]^{(k)} \equiv 1$ or $f = g$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$ where $R(x, y) = x^n (x - 1)^m - y^n (y - 1)^m$.

In 2011, Xiao Bin Zhang, JunFeng Xu[6] considered more general differential polynomial and obtained the following theorem:

Theorem E. Let $f$ and $g$ be two non constant meromorphic functions and $a(z)(\neq 0, \infty)$ be small function with respect to $f$. Let $n, k$ and $m$ be three positive integers with $n > 3k + m + 7$ and $P(\omega) = a_m \omega^m + a_{m-1} \omega^{m-1} + \cdots + a_0$ where $a_m \neq 0, a_i \cdots a_{m-l}, a_m \neq 0$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM, $f$ and $g$ share $\infty$ IM, then

i) $f(z) = tz$ for a constant $t$ such that $t^d = 1$.

where $d = \text{GCD}(n + m, \ldots, n + m - i, \ldots, n), a_{m-i} \neq 0$, for some $i = 0, 1, \ldots, m$.

ii) $f$ and $g$ satisfy the algebraic equation $R(f, g) = 0$.

where $R(\omega_1, \omega_2) = \omega_1^m (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_0) - \omega_2^m (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_0)$.

iii) $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = a^2$.

In 2009, using the notion of weighted sharing of values, Hong yan Xu and Ting Bin Cao[7] obtained following result.

Theorem F. Let $f$ and $g$ be two nonconstant entire functions and let $m, n$ and $k$ be three positive integers. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share

i) $(1, 0)$ with $n \geq 5m + 5k + 8$

ii) $(1, 1)$ with $n \geq \frac{9}{2} m + 4k + 9$

iii) $(1, 2)$ with $n \geq 3m + 3k + 5$

(1) when $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, then either $f = tg$, for a constant $t$ such that $t^d = 1$ where $d = (n + m, \ldots, n + m - i, \ldots, n), a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$ or $f$ and $g$ satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^m (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_0) - \omega_2^m (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_0)$.
(2) When $P(z) = 0$, then either $f = \frac{c_1}{\sqrt{c_2}e^{cz}}$, $g = \frac{c_2}{\sqrt{c_1}e^{-cz}}$. where $c_1$, $c_2$ and $c$ are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for some constant $t$ such that $t^n = 1$.

In this paper with the notion of weighted sharing of values, we investigate result for meromorphic function.

**Theorem 1.** Let $f$ and $g$ be two nonconstant transcendental meromorphic functions and let $n(\geq 1), k(\geq 1), l(\geq 0)$ be three integers. Let $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots, a_{m-1}, a_m \neq 0$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ and $f$ then either

i) $l \geq 2$ and $n > 3k + 2m^* + m + 8$

ii) $l = 1$ and $n > 5k + 2m^* + m + 11$

iii) $l = 0$ and $n > 9k + 2m^* + 4m + 14$

then either

$f = tg$, for a constant $t$ such that $t^n = 1$ where $d = (n + m, n + m - i, \ldots, n), a_{m-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m$ or $f$ and $g$ satisfy the algebraic equation $R(f, g) = 0$,

where $R(\omega_1, \omega_2) = \omega_1^l(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^l(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

**Theorem 2.** Let $f$ and $g$ be two nonconstant entire functions and $n$, $m$ and $k$ be three positive integers. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ and $f$

i) $l \geq 2$ and $n > 2k + m + 2m^* + 3$

ii) $l = 1$ and $n > 3k + 3m + 2m^* + 5$

iii) $l = 0$ and $n > 5k + 4m + 2m^* + 7$

then conclusion of Theorem 1 still holds.

In 2004, Lin and Yi [12] proved the following theorems.

**Theorem G.** Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 12$ an integer. If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share the value 1 CM, then $g = (n + 2)(1 - h^{n+1})/(n + 1)(1 - h^{n+2})$,

$f = (n + 2)h(1 - h^{n+1})/(n + 1)(1 - h^{n+2})$, where $h$ is a nonconstant meromorphic function.

