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UNIQUENESS OF MEROMORPHIC FUNCTIONS

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ABSTRACT

In this paper,we investigate the uniqueness of meromorphic functions concerning differential polynomials with weighted sharing method.Also study the uniqueness of meromorphic functions sharing a small function and a positive answer is given to the open problem possed by Dyavanal[11].

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1. INTRODUCTION AND MAIN RESULTS

In this paper, meromorphic function means meromorphic in the complex plane.We adopt the standard notations in Nevanlinna theory of meromorphic functions as explained in [1,2]. Let E denote any set of positive real numbers of finite linear measure,not necessarily the same at each occurence.For a nonconstant meromorphic function f,we denote $T(r,f)$ the Nevanlinna characteristic of $f$and $S(r,f)$ any quantity satisfying

$S(r,f) = o\{T(r,f)\}\ (r \to \infty, r \neq E)$.

Let f and g be two nonconstant meromorphic functions,and let a be a finite value. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with same multiplicities.Similiarly,we say that f and g share the value a IM,provided that $f-a$ and $g-a$ have the same zeros with ignoring multiplicities.

For convenience ,we give following notations and definitions .

For any constant a,we denote by $N_{k)}(r,\frac{1}{f-a})$ the counting function for zeros of $f(z)-a$ with multiplicity no more than k and $\overline{N}_{k)}(r,\frac{1}{f-a})$ the corresponding for which multiplicity is not counted.Let $N_{(k}(r,\frac{1}{f-a})$ be the counting function for zeros of $f(z)-a$ with multiplicity atleast k and $\overline{N}_{(k}(r,\frac{1}{f-a})$ the corresponding for which the multiplicity is not counted.

Set $N_k(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}_{(2}(r,\frac{1}{f-a}) + \cdots + \overline{N}_{(k}(r,\frac{1}{f-a})$

We define,

$$\delta_k(a,f) = 1 - limsup_{r\to\infty}\frac{N_k(r,\frac{1}{f-a})}{T(r,f)}$$

$$\Theta(a,f) = 1 - limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}$$

Let l be non-negative integer or ∞. For any $a \in C \cup \infty$, we denote by $E_l(a,f)$ the set of all a-points of $f(z)$ where an a-points of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$.If $E_l(a,f) = E_l(a,g)$, we say that f and g share the value a with weight l. When $l = 0, f$ and g share 1 IM.[8]

In 2007, Bhoosnurmath and Dyavanal[3] proved the following theorem.

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Theorem A. Let f and g be two nonconstant meromorphic functions, and n, k be two positive integers with $n > 3k + 8$.If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM then either $f = tg$ for some n^{th} root of unity or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$

Theorem B. Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{3}{n+1}$ and let n, k be two positive integers with $n \geq 3k + 13$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM,then $f(z) \equiv g(z)$

In 2008, A Banerjee[4] proved the following theorem.

Theorem C. Let f and g be two trancedental meromorphic functions and let n,k be two positive integers with $n > 9k + 14$.Suppose $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a nonzero constant b IM,then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$ or $f = tg$ for some n^{th} root of unity.

In 2010, Pulak Sahoo[5] obtained the following result.

Theorem D. Let f and g be two trancedental meromorphic functions and let $n(\geq 1)$,$k(\geq 1)$ and $m(\geq 0)$ be three integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 IM.Then one of the following holds:

i) when $m = 0$,if $f(z) \neq \infty$, $g(z) \neq \infty$ and $n > 9k + 14$, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c, c_1$and c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

ii) when $m = 1$, $n > 9k + 20$ and $\Theta(\infty, f) > \frac{2}{n}$, the either $[f^n(f-1)^m]^{(k)}[g^n(\mathrm{g}-1)^m]^{(k)} \equiv 1$ or $f = g$.

iii) when $m \geq 2$, $n > 9k + 4m + 16$ then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f = g$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$ where $R(x, y) = x^n(x-1)^m - y^n(y-1)^m$.

