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ON SZEGED INDICES RELATED TO TENSOR PRODUCT OF STANDARD GRAPHS OF SAME CATEGORY

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ABSTRACT

A recently introduced graph – invariant is 'Szeged index' of a graph and it has considerable applications in molecular chemistry. In this paper, the Szeged indices related to the tensor product of standard graphs of same category are calculated.

Keywords: *Molecular graph, connected graph, Wiener Number, Szeged Index.*

1. INTRODUCTION

An important concept of a molecular graph associated with alkanes or more generally of a simple, connected graph is termed as the Wiener number (see [6]). A refined concept of this is coined as Szeged Index (see [2 & 3]). As in the case of Wiener number, no standard formula is available to calculate Szeged Index of a (connected) graph. In the succeeding articles, the Szeged indices of $K_m \Lambda K_n$, $C_m \Lambda C_n$ and $P_m \Lambda P_n$ (whenever possible) are obtained. Some interesting observations are also made.

For the standard notations and results, we refer Bondy and Murthy (see [1]). For any positive integer n, N_n stands for the set of all positive integers $\le$ n.

For ready reference, we give the following:

Definition 1.1[6]: G is a connected graph. Then the Wiener Number W (G) of G is defined to be $\frac{1}{2}\sum_{u,v\in V} d(u,v)$, where V = V(G) is the vertex set of G and d(u, v) = d_G(u, v) is the shortest distance between the vertices u, v of G.

Definition 1.2 [3]: Let G be a simple graph. Let e = uv be any edge of G. Denote

$N_1 [e \mid G] = \{ w \in V (G): d (w, u)) < d (w, v))\}$ (w is closer to u than to v in G),

$N_2 [e \mid G] = \{ w \in V (G): d (w, v)) < d (w, u))\}$ (w is closer to v than to u in G);

and
$n_1 (e|G) = \mid N_1 (e|G) \mid$, $n_2 (e|G) = \mid N_2 (e|G) \mid$ (| | denotes the cardinality function).

The Szeged index of G, denoted by Sz(G), is defined to be $\sum_{e\in E(G)} n_1(e/G).n_2(e/G)$ (E(G) being the edge-set of G).

(when there is only one graph G under consideration instead of e|G, we write 'e' only)

Consequences 1.3 [4]:

a) For the complete graph K_n (n $\ge$ 2), $Wz(K_n) = n(n-1)/2$.
b) For the path P_n (n $\ge$ 2), $Sz(P_n) = n(n^2 – 1)/6$.
c) For the cycle C_n (n $\ge$ 3), $Sz(C_n) = n\,[n/2]^2$ ([ ] denotes the integer part).
d) For the graph $K_{1,n}$, $Sz(K_{1,n}) = n^2$.
e) For the complete bipartite graph $K_{m,n}$, $Sz(K_{m,n}) = (mn)^2$.
f) For the wheel $K_1 \vee C_n$ (n $\ge$ 3), $Sz(K_1 \vee C_n) = n(n-2) + [n/2]^2$.

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Definition 1.4 [5]: G,H are disjoint, simple graphs. The tensor product of G and H, denoted by G Λ H (isomorphic to H Λ G) is the graph whose vertex set is V(G) x V(H) and edge set being, the set of all edges of the form $(u, v)\,(u', v')$, where $u, u' \in V(G)$, $v, v' \in V(H)$, $uu' \in E(G)$ and $vv' \in E(H)$.

OBSERVATION 1.5:

a) If one of G,H is an empty graph (i.e has no edges) then G Λ H is also empty.
b) If G, H are finite, simple graphs with m, n vertices respectively, then G Λ H is a finite, simple graph with mn vertices. Further, if $u \in V(G)$ and $v \in V(H)$ then $\deg_{G\Lambda H}(u, v) = \{\deg_G(u)\} \times \{\deg_H(v)\}$.

Result 1.6 [5]: G_1, G_2 are connected graphs, then $G_1 \Lambda G_2$ is connected if and only if (iff) either G_1 or G_2 contains an odd cycle.

Result 1.7 [5]: If G_1, G_2 are connected graph with no odd cycle, then $G_1 \Lambda G_2$ has exactly two components.

2. RESULTS RELATED TO $K_m \Lambda K_n$ (m, n being positive integers)

Initially we have,

Observation 2.1: If atleast one of m, n is 1, then $K_m \Lambda K_n$ is an empty graph. So we consider $m,n \geq 2$ and take disjoint graphs K_m, K_n.