**Theorem H.** Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 13$ an integer. If $f^n(f - 1)^2f'$ and $g^n(g - 1)^2g'$ share the value 1 CM, then $f(z) = g(z)$.


**Theorem I.** Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least $s$, where $s$ is a positive integer. Let $n \geq 2$ be an integer satisfying $(n + 1)s \geq 12$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$ for some $(n+1)$ th root of unity $d$ or $f(z) = c_2e^{cz}$ and $g(z) = c_1e^{cz}$, where $c_1$ and $c_2$ are constants satisfying $(c_1c_2)^{(n+1)s}c_1^2 = 1$

**Theorem J.** Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least $s$, where $s$ is a positive integer. Let $n$ be an integer satisfying

$(n - 2)s \geq 10$.

If $f^n f'$ and $g^n g'$ share the value 1 CM, then $g = (n + 2)(1 - h^{n+1})/(n + 1)(1 - h^{n+2})$,

$f = (n + 2)h(1 - h^{n+1})/(n + 1)(1 - h^{n+2})$, where $h$ is a non-constant meromorphic function.

**Theorem K.** Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least $s$, where $s$ is a positive integer. Let $n$ be an integer satisfying $(n - 3)s \geq 10$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then $f \equiv g$.

At the end of this paper [11], she posed the question: Can the differential polynomials in theorems I, J and K be replaced by the differential polynomials of the form $[f^n]^{(k)}$ and $[f^n f - 1]^{(k)}$?
In this paper we consider more general differential polynomial of the form \( f^n P(f)^{(k)} \), where \( P(f) \) is as defined in Theorem 1, and give answer to open question (4.4) of [11]

**Theorem 3.** Let \( f \) and \( g \) be trancendental meromorphic functions, whose zeros and poles are of multiplicity at least \( s \). where \( s \) is a positive integer.\( a(z) (\neq 0, \infty) \) be a small function with respect to \( f \) with finitely many zeros and poles. Let \( n, k \) and \( m \) be three positive integers satisfying \( (n-m)s > 3k +7 \). If \( [f^n P(f)]^{(k)} \) and \( [g^n P(g)]^{(k)} \) share \( a \) CM and \( f \) and \( g \) share \( \infty \) IM, then one of the following cases holds:

i)\( f(z) = tg(z) \) for a constant \( t \) such that \( t^d = 1 \)

where \( d = (n+m, ..., n+m-i, ..., n) a_{m-i} \neq 0 \) for some \( i = 0, 1, ..., m \)

ii) \( f \) and \( g \) satisfy the algebraic equation \( R(f,g) = 0 \),

where, \( R(\omega_1, \omega_2) = a_{n}^{\omega_2}(a_{n} \omega_1^{m} + a_{n-1} \omega_1^{m-1} + \cdots + a_0) - \omega_2^{\omega_2}(a_{n} \omega_2^{m} + a_{n-1} \omega_2^{m-1} + \cdots + a_0) \)

**Remark:** We set \( P(z) = (z - 1)^m \). With \( a_m = 1, a_0 = -1 \) and under condition (ii) of theorem 3, we have following important results.

i) When \( m = 0, ns > 3k + 7 \) and if \( [f^n]^{(k)} \) and \( [g^n]^{(k)} \) share a CM and \( \infty \) IM then either \( f(z) = c_1 e^{cz} \) and \( g(z) = c_2 e^{-cz} \), where \( c_1 \) and \( c_2 \) are constants satisfying \((-1)^k(c_1 c_2)^{n} (nc)^{k^3} = 1 \) or \( f = tg \) for a constant \( t \) such that \( t^n = 1 \).

ii) When \( m = 1, (n-1)s > 3k + 7 \) and if \( [f^n(f - 1)]^{(k)} \) and \( [g^n(g - 1)]^{(k)} \) share a CM and \( \infty \) IM then \( f \equiv g \).

iii) When \( m \geq 2, (n-2)s > 3k + 7 \) and if \( [f^n(f - 1)^m]^{(k)} \) and \( [g^n(g - 1)^m]^{(k)} \) share a CM and \( \infty \) IM then \( f \) and \( g \) satisfy the algebraic equation \( R(f,g) = 0 \), where \( R(\omega_1, \omega_2) = a_{n}^{\omega_2}(\omega_1 - 1)^m - \omega_2^{\omega_2}(\omega_2 - 1)^m \).