In 2011, Xiao Bin Zhang,JunFeng Xu[6] considered more general differential polynomial and obtained the following theorem:

Theorem E. Let f and g be two non constant meromorphic functions and $a(z)(\neq 0, \infty)$ be small function with respect to f. Let n, k and m be three positive integers with $n > 3k + m + 7$ and $P(\omega) = a_m \omega^m + a_{m-1}\omega^{m-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots a_{m-1}, a_m \neq 0$ are complex constants.If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM, f and g share ∞ IM, then

i) $f(z) = tz$ for a constant t such that $t^d = 1$,

where $d = GCD(n + m, \cdots, n + m - i, \cdots, n), a_{m-i} \neq 0$, for some $i = 0,1, \cdots, m$

ii) f and g satisfy the algebraic equation $R(f, g) = 0$,

where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

iii)$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = a^2$.

In 2009, using the notion of weighted sharing of values, Hong yan Xu and Ting Bin Cao[7] obtained following result.

Theorem F. Let f and g be two nonconstant entire functions and let m, n and k be three positive integers. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share

i) $(1,0)$ with $n \geq 5m + 5k + 8$

ii) $(1,1)$ with $n \geq \frac{9}{2}m + 4k + \frac{9}{2}$

iii) $(1,2)$ with $n \geq 3m + 3k + 5$

(1) when $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, then either $f = tg$, for a constant t such that $t^d = 1$ where $d = (n + m, \ldots, n + m - i, \ldots, n), a_{m-i} \neq 0$ for some $i = 0,1,2, \ldots, m$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where, $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

(2) When $P(z) = 0$,then either $f = \frac{c_1}{\sqrt[n]{c_0}e^{cz}}$, $g = \frac{c_2}{\sqrt[n]{c_0}e^{-cz}}$. where c_1, c_2 and c are three constants satisfying

$(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for some constant t such that $t^n = 1$.

In this paper with the notion of weighted sharing of values,we investigate result for meromorphic function.

Theorem 1. Let f and g be two nonconstant transcendental meromorphic functions and let $n(\geq 1), k(\geq 1), l(\geq 0)$ be three integers.Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots a_{m-1}, a_m \neq 0$ are complex constants.If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1, l)$ and if

i) $l \geq 2$ and $n > 3k + 2m^* + m + 8$

ii) $l = 1$ and $n > 5k + 2m^* + m + 11$

iii) $l = 0$ and $n > 9k + 2m^* + 4m + 14$

then either
$f = tg$, for a constant t such that $t^d = 1$ where $d = (n + m, \ldots, n + m - i, \ldots, n), a_{m-i} \neq 0$ for some $i = 0,1,2, \ldots, m$ or f and g satisfy the algebraic equation $R(f, g) = 0$,

where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

Theorem 2. Let f and g be two nonconstant entire functions and n, m and k be three positive integers.If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1, l)$ and if

i) $l \geq 2$ and $n > 2k + m + 2m^* + 3$
ii) $l = 1$ and $n > 3k + 3m + 2m^* + 5$
iii) $l = 0$ and $n > 5k + 4m + 2m^* + 7$

then conclusion of Theorem 1 still holds.

In 2004, Lin and Yi[12] proved the following theorems.

Theorem G. Let f and g be two noncontant meromorphic funcitons,$n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $g = (n+2)(1-h^{n+1})/(n+1)(1-h^{n+2})$,

$f = (n+2)h(1-h^{n+1})/(n+1)(1-h^{n+2})$, where h is a nonconstant meromorphic function.

Theorem H. Let f and g be two noncontant meromorphic funcitons,$n \geq 13$ an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share the value 1 CM, then $f(z) = g(z)$.

In 2011, Renukadevi S Dyavanal [11] obtained following results.

Theorem I. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If f^nf' and g^ng' share the value 1 CM, then either $f = dg$ for some (n+1) th root of unity d or $f(z) = c_2e^{-cz}$ and $g(z) = c_1e^{cz}$, where c, c_1 and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$

Theorem J. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let n be an integer satisfying

$(n-2)s \geq 10$.