Notation 2.2: Denote $V(K_m) = \{u_1, u_2,..., u_m\}$ and $V(K_n) = \{v_1,v_2,..., v_n\}$; then $V(K_m \Lambda K_n) = \{(u_i, v_j) : i = 1,2,..., m ; j = 1,2,..., n\}$ and the edge set being the set of all edges of the form $(u_i, v_j), (u_{i'}, v_{j'})$ where $i, i' \in \{1,2,..., m\}$; $j, j' \in \{1,2,..., n\}$ and $i' \neq i$, $j' \neq j$.

Result 2.3[4]: For $m, n \geq 2$, $K_m \Lambda K_n$ is a simple, finite and (m-1) (n-1) – regular graph with mn vertices and ½ mn (m-1) (n-1) edges.

Observation 2.4 [5]: $K_2 \Lambda K_2$ is a disconnected graph with two components (clearly it is 1-regular bipartite graph with four vertices).

A diagrammatic representation of $K_2 \Lambda K_2$ (in the usual notation) is

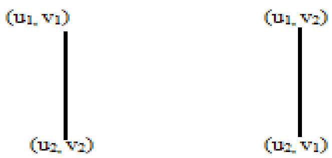

Figure – 1

Theorem 2.5: $K_2 \Lambda K_n$ ($n \geq 2$) is a connected, bipartite graph with $Sz(K_2 \Lambda K_n) = (n-1)\, n^3$. (It is a (n-1)-regular graph with 2n vertices and n (n-1) edges).

Proof: K_2, K_n are disjoint, connected graphs and K_n contains the odd cycle K_3; by result (1.6) follows that $K_2 \Lambda K_n$ is connected. Let $V(K_2) = \{u_1, u_2\}$ and $V(K_n) = \{v_1, v_2, ..., v_n\}$. Now $\{V_1, V_2\}$, where $V_1 = \{(u_1, v_j) : j = 1,2, ..., n\}$ and $V_2 = \{(u_2, v_j) : j = 1,2,..., n\}$ is a bipartition of the vertex set of $K_2 \Lambda K_n$. Thus $K_2 \Lambda K_n$ is a bipartite graph. By result (2.3), clearly this is a (n-1)-regular graph with 2n vertices and hence with n(n-1) edges. A diagrammatic representation of this graph is

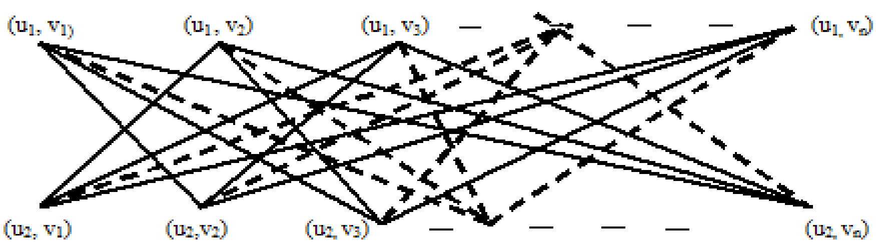

Figure – 2

For, $1 \le j < j' \le n$,

$N_1 [(u_1, v_j) (u_2, v_{j'})/(K_2 \Lambda K_n)] = \{(u_1, v_j), (u_1, v_{j'})\} \cup \{(u_2, v_k) : k \in N_n - \{j, j'\}\}$

and

$N_2 [(u_1, v_j) (u_2, v_{j'})/(K_2 \Lambda K_n)] = \{(u_1, v_k) : k = N_n - \{j, j'\}\} \cup \{(u_2, v_j), (u_2, v_{j'})\}$.

So $n_1 [(u_1, v_j) (u_2, v_{j'})] = 2 + (n-2) = n$ and $n_2[(u_1, v_j) (u_2, v_{j'})] = (n-2) + 2 = n$.

By definition,

$$\begin{aligned} Sz(K_2 \Lambda K_n) &= \sum_{j=1}^{n} \sum_{\substack{j'=1 \\ j' \neq j}}^{n} n_1[(u_{1,} v_j)(u_2, v_{j'})] n_2[(u_1, v_j)(u_2, v_{j'})] \\ &= \sum_{j=1}^{n} \sum_{\substack{j'=1 \\ j' \neq j}}^{n} (n)(n) \\ &= n^2 (n) (n-1) = (n-1)\, n^3 \end{aligned}$$

This completes the poof of the theorem.

OPEN PROBLEM 2.6: To find the Sz $(K_m \wedge K_n)$ for m, n ≥ 3.

3. RESULTS RELATED TO $C_m \wedge C_n$ (m, n being integers ≥ 3).

Since $C_3 = K_3$, it follows that $C_3 \wedge C_3 = K_3 \wedge K_3$ and this is discussed in **§**2.