**Remarks (i), (ii) and (iii) give answers to open problem (4.4) of [11].

2. LEMMAS

In order to prove our results, we need the following lemmas.

**Lemma 1 [1].** Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \), where \( a_n (\neq 0) \), \( a_{n-1}, ..., a_0 \) are constants. If \( f(z) \) is a meromorphic function, then

\[
T(r, P(f)) = nT(r, f) + S(r, f).
\]

**Lemma 2 [12].** Let \( f(z) \) a nonconstant meromorphic and \( p, k \) be two positive integer. Then

\[
N_p(r, f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 1) + S(r, f)
\]

\[
N_p(r, f^{(k)}) \leq k\overline{N}(r, f) + N_{p+k}(r, 1) + S(r, f)
\]

**Lemma 3 [2].** Let \( f(z) \) be nonconstant meromorphic functions and \( k \) be a positive integer.Suppose that \( f^{(k)} \neq 0 \), then

\[
N(r, f^{(k)}) \leq N(r, f) + k\overline{N}(r, f) + S(r, f)
\]

**Lemma 4 [6].** Let \( f(z) \) and \( g(z) \) two be nonconstant meromorphic function and \( n, k \) be two positive integers and \( a \) be a finite nonzero constant. If \( f(z) \) and \( g(z) \) share \( a \) CM and \( \infty \) IM, then one of the following cases holds:

i) \( \overline{T}(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + 3\overline{N}(r, f) + S(r, f) + S(r, g) \) the same inequality holds for \( T(r,g) \);

ii) \( \overline{T}(r,f) = 0 \) if \( g = a^2 \);

iii) \( f \equiv g \).

**Lemma 5 [13].** Let \( f(z) \) and \( g(z) \) two be nonconstant meromorphic functions, \( k(\geq 1), l(\geq 0) \) be two integers.Suppose that \( f^{(k)} \) and \( g^{(k)} \) share \( (1, l) \). If one of the following conditions holds,

i) \( l \geq 2 \) and \( \Delta_l = 2\theta(\infty, f) + (k + 2)\theta(\infty, g) + \theta(0, f) + \theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > (k + 7) \)
Therefore, $\Delta_2 = (k + 3)\theta(\infty, f) + (k + 2)\theta(\infty, g) + \theta(0, f) + \theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 2k + 9$

\[\text{ii) } l = 0 \text{ and } \Delta_2 = (2k + 4)\theta(\infty, f) + (2k + 3)\theta(\infty, g) + \theta(0, f) + \theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 4k + 13 \text{ then either } f^{(k)}g^{(k)} \equiv 1 \text{ or } f(z) = g(z).\]

Taking $N(r, f) = N(r, g) = 0$ and proceeding as in lemma 6[12], we get following lemma.

**Lemma 6.** Let $f(z)$ and $g(z)$ two be nonconstant entire functions, $k(\geq 1), l(\geq 0)$ be two integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share (1,1). If one of the following conditions holds,

\[\text{i) } l \geq 2 \text{ and } \Delta_2 = \theta(0, f) + \theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 3\]

\[\text{ii) } l = 1 \text{ and } \Delta_2 = \theta(0, f) + \theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 4\]

\[\text{iii) } l = 0 \text{ and } \Delta_2 = \theta(0, f) + \theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 6 \text{ then either } f^{(k)}g^{(k)} \equiv 1 \text{ or } f(z) = g(z).\]

**Lemma 7[6].** Let $f$ and $g$ be two nonconstant meromorphic functions, let $n$ and $k$ be two integers with $n > k + 2$, let $P(z) = a_mz^n + a_{m-1}z^{n-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots, a_{m-1}, a_m \neq 0$ are constants, and let $a(z)(\neq 0, \infty)$ be small function with respect to $f$ with finitely many zeros and poles.