If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $g = (n+2)(1-h^{n+1})/(n+1)(1-h^{n+2})$, $f = (n+2)h(1-h^{n+1})/(n+1)(1-h^{n+2})$,where h is a non-constant meromorphic function.

Theorem K. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let n be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share the value 1 CM, then $f \equiv g$.

At the end of this paper[11],she posed the question:Can the differential polynomials in theorems I, J and K be replaced by the differential polynomials of the form $[f^n]^{(k)}$ and $[f^n(f-1]^{(k)}$?

In this paper we consider more general differential polynomial of the form $[f^n P(f)]^{(k)}$, where $P(f)$ is as defined in Theorem 1, and give answer to open question(4.4) of [11]

Theorem 3. Let f and g be trancendental meromorphic functions,whose zeros and poles are of multiplicity atleast s. where s is a positive integer.$a(z)(\neq 0,\infty)$ be a small function with respect to f with finitely many zeros and poles. Let n, k and m be three positive integers satisfying $(n-m)s > 3k+7$.If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM and f and g share ∞ IM, then one of the following cases holds:

i)$f(z) = tg(z)$ for a constant t such that $t^d = 1$

where $d = (n+m, \dots, n+m-i, \dots, n)$,$a_{m-i} \neq 0$ for some $i = 0,1,\dots,m$

ii)f and g satisfy the algebraic equation $R(f,g) = 0$,

where, $R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

Remark:We set $P(z) = (z-1)^m$. With $a_m = 1, a_0 = -1$ and under condition (ii) of theorem 3, we have following important results.

i) When $m = 0$, $ns > 3k+7$ and if $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a CM and ∞ IM then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$,where $c, c_1$and c_2 are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

ii) When $m = 1$, $(n-1)s > 3k+7$ and if $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share a CM and ∞ IM then $f \equiv g$.

iii) When $m \geq 2$, $(n-2)s > 3k+7$ and if $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share a CM and ∞ IM then f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1,\omega_2) = \omega_1^n(\omega_1-1)^m - \omega_2^n(\omega_2-1)^m$.

Remarks (i),(ii) and (iii) give answers to open problem (4.4) of [11].

2. LEMMAS

In order to prove our results, we need the following lemmas.

Lemma1 [1]. Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $a_n(\neq 0)$, $a_{n-1},\dots,a_0$ are constants. If $f(z)$ is a meromorphic function, then

$T(r,P(f)) = nT(r,f) + S(r,f)$.

Lemma 2[12]. Let $f(z)$ a nonconstant meromorphic and p,k be two positive integer. Then

$$N_p(r,\frac{1}{f^{(k)}}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f)$$
$$N_p(r,\frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f)$$

Lemma 3 [2]. Let $f(z)$ be nonconstant meromorphic functions and k be a positive integer.Suppose that $f^{(k)} \neq 0$, then

$$N(r,\frac{1}{f^{(k)}}) \leq N(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f)$$

Lemma 4[6]. Let $f(z)$ and $g(z)$ two be nonconstant meromorphic function and n,k be two positive integers and a be a finite nonzero constant.If $f(z)$ and $g(z)$ share a CM and ∞IM,then one of the following cases holds:

i)$T(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + 3\overline{N}(r,f) + S(r,f) + S(r,g)$the same inequality holds for T(r,g);
ii)$fg = a^2$,
iii)$f \equiv g$.

Lemma 5[13]. Let $f(z)$ and $g(z)$ two be nonconstant meromorphic functions, $k(\geq 1), l(\geq 0)$ be two integers.Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1,l)$. If one of the following conditions holds,

i) $l \geq 2$ and $\Delta_1 = 2\Theta(\infty,f) + (k+2)\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > (k+7)$

ii) $l = 1$ and $\Delta_2 = (k+3)\Theta(\infty, f) + (k+2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 2k + 9$
iii) $l = 0$ and $\Delta_3 = (2k+4)\Theta(\infty, f) + (2k+3)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 4k + 13$ then either $f^{(k)}g^{(k)} \equiv 1$ or $f(z) = g(z)$.