Hence, we consider the case when one of m, n is ≥ 4. As usual, we consider disjoint graphs C_m & C_n and denote

$V (C_m) = \{u_1, u_2, \ldots, u_m\}$ & $V(C_n) = \{v_1, v_2, \ldots, v_n\}$.

Result 3.1[4]: $C_m \wedge C_n$ is a simple, 4-regular graph with mn vertices (and hence with 2mn edges).

Result 3.2[4]: $C_m \wedge C_n$ is a bipartite graph iff atleast one of m, n is even.

Remark 3.3[4]: Since C_m, C_n are connected graphs, by result (1.6), it follows that $C_m \wedge C_n$ is connected iff C_m or C_n contains an odd cycle $\Leftrightarrow$ atleast one of m, n is odd. Since C_m, C_n are connected graphs, when none of them contains an odd cycle, by result (1.7), it follows that $C_m \wedge C_n$ has exactly two components.

Thus, when both m and n are even $C_m \wedge C_n$ is a disconnected graph with two components.

For convenience, we first prove the following.

Theorem 3.4: For the disjoint graphs C_3, C_4, $Sz(C_3 \wedge C_4) = 2(3x4)(3x\left[\frac{4}{2}\right])^2 = 864 = 108(2)^3$.

Proof: Let $V (C_3) = \{u_1, u_2, u_3\}$ and $V(C_4) = \{v_1,v_2,v_3,v_4\}$.

So **V(**$C_3 \wedge C_4) = \{(u_i, v_j) : i=1, 2, 3; j=1,2,3,4\}$.

By the previous results, it follows that $C_3 \wedge C_4$ is a simple, connected, 4-regular, bipartite graph with 12 vertices and 24 edges.

A bipartition of this graph is {X, Y} where $X = \{(u_i, v_{2j-1})$: i= 1, 2, 3; j = 1, 2} and

$Y = \{(u_i, v_{2j}) : i = 1,2,3 ; j = 1,2\}$.

Further, the vertex (u_i, v_{2j-1}) of X is adjacent with $(u_{i'}, v_{2j})$ and $(u_{i'}, v_{2j-2})$ where $1 \le i \ne i^1 \le 3$ and $1 \le j \le 2$ with the convention $v_0 = v_4$.

A diagrammatic representation of $C_3 \wedge C_4$ (i.e $C_4 \Lambda C_3$) is

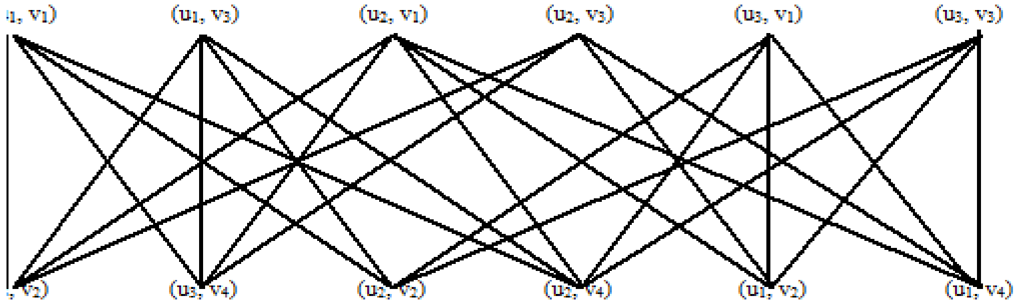

Figure - 3

In the usual notation, we have

I (i) (a) the edge (u_1, v_1) (u_2, v_2):

$$N_1\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1),\\ (u_2,v_1)\ (u_2,v_3)\ (u_2,v_4),\\ (u_3,v_2)\ (u_3,v_4)\},\end{array}$$

$$N_2\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2)\ (u_1,v_3)\ (u_1,v_4),\\ (u_2,v_2),\\ (u_3,v_1),\ (u_3,v_3)\}.\end{array}$$

This implies $n_1[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_4)]$= 1+3+2 = 6 and

$n_2[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_4)]$=3+1+2 = 6.