If $[f^nP(f)][g^nP(g)]^{(k)} = a^2$ and $f$ and $g$ share $\infty$ IM, then $P(z)$ is reduced to a nonzero monomial, namely, $P(z) = a_iz^i \neq 0$ for some $i = 0, 1, \ldots, m$.

### 3. PROOF OF THEOREMS

**Proof of theorem 1.**

Let $F = f^nP(f)$ and $G = g^nP(g)$.

Then we have,

\[\theta(0, f) = 1 - \lim_{r \to \infty} \frac{N(r, 1/F)}{T(r, F)} = 1 - \lim_{r \to \infty} \frac{N(r, 1/f^n) + N(r, 1/P(f))}{T(r, F)} \geq 1 - \frac{(1 + m^*)T(r, f)}{(n + m)T(r, f)},\]

Therefore,

\[\theta(0, f) \geq \frac{n + m - 1 - m^*}{n + m}\]

where $m^* = 0$ if $m = 0$ and $m^* = 1$ if $m \geq 1$.

Similarly,

\[\theta(0, G) \geq \frac{n + m - 1 - m^*}{n + m}\]

Next we have,

\[\delta_{k+1}(0, f) = 1 - \lim_{r \to \infty} \frac{N_{k+1}(r, 1/F)}{T(r, F)} \geq 1 - \lim_{r \to \infty} \frac{(k+1)T(r, 1/f^n) + N_{k+1}(r, 1/P(f))}{(n + m)T(r, f)} \geq 1 - \lim_{r \to \infty} \frac{(k+1)T(r, 1/F)}{(n + m)T(r, f)} \geq 1 - \frac{m + k + 1}{n + m}\]

Therefore,

\[\delta_{k+1}(0, f) \geq \frac{n - k - 1}{n + m}\]

Similarly, $\delta_{k+1}(0, G) \geq \frac{n - k - 1}{n + m}$.

We have,

\[\theta(\infty, f) = 1 - \lim_{r \to \infty} \frac{T(r, F)}{T(r, F)} = 1 - \lim_{r \to \infty} \frac{T(r, f)}{(n + m)T(r, f)} \geq 1 - \frac{T(r, f)}{(n + m)T(r, f)}\]

Therefore,

\[\theta(\infty, f) \geq \frac{n + m - 1}{n + m} \leq \text{Proof of theorem 1.}\]
Since $F^{(k)}$ and $G^{(k)}$ share $(1, l)$ we consider following three cases.

**Case 1:** Let $l \geq 2$

$$\delta_1 = (k + 2)\theta(\infty, G) + (k + 3)\theta(\infty, F) + \theta(0, F) + \theta(0, G) + 2\delta_{k+1}(0, F) + \delta_{k+1}(0, G)$$

$$\geq (k + 4)\left(\frac{n + m - 1}{n + m}\right) + 2\left(\frac{n + m - 1 - m^*}{n + m}\right) + 3\left(\frac{n + m - 1}{n + m}\right)$$

$$= (k + 4)\left(1 - \frac{1}{n + m}\right) + 2\left(1 - \frac{1}{n + m}\right) + 3\left(\frac{n + m - 1}{n + m}\right)$$

$$= (k + 6) - \left(\frac{k + 4}{n + m} + 2 + 2m^* + \frac{n + m - 1}{n + m}\right)$$

$$= (k + 6) - \left(\frac{3k + 2m^* + 2 + 2m^* + 8}{n + m}\right)$$

From (i) of lemma (5), we have $n + m \leq 3k + 2m^* + 2m + 8$ i.e $n \leq 3k + 2m^* + m + 8$

which contradicts our hypothesis that $n > 3k + 2m^* + m + 8$.