Taking $N(r, f) = N(r, g) = 0$ and proceeding as in lemma 6[12], we get following lemma.

Lemma 6. Let $f(z)$ and $g(z)$ two be nonconstant entire functions, $k(\geq 1), l(\geq 0)$ be two integers.Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1, l)$. If one of the following conditions holds,
i)$l \geq 2$ and $\Delta_4 = \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 3$
ii)$l = 1$ and $\Delta_5 = \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 4$
iii)$l = 0$ and $\Delta_6 = \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 6$ then either $f^{(k)}g^{(k)} \equiv 1$ or $f(z) = g(z)$.

Lemma 7[6]. let f and g be two nonconstant meromorphic functions, let n and k be two integers with $n > k + 2$, let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots a_{m-1}, a_m \neq 0$ are constants, and let $a(z)(\neq 0, \infty)$ be small function with respect to f with finitely many zeros and poles.

If $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = a^2$ and f and g share ∞ IM,then $P(z)$ is reduced to a nonzero monomial,namely,$P(z) = a_i z^i \neq 0$ for some i = 0,1,... , m.

3. PROOF OF THEOREMS

Proof of theorem 1.

Let $F = f^n P(f)$ and $G = g^n P(g)$

Then we have,

$$\Theta(0, f) = 1 - \overline{lim}_{r\to\infty} \frac{\overline{N}(r, 1/F)}{T(r, F)}$$
$$= 1 - \overline{lim} r \to \infty \frac{\overline{N}(r, 1/f^n) + \overline{N}(r, 1/P(f))}{T(r, F)}$$
$$\geq 1 - \frac{(1 + m^*)T(r, f)}{(n + m)T(r, f)},$$

Therefore, $\Theta(0, f) \geq \frac{n+m-1-m^*}{n+m}$

where $m^* = 0$ if $m = 0$ and $m^* = 1$ if $m \geq 1$

Similarly, $\Theta(0, G) \geq \frac{n+m-1-m^*}{n+m}$

Next we have ,

$$\delta_{k+1}(0, f) = 1 - \overline{lim}_{r\to\infty} \frac{N_{k+1}(r, 1/F)}{T(r, F)}$$
$$\geq 1 - \overline{lim}_{r\to\infty} \frac{(k+1)\overline{N}(r, 1/f) + N_{k+1}(r, 1/P(f))}{(n+m)T(r, f)}$$
$$\geq 1 - \overline{lim}_{r\to\infty} \frac{(k+1)\overline{N}\left(r, \frac{1}{f}\right) + mT(r, f)}{(n+m)T(r, f)}$$
$$\geq 1 - \frac{m+k+1}{n+m}$$

Therefore, $\delta_{k+1}(0, f) \geq \frac{n-k-1}{n+m}$

Similarly, $\delta_{k+1}(0, G) \geq \frac{n-k-1}{n+m}$

We have,
$$\Theta(\infty, f) = 1 - \overline{lim}_{r\to\infty} \frac{\overline{N}(r, F)}{T(r, F)}$$
$$= 1 - \overline{lim} r \to \infty \frac{\overline{N}(r, f)}{T(r, F)}$$
$$\geq 1 - \frac{T(r, f)}{(n+m)T(r, f)},$$

Therefore, $\Theta(\infty, f) \geq \frac{n+m-1}{n+m}$

Since $F^{(k)}$ and $G^{(k)}$ share $(1,l)$ we consider following three cases.

Case 1: Let $l \geq 2$,

$$\begin{aligned}\Delta_1 &= (k+2)\Theta(\infty,G) + 2\Theta(\infty,F) + \Theta(0,F) + \Theta(0,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G)\\ &\geq (k+4)\left(\frac{n+m-1}{n+m}\right) + 2\left(\frac{n+m-1-m^*}{n+m}\right) + 2\left(\frac{n-k-1}{n+m}\right)\\ &= (k+4)\left(1-\frac{1}{n+m}\right) + 2\left(1-\frac{1+m^*}{n+m}\right) + 2\left(\frac{n-k-1}{n+m}\right)\\ &= (k+6) - \left(\frac{k+4}{n+m} + \frac{2+2m^*}{n+m} - \frac{2(n-k-1)}{n+m}\right)\\ &= (k+8) - \left(\frac{3k+2m^*+2m+8}{n+m}\right)\end{aligned}$$

from(i) of lemma(5), we have $n+m \leq 3k+2m^*+2m+8$ i.e $n \leq 3k+2m^*+m+8$

which contradicts our hypothesis that $n > 3k+2m^*+m+8$.