I (i) (b) the edge (u_1, v_1) (u_3, v_2):

$$N_1\left[(u_1,v_1)(u_3,v_2)\,|\,(C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1),\\ (u_2,v_2)\ (u_2,v_4)\ ,\\ (u_3,v_1)\ (u_3,v_3)\ (u_3,v_4)\},\end{array}$$

$$N_2\left[(u_1,v_1)(u_3,v_2)\,|\,(C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2)\ (u_1,v_3)\ (u_1,v_4),\\ (u_2,v_1)\ (u_2,v_3),\\ (u_3,v_2)\}.\end{array}$$

(Observe that these are obtained by interchanging u_2 and u_3 in I (i) (a))

I (i) (c) The edge (u_1,v_1) (u_2,v_4):

$$N_1\left[(u_1,v_1)(u_2,v_4)\,|\,(C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1),\\ (u_2,v_1)\ (u_2,v_2)\ (u_2,v_3),\\ (u_3,v_2)\ (u_3,v_4)\},\end{array}$$

$$N_2\left[(u_1,v_1)(u_2,v_4)\,|\,(C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2)\ (u_1,v_3)\ (u_1,v_4),\\ (u_2,v_4),\\ (u_3,v_1)\ (u_3,v_3)\}.\end{array}$$

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(Since v_2 and v_4 are adjacent with the same vertices v_1 and v_3, the above sets are obtained by interchanging v_2 and v_4 in I(i) (a)

I (i) (d) The edge $(u_1, v_1)\ (u_3, v_4)$

$$N_1\left[(u_1,v_1)(u_3,v_4) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_1),\ (u_2,v_2)\ (u_2,v_4),\ (u_3,v_1)\ (u_3,v_2)\ (u_3,v_3)\},$$

$$N_2\left[(u_1,v_1)(u_2,v_2) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_2)\ (u_1,v_3)\ (u_1,v_4),\ (u_2,v_1)\ (u_2,v_3),\ (u_3,v_4)\}.$$

(These sets are obtained by interchanging u_2 and u_3 in I (i) (c))

I (ii) (a) The edge $(u_1, v_3)\ (u_2, v_2)$:

$$N_1\left[(u_1,v_3)(u_2,v_2) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_3),\ (u_2,v_1)\ (u_2,v_3)\ (u_2,v_4),\ (u_3,v_2)\ (u_3,v_4)\},$$

$$N_2\left[(u_1,v_3)(u_2,v_2) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_1)\ (u_1,v_2)\ (u_1,v_4),\ (u_2,v_2),\ (u_3,v_1)\ (u_3,v_3)\}.$$

(These sets are obtained by interchanging v_1 and v_3 in I (i) (a))

I (ii) (b) The edge $(u_1, v_3)\ (u_3, v_2)$:

$$N_1\left[(u_1,v_3)(u_3,v_2) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_3),\ (u_2,v_2)\ (u_2,v_4),\ (u_3,v_1)\ (u_3,v_3)\ (u_3,v_4)\},$$

$$N_2\left[(u_1,v_3)(u_3,v_2) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_1)\ (u_1,v_2)\ (u_1,v_4),\ (u_2,v_1)\ (u_2,v_3),\ (u_3,v_2)\}.$$

(These sets are obtained from the above by interchanging u_2 and u_3).

I (ii) (c) The edge $(u_1, v_3)\ (u_2, v_4)$:

$$N_1\left[(u_1,v_3)(u_2,v_4) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_3),\ (u_2,v_1)\ (u_2,v_2)\ (u_2,v_3),\ (u_3,v_2)\ (u_3,v_4)\},$$

$$N_2\left[(u_1,v_3)(u_2,v_4) \mid (C_3 \wedge C_4)\right] = \{(u_1,v_1)\ (u_1,v_2)\ (u_1,v_4),\ (u_2,v_4),\ (u_3,v_1)\ (u_3,v_3)\}.$$

(These sets are obtained by interchanging v_2 and v_4 in I (ii) (a)).

I (ii) (d) The edge $(u_1, v_3)\ (u_3, v_4)$:

$$N_1\left[(u_1,v_3)(u_3,v_4) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_3),\\ (u_2,v_2)\ (u_2,v_4),\\ (u_3,v_1)\ (u_3,v_2)\ (u_3,v_3)\},\end{array}$$

$$N_2\left[(u_1,v_3)(u_3,v_4) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_2)\ (u_1,v_4),\\ (u_2,v_1)\ (u_2,v_3),\\ (u_3,v_4)\}.\end{array}$$

(These sets are obtained from the above by interchanging u_2 and u_3).

II (i) (a) The edge $(u_2, v_1)\ (u_1, v_2)$:

$$N_1\left[(u_2,v_1)(u_1,v_2) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_3)\ (u_1,v_4),\\ (u_2,v_1),\\ (u_3,v_2)\ (u_3,v_4)\},\end{array}$$

$$N_2\left[(u_2,v_1)(u_1,v_2) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2),\\ (u_2,v_2)\ (u_2,v_3)\ (u_2,v_4),\\ (u_3,v_1)\ (u_3,v_3)\}.\end{array}$$

(These sets are obtained from I(i) (a) by interchanging u_1 and u_2).