By lemma (5), we have either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

**Case 2:** Let $l = 1$

$$\delta_2 = (k + 2)\theta(\infty, G) + (k + 3)\theta(\infty, F) + \theta(0, F) + \theta(0, G) + 2\delta_{k+1}(0, F) + \delta_{k+1}(0, G)$$

$$\geq (2k + 5)\left(\frac{n + m - 1}{n + m}\right) + 2\left(\frac{n + m - 1 - m^*}{n + m}\right) + 3\left(\frac{n + m - 1}{n + m}\right)$$

$$= (2k + 5)\left(1 - \frac{1}{n + m}\right) + 2\left(1 - \frac{1}{n + m}\right) + 3\left(\frac{n + m - 1}{n + m}\right)$$

$$= (2k + 7) - \left(\frac{2k + 5}{n + m} + 2 + 2m^* + \frac{3(n - k - 1)}{n + m}\right)$$

$$= (2k + 7) - \left(\frac{5k + 2m^* + 3m + 10}{n + m}\right)$$

From (ii) of lemma (5), we have $n \leq 5k + 2m^* + 2m + 10$

which contradicts our hypothesis that $n > 5k + 2m^* + m + 10$.

By lemma (5), either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

**Case 3:** Let $l = 0$

$$\delta_3 = (k + 2)\theta(\infty, F) + (k + 3)\theta(\infty, G) + \theta(0, F) + \theta(0, G) + 3\delta_{k+1}(0, F) + 2\delta_{k+1}(0, G)$$

$$\geq (4k + 7)\left(\frac{n + m - 1}{n + m}\right) + 2\left(\frac{n + m - 1 - m^*}{n + m}\right) + 5\left(\frac{n + m - 1}{n + m}\right)$$

$$= (4k + 7)\left(1 - \frac{1}{n + m}\right) + 2\left(1 - \frac{1}{n + m}\right) + 5\left(\frac{n + m - 1}{n + m}\right)$$

$$= (4k + 9) - \left(\frac{4k + 7}{n + m} + 2 + 2m^* + \frac{5(n - k - 1)}{n + m}\right)$$

$$= (4k + 9) - \left(\frac{9k + 2m^* + 5m + 14}{n + m}\right)$$

From (iii) of lemma (5), we have $n \leq 9k + 2m^* + 4m + 14$

which contradicts our hypothesis that $n > 9k + 2m^* + m + 14$.

By lemma (5), either $F^{(k)}G^{(k)} \equiv 1$, or $F \equiv G$.

Suppose $F^{(k)}G^{(k)} \equiv 1$ then by lemma (7), $P(z)$ as defined in Theorem 1 reduces to a nonzero monomial. That is

$$P(z) = a_{z^i} \neq 0 \text{ for some } i = 0, 1, 2, ..., m.$$  

By hypothesis of theorem (1), we arrive at a contradiction.

Hence we deduce that $F(z) \equiv G(z)$, that is

$$f^n(a_m f^m + a_{m-1} f^{m-1} + ... + a_0) = g^n(a_m g^m + a_{m-1} g^{m-1} + ... + a_0)$$

Let $h = f/g$. If $h$ is a constant then substituting $f = gh$, we deduce,

$$a_m g^n + (h + m - 1) + a_{m-1} g^{n+m-1} + n+m-1 + ... + a_0 g^n (h^n - 1) = 0$$

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which implies that $h^{d} = 1$ where $d = (n + m, ..., n + m - i, ..., n) \cdot a_{m-i} \neq 0$ for some $i = 0,1, ..., m$

Thus $f(z) = t g(z)$ for a constant $t$ such that $t^{d} = 1$,

where $d = (n + m, ..., n + m - i, ..., n) \cdot a_{m-i} \neq 0$, for some $i = 0,1, ..., m$.

If $h$ is not a constant then $f$ and $g$ satisfy the algebraic equation $R(f,g) = 0$,

where $R(\omega_{1}, \omega_{2}) = \omega_{1}^{2}(a_{m}a_{1}^{m} + a_{m-1} \omega_{1}^{m-1} + \cdots + a_{0}) - \omega_{2}^{2}(a_{m}a_{2}^{m} + a_{m-1} \omega_{2}^{m-1} + \cdots + a_{0})$

This proves the theorem.