By lemma (5),we have either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$

Case 2: Let $l = 1$

$$\begin{aligned}\Delta_2 &= (k+2)\Theta(\infty,G) + (k+3)\Theta(\infty,F) + \Theta(0,F) + \Theta(0,G) + 2\delta_{k+1}(0,F) + \delta_{k+1}(0,G)\\ &\geq (2k+5)\left(\frac{n+m-1}{n+m}\right) + 2\left(\frac{n+m-1-m^*}{n+m}\right) + 3\left(\frac{n-k-1}{n+m}\right)\\ &= (2k+5)\left(1-\frac{1}{n+m}\right) + 2\left(1-\frac{1+m^*}{n+m}\right) + 3\left(\frac{n-k-1}{n+m}\right)\\ &= (2k+7) - \left(\frac{2k+5}{n+m} + \frac{2+2m^*}{n+m} - \frac{3(n-k-1)}{n+m}\right)\\ &= (2k+10) - \left(\frac{5k+2m^*+3m+10}{n+m}\right)\end{aligned}$$

From (ii) of lemma(5), we have $n \leq 5k+2m^*+2m+10$

which contradicts our hypothesis that $n > 5k+2m^*+m+10$.

By lemma (5),either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Case 3: let $l = 0$

$$\begin{aligned}\Delta_3 &= (2k+4)\Theta(\infty,F) + (2k+3)\Theta(\infty,G) + \Theta(0,F) + \Theta(0,G) + 3\delta_{k+1}(0,F) + 2\delta_{k+1}(0,G)\\ &\geq (4k+7)\left(\frac{n+m-1}{n+m}\right) + 2\left(\frac{n+m-1-m^*}{n+m}\right) + 5\left(\frac{n-k-1}{n+m}\right)\\ &= (4k+7)\left(1-\frac{1}{n+m}\right) + 2\left(1-\frac{1+m^*}{n+m}\right) + 5\left(\frac{n-k-1}{n+m}\right)\\ &= (4k+9) - \left(\frac{4k+7}{n+m} + \frac{2+2m^*}{n+m} - \frac{5(n-k-1)}{n+m}\right)\\ &= (4k+14) - \left(\frac{9k+2m^*+5m+14}{n+m}\right)\end{aligned}$$

From (iii) of lemma(5), we have $n \leq 9k+2m^*+4m+14$

which contradicts our hypothesis that $n > 5k+2m^*+m+10$.

By lemma (5), either $F^{(k)}G^{(k)} \equiv 1$, or $F \equiv G$.

Suppose $F^{(k)}G^{(k)} \equiv 1$ then by lemma (7), $P(z)$ as defined in Theorem 1 reduces to a nonzero monomial. That is

$P(z) = a_i z^i \neq 0$ for some $i = 01,2,\dots,m$.

By hypothesis of theorem (1), we arrive at a contradiction.

Hence we deduce that $F(z) \equiv G(z)$, that is

$f^n(a_m f^m + a_{m-1}f^{m-1} \dots + a_0) = g^n(a_m g^m + a_{m-1}g^{m-1} \dots + a_0)$

Let $h = f/g$.If h is a constant then substituting $f = gh$, we deduce,

$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1}g^{n+m-1}(h^{n+m-1} - 1) + \cdots + a_0 g^n(h^n - 1) = 0$

which implies that $h^d = 1$ where $d = (n+m, \dots, n+m-i, \dots, n), a_{m-i} \neq 0$ for some $i = 0,1, \dots, m$

Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$,

where $d = (n+m, \dots, n+m-i, \dots, n), a_{m-i} \neq 0$, for some $i = 0,1, \dots, m$.