II (i) (b) The edge $(u_2, v_1)\ (u_3, v_2)$:

$$N_1\left[(u_2,v_1)(u_3,v_2) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2)\ (u_1,v_4),\\ (u_2,v_1),\\ (u_3,v_1)\ (u_3,v_3)\ (u_3,v_4)\},\end{array}$$

$$N_2\left[(u_2,v_1)(u_3,v_2) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_3),\\ (u_2,v_2)\ (u_2,v_3)\ (u_2,v_4),\\ (u_3,v_2)\}.\end{array}$$

(These sets are obtained from the above by interchanging u_1 and u_3).

II (i) (c) The edge $(u_2, v_1)\ (u_1, v_4)$:

$$N_1\left[(u_2,v_1)(u_1,v_4) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_2)\ (u_1,v_3),\\ (u_2,v_1),\\ (u_3,v_2)\ (u_3,v_4)\},\end{array}$$

$$N_2\left[(u_2,v_1)(u_1,v_4) \mid (C_3 \wedge C_4)\right] = \begin{array}{l}\{(u_1,v_4),\\ (u_2,v_2)\ (u_2,v_3)\ (u_2,v_4),\\ (u_3,v_1)\ (u_3,v_3)\}.\end{array}$$

(These sets are obtained from II(i) (a) by interchanging v_2 and v_4).

II (i) (d) The edge $(u_2, v_1)\,(u_3, v_4)$:

$$N_1\left[(u_2, v_1)(u_3, v_4) \,|\, (C_3 \wedge C_4)\right] = \begin{array}{l} \{(u_1, v_2)\,(u_1, v_4), \\ (u_2, v_1), \\ (u_3, v_1)\,(u_3, v_2)\,(u_3, v_3)\}, \end{array}$$

$$N_2\left[(u_2, v_1)(u_3, v_4) \,|\, (C_3 \wedge C_4)\right] = \begin{array}{l} \{(u_1, v_1)\,\,(u_1, v_3), \\ (u_2, v_2)\,(u_2, v_3)\,(u_2, v_4), \\ (u_3, v_4)\}. \end{array}$$

(These sets are obtained from the above by interchanging u_1 and u_3).

III (i) (a) The edge $(u_3, v_1)\,(u_1, v_2)$:

$$N_1\left[(u_3, v_1)(u_1, v_2) \,|\, (C_3 \wedge C_4)\right] = \begin{array}{l} \{(u_1, v_1)\,(u_1, v_3)\,(u_1, v_4), \\ (u_2, v_2)\,(u_2, v_4), \\ (u_3, v_1)\}, \end{array}$$

$$N_2\left[(u_3, v_1)(u_1, v_2) \,|\, (C_3 \wedge C_4)\right] = \begin{array}{l} \{(u_1, v_2), \\ (u_2, v_1)\,(u_2, v_3), \\ (u_3, v_2)\,\,(u_3, v_3)\,(u_3, v_4)\}. \end{array}$$

(These sets are obtained from II(i) (a) by interchanging u_2 and u_3).

III (i) (b) The edge $(u_3, v_1)\,(u_2, v_2)$:

$$N_1\left[(u_3, v_1)(u_2, v_2) \,|\, (C_3 \wedge C_4)\right] = \begin{array}{l} \{(u_1, v_2)\,(u_1, v_4), \\ (u_2, v_1)\,\,(u_2, v_3)\,(u_2, v_4), \\ (u_3, v_1)\}, \end{array}$$

$$N_2\left[(u_3, v_1)(u_2, v_2) \,|\, (C_3 \wedge C_4)\right] = \begin{array}{l} \{(u_1, v_1)\,(u_1, v_3), \\ (u_2, v_2), \\ (u_3, v_2)\,\,(u_3, v_3)\,(u_3, v_4)\}. \end{array}$$

(These sets are obtained from the above by interchanging u_1 and u_2).

III (i) (c) The edge $(u_3, v_1)\,(u_1, v_4)$

$$N_1\left[(u_3, v_1)(u_1, v_4) \,|\, (C_3 \wedge C_4)\right] = \begin{array}{l} \{(u_1, v_1)\,(u_1, v_2)\,(u_1, v_3), \\ (u_2, v_2)\,\,(u_2, v_4), \\ (u_3, v_1)\}, \end{array}$$

$$N_2\left[(u_3,v_1)(u_1,v_4)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_4),\\ (u_2,v_1)\ (u_2,v_3),\\ (u_3,v_2)\ \ (u_3,v_3)\ (u_3,v_4)\}.\end{array}$$

(These sets are obtained from III (i) (a) by interchanging v_2 and v_4).