**Proof of theorem 2.**

Since $f$ and $g$ are entire functions $N(r, f) = N(r, g) = 0$. Proceeding as in theorem 1 and using lemma (5), we easily prove theorem 2.

**Proof of theorem 3.**

Let $F = [f^{n} P(f)]^{(k)}$, $G = [g^{n} P(g)]^{(k)}$, $F_{1} = F/a$, $G_{1} = G/a$, $F_{i} = f^{n} P(f)$, $G_{i} = g^{n} P(g)$

then by hypothesis $F_{1}$ and $G_{1}$ share 1 CM.

By case(i) of lemma (4), we have

$$T(r,F) \leq N_{2}(r, \frac{1}{G}) + N_{2}(r, \frac{1}{G}) + 3\overline{N}(r, F) + S(r, F) + S(r, G) \tag{1}$$

By lemma (2), with $p = 2$, we obtain,

$$T(r, F^{*}) \leq T(r, F) - N_{2}(r, \frac{1}{G}) + N_{k+2}(r, \frac{1}{F}) + S(r, F) \tag{2}$$

$$N_{2}(r, \frac{1}{G}) \leq N_{k+2}(r, \frac{1}{F_{1}}) + k\overline{N}(r, G) + S(r, G) \tag{3}$$

By (1) and (2), we have

$$T(r, F^{*}) \leq N_{2}\left(r, \frac{1}{G}\right) + 3\overline{N}(r, F) + N_{k+2}\left(r, \frac{1}{F_{1}}\right) + S(r, F) + S(r, G)$$

using (3), we get

$$T(r, F^{*}) \leq N_{k+2}(r, \frac{1}{G}) + k\overline{N}(r, G) + 3\overline{N}(r, F) + N_{k+2}(r, \frac{1}{F_{1}}) + S(r, F) + S(r, G)$$

$$\leq (k + 2)\overline{N}\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{F_{1}}\right) + k\overline{N}(r, G) + 3\overline{N}(r, F) + (k + 2)\overline{N}\left(r, \frac{1}{F_{1}}\right) + N\left(r, \frac{1}{F_{1}}\right) + S(r, F) + S(r, G)$$

By our assumption, zeros and poles are of multiplicities at least $s$, that is, $\overline{N}(r, G) \leq \frac{1}{s}N(r, G) \leq \frac{1}{s}T(r, G)$, and we deduce the above inequality as,

$$T(r, F^{*}) \leq \left(\frac{k + 2}{s}\right)T(r, g) + mT(r, g) + \frac{k}{s}T(r, g) + \frac{3}{s}T(r, f) + \left(\frac{k + 2}{s}\right)T(r, f) + mT(r, f) + S(r, F) + S(r, G)$$

$$\leq \left(\frac{k + 2}{s} + \frac{3}{s} + m\right)T(r, f) + \left(\frac{k + 2}{s} + \frac{3}{s} + m\right)T(r, g) + S(r, F) + S(r, G)$$

$$(n + m)T(r, f) \leq \left(\frac{ms + k + 5}{s}\right)T(r, f) + \left(\frac{2k + ms + 2}{s}\right)T(r, g) + S(r, F) + S(r, G)$$

$$\frac{(ns - k - 5)}{s}T(r, f) \leq \frac{(2k + ms + 2)}{s}T(r, g) + S(r, F) + S(r, G)$$

Similarly,

$$\frac{(ns - k - 5)}{s}T(r, g) \leq \frac{(2k + ms + 2)}{s}T(r, f) + S(r, F) + S(r, G)$$

$$\frac{(ns - k - 5)}{s}(T(r, f) + T(r, g)) \leq (2k + ms + 2)(T(r, f) + T(r, g) + S(r, f) + S(r, g))$$
\[(ns - ms - 3k - 7)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)\]

which contradicts \((n - m)s > 3k + 7\)

Therefore by Lemma(4), either \(F^{(k)}G^{(k)} \equiv 1\) or \(F \equiv G\).

Proceeding as in proof of theorem 1 we obtain theorem 3.

6. ACKNOWLEDGMENT

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