If h is not a constant then f and g satisfy the algebraic equation $R(f,g) = 0$,

where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

This proves the theorem.

Proof of theorem 2.
Since f and g are entire functions $N(r,f) = N(r,g) = 0$. Proceeding as in theorem 1 and using lemma (5) ,we easily prove theorem 2.

Proof of theorem 3.

Let $F = [f^n P(f)]^{(k)}$, $G = [g^n P(g)]^{(k)}$, $F_1 = F/a$, $G_1 = G/a$,
$F^* = f^n P(f)$, $G^* = g^n P(g)$

then by hypothesis F_1 and G_1 share 1 CM.

By case(i) of lemma (4),we have

$$T(r,F) \leq N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + 3\overline{N}(r,F) + S(r,F) + S(r,G) \tag{1}$$

By lemma(2),with $p = 2$,we obtain,

$$T(r,F^*) \leq T(r,F) - N_2(r,\tfrac{1}{F}) + N_{k+2}(r,\tfrac{1}{F^*}) + S(r,F) \tag{2}$$

$$N_2(r,\tfrac{1}{G}) \leq N_{k+2}(r,\tfrac{1}{G^*}) + k\overline{N}(r,G) + S(r,G) \tag{3}$$

By (1) and (2),we have

$$T(r,F^*) \leq N_2\left(r,\frac{1}{G}\right) + 3\overline{N}(r,F) + N_{k+2}\left(r,\frac{1}{F^*}\right) + S(r,F) + S(r,G)$$

using (3), we get

$$\begin{aligned} T(r,F^*) &\leq N_{k+2}(r,\frac{1}{G^*}) + k\overline{N}(r,G) + 3\overline{N}(r,F) + N_{k+2}(r,\frac{1}{F^*}) + S(r,F) + S(r,G) \\ &\leq (k+2)\overline{N}\left(r,\tfrac{1}{g}\right) + N\left(r,\tfrac{1}{P(G)}\right) + k\overline{N}(r,G) + 3\overline{N}(r,F) + (k+2)\overline{N}\left(r,\tfrac{1}{f}\right) + N(r,\tfrac{1}{P(f)}) + S(r,F) + S(r,G) \end{aligned}$$

By our assumption,zeros and poles are of multiplicities at least s, that is,
$\overline{N}(r,g) \leq \frac{1}{s}N(r,g) \leq \frac{1}{s}T(r,g)$ and we deduce the above inequality as,

$$\begin{aligned} T(r,F^*) &\leq \left(\frac{k+2}{s}\right)T(r,g) + mT(r,g) + \frac{k}{s}T(r,g) + \frac{3}{s}T(r,f) + (\frac{k+2}{s})T(r,f) + mT(r,f) + S(r,F) + S(r,G) \\ &\leq \left(\tfrac{k+2}{s} + \tfrac{3}{s} + m\right)T(r,f) + \left(\tfrac{k+2}{s} + \tfrac{k}{s} + m\right)T(r,g) + S(r,F) + S(r,G) \end{aligned}$$

$$(n+m)T(r,f) \leq \left(\frac{ms+k+5}{s}\right)T(r,f) + \left(\frac{2k+ms+2}{s}\right)T(r,g) + S(r,F) + S(r,G)$$

$$(ns-k-5)T(r,f) \leq (2k+ms+2)T(r,g) + S(r,F) + S(r,G)$$

Similarly,

$$(ns-k-5)T(r,g) \leq (2k+ms+2)T(r,g) + S(r,F) + S(r,G)$$

$$(ns-k-5)(T(r,f) + T(r,g)) \leq (2k+ms+2)(T(r,f) + T(r,g) + S(r,f) + S(r,g)$$

$(ns - ms - 3k - 7)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g)$

which contradicts $(n-m)s > 3k + 7$

Therefore by Lemma(4), either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Proceeding as in proof of theorem 1 we obtain theorem 3.

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