III (i) (d) The edge $(u_3,v_1)\ (u_2,v_4)$:

$$N_1\left[(u_3,v_1)(u_2,v_4)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2)\ (u_1,v_4),\\ (u_2,v_1)\ (u_2,v_2)\ (u_2,v_3),\\ (u_3,v_1)\},\end{array}$$

$$N_2\left[(u_3,v_1)(u_2,v_4)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_3),\\ (u_2,v_4),\\ (u_3,v_2)\ \ (u_3,v_3)\ (u_3,v_4)\}.\end{array}$$

(These sets are obtained from the above by interchanging u_2 and u_1).

III (ii) (a) The edge $(u_3,v_3)\ (u_1,v_2)$:

$$N_1\left[(u_3,v_3)(u_1,v_2)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ \ (u_1,v_3)\ (u_1,v_4),\\ (u_2,v_2)\ (u_2,v_4),\\ (u_3,v_3)\},\end{array}$$

$$N_2\left[(u_3,v_3)(u_1,v_2)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2),\\ (u_2,v_1)\ (u_2,v_3),\\ (u_3,v_1)\ (u_3,v_2)\ \ (u_3,v_4)\}.\end{array}$$

(These sets are obtained from III (i) (a) by interchanging v_1 and v_3).

III (ii) (b) The edge $(u_3,v_3)\ (u_2,v_2)$:

$$N_1\left[(u_3,v_3)(u_2,v_2)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2)\ (u_1,v_4),\\ (u_2,v_1)\ (u_2,v_3)\ (u_2,v_4),\\ (u_3,v_3)\},\end{array}$$

$$N_2\left[(u_3,v_3)(u_2,v_2)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_3),\\ (u_2,v_2),\\ (u_3,v_1)\ (u_3,v_2)\ \ (u_3,v_4)\}.\end{array}$$

(These sets are obtained from the above by interchanging u_1 and u_2).

III (ii) (c) The edge $(u_3,v_3)\ (u_1,v_4)$:

$$N_1\left[(u_3,v_3)(u_1,v_4)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_2)\ (u_1,v_3),\\ (u_2,v_2)\ (u_2,v_4),\end{array}$$

$(u_3,v_3)\}$,

$$N_2\left[(u_3,v_3)(u_1,v_4)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_4),\\ (u_2,v_1)\ (u_2,v_3),\\ (u_3,v_1)\ (u_3,v_2)\ \ (u_3,v_4)\}.\end{array}$$

(These sets are obtained from III(ii) (a) by interchanging v_2 and v_4).

III(ii) (d) The edge $(u_3,v_3)\ (u_2,v_4)$:

$$N_1\left[(u_3,v_3)(u_2,v_4)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_2)\ (u_1,v_4),\\ (u_2,v_1)\ (u_2,v_2)\ (u_2,v_3),\\ (u_3,v_3)\},\end{array}$$

$$N_2\left[(u_3,v_3)(u_2,v_4)\,|\,(C_3\wedge C_4)\right] = \begin{array}{l}\{(u_1,v_1)\ (u_1,v_3),\\ (u_2,v_4),\\ (u_3,v_1)\ (u_3,v_2)\ \ (u_3,v_4)\}.\end{array}$$

(These sets are obtained from the above by interchanging u_1 and u_2).

Thus, we observe that for any edge e of $C_3\wedge C_4$, n_1[e | $(C_3\wedge C_4)$] = 6 = n_2 [e | $(C_3\wedge C_4)$].

Since, there are 24 edges, we get that $Sz(C_3 \wedge C_4) = \sum_{e\in E(C_3\wedge C_4)} n_1(e).n_2(e)$

$= (6.\ 6)\ |E(C_3 \wedge C_4)|$

$= 36 \times 24 = 864 = 2(3 \times 4)\left(3\left[\frac{4}{2}\right]\right)^2 = 108(2)^3$

We extend this theorem by replacing C_4 by C_{2n} ($n \geq 3$).

Theorem 3.5: For the disjoint graph C_3, C_{2n} ($n \geq 3$), $Sz(C_3 \wedge C_{2n}) = 2(3 \times 2n)\left(3\left[\frac{2n}{2}\right]\right)^2 = 108\, n^3$.

Proof: In the usual notation, Let $V(C_3) = \{u_1, u_2, u_3\}$ and $V(C_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$.

Now, $V(C_3 \wedge C_{2n}) = \{(u_i, v_j) : i = 1,2,3\ ;\ j = 1, 2, \ldots, 2n\}$.

Clearly, $(C_3 \wedge C_{2n})$ is a simple, connected, 4-regular graph with 6n vertices and so with 12n edges. Further it is a bipartite graph with a bipartition {X, Y}, where $\{(u_i, v_{2j-1}) : i = 1,2,3\ ;\ j = 1,2, \ldots, n\}$ and $Y = \{(u_i, v_{2j}) : i = 1,2,3\ ;\ j = 1,2, \ldots, n\}$.

So $|X| = |Y| = 3n$.

Further, in this graph the vertex (u_i, v_{2j-1}) is adjacent with $(u_{i'}, v_{2j})$ and $(u_{i'}, v_{2j-2})$, for i = 1, 2, 3, $i' \in \{1, 2, 3\} - \{i\}$, j = 1, 2, …, n with the convention $v_0 = v_{2n}$.

Since v_{2j-1} is adjacent with v_{2j-1} & v_{2j-3} and v2j is adjacent with v_{2j-1} & v_{2j+1} in C_{2n}, the sets $N_k\,[(u_i, v_{2j-1})\,(u_{i'}, v_{2j-2})]$ can be obtained from $N_k\,[(u_i, v_{2j-1})\,(u_{i'}, v_{2j})]$ by interchanging v_{2j} & v_{2j-2} and v_{2j+1} & v_{2j-3} with the convention $v_{-1} = v_{2n-1}$ and $v_{2n+1} = v_1$.

Hence follows that for any edges e_1, e_2 of this graph, $N_1(e_1) = N_1(e_2)$ and $N_2(e_1) = N_2(e_2)$.

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So we calculate these numbers for the edge (u_1, v_1), (u_2, v_2).

$N_1\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3\wedge C_{2n})\right]$

$= (u_1,v_1)\} \bigcup \{ (u_1,v_j) : j = n+2\ldots 2n-1\} \bigcup \{ (u_2,v_1),\ (u_2,v_3) \} \bigcup \{ (u_2,v_j) : j = n+2, \ldots, 2n\}$

$\bigcup \{ (u_3,v_2) \} \bigcup \{ (u_3,v_j) : j = n+2\ldots 2n\}$.

$(\Rightarrow n_1\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3\wedge C_{2n})\right] = 1+ (2n-n-2) + 2+ (2n+1-n-2) + 1+(2n+1-n-2)$

$= (n-1) + (2+n-1) + n = 3n)$

and

$N_2\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3\wedge C_{2n})\right] =$

$\{ (u_1,v_j) : j=2, \ldots, (n+1)\} \bigcup \{ (u_1,v_{2n}) \} \bigcup \{ (u_2,v_2) \} \bigcup \{ (u_2,v_j) : j = 4, \ldots, n+1\} \bigcup \{ (u_3,v_1) \} \bigcup \{ (u_3,v_j) : j = 3, \ldots, (n+1)\}$

$(\Rightarrow n_2\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3\wedge C_{2n})\right] = (n+2-2)+1+1+(n+2-4)+1+(n+2-3)$

$= (n+1) + (1+n-2) + (1+n-1) = 3n)$.

Thus for any edge of ($C_3 \wedge C_{2n}$), $n_1[e \,|\, (C_3 \wedge C_{2n})] = 3n = n_2[e \,|\, (C_3 \wedge C_{2n})]$.

$$\therefore Sz(C_3 \wedge C_{2n}) = \sum_{e\in E(C_3\wedge C_4)} n_1(e).n_2(e)$$

$$= |(C_3 \wedge C_{2n})|\,(3n)^2 = (12n)(3n)^2.$$

$$= (2)\,(3)(2n)\left(3\left[\frac{2n}{2}\right]\right)^2$$

$$= 108n^3.$$

Corollary 3.6: Taking n = 2 in Th.(3.5), we get the result given in Th.(3.4) (Here the convention is that the set $\{a_j : j = r, \ldots, s\} = \varnothing$ if $s < r$).

Theorem 3.7: For any integer $n \geq 2$, $Sz(C_3 \wedge C_{2n+1}) = 2(3)(2n+1)\ \{3.\left[\frac{2n+1}{2}\right]\}^2$

$= 54(2n+1)$ n2.

Proof: By results (3.1) & (3.2), it follows that $C_3 \wedge C_{2n+1}$ is a simple, connected and 4-regular graph with 3(2n+1) vertices and 6(2n+1) edges. Further it is not bipartite.

Let $V(C_3) = \{u_1, u_2, u_3\}$ and $V(C_{2n+1}) = \{v_1, v_2, \ldots, v_{2n+1}\}$. The edge set of this graph is

$\{ (u_1,v_{2j-1})\ (u_i,v_{2j-2}) : i = 2, 3\,;\ j \in N_{n+1}\} \bigcup \{ (u_1,v_{2j-1})\ (u_i,v_{2j}) : i = 2, 3;\ j \in N_{n+1}\} \bigcup \{ (u_1,v_{2j})\ (u_i,v_{2j-1}) :$ $i=2, 3;\ j \in N_n\} \bigcup \{ (u_1,v_{2j})\ (u_i,v_{2j+1}) : i=2, 3;\ j \in N_n\} \bigcup \{ (u_2,v_j)\ (u_3,v_{j'}) \}$: $j \in N_{2n+1}$ and $j^1 = j-1$ or $j+1$ (with the convention $v_0 = v_{2n+1}$, $v_{2n+2} = v_1$).

As in the proof of Th.(3.5) it follows that for any edge e of $C_3 \wedge C_{2n+1}$, we get that $n_1(e)$ as well as $n_2(e)$ are the same number.

So we calculate these sets for the edge (u_1, v_1) (u_2, v_2).

$N_1\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3\wedge C_{2n})\right] =$

$\{ (u_1,v_1) \} \bigcup \{ (u_1,v_j) : j = n+3, \ldots, 2n\} \bigcup \{ (u_2,v_1),\ (u_2,v_3) \} \bigcup \{ (u_2,v_j) : j = n+3 \ldots, 2n+1\}$

$\bigcup \{ (u_3,v_2) \} \bigcup \{ (u_3,v_j) : j = n+3, \ldots, (2n+1)\}$.

($\Rightarrow$ $n_1\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_{2n})\right]$ = (n-1) + (n+1) + n = 3n)
and
$N_2\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_{2n})\right]$ = { (u_1,v_j) : j=2… (n+1)} $\bigcup$ { (u_1,v_{2n+1}) } $\bigcup$ { (u_2,v_2) } $\bigcup$ { (u_2,v_j) : j = 4, …, (n+1)} $\bigcup$ { (u_3,v_1) } $\bigcup$ { (u_3,v_j) : j = 3, …, (n+1)}

($\Rightarrow$ $n_2\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_{2n})\right]$ = (n+1) + (n-1) + n = 3n).

Observe that the vertices (u_i, v_{2n+2}), i=1, 2 are at the same distance from (u_1, v_1)as well as (u_2, v_2). Hence these vertices are not entering either in $N_1\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_{2n})\right]$ or in $N_2\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_{2n})\right]$. So is the case with the remaining edges as well.

Thus $n_1\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_{2n})\right]$ = 3n = $n_2\left[(u_1,v_1)(u_2,v_2)\,|\,(C_3 \wedge C_{2n})\right]$ and this is true for all the (12n + 6) edges of $C_3 \Lambda C_{2n+1}$. So

$$Sz(C_3 \Lambda C_{2n+1}) = \sum_{e \in E(C_3 \wedge C_{2n+1})} n_1(e).n_2(e)$$
$$= (12n+6)\,(3n)^2$$
$$= 2(3)(2n+1)\,\{3.\left[\frac{2n+1}{2}\right]\}^2 = 54(2n+1)n^2.$$

Corollary 3.9: For the graph $C_3 \Lambda C_5$, $Sz(C_3 \Lambda C_5) = 1080 = 54(4+1)2^2$.

A diagrammatic representation of $C_3 \Lambda C_5$ is given under.

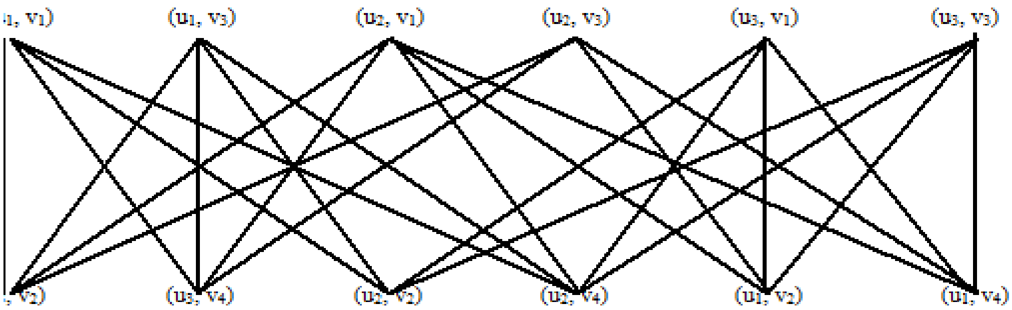

Figure - 3

OPEN PROBLEM 3.10: If m, n are integers $\geq 4$and one of them is odd, then to find the Szeged index of $C_m \Lambda C_n$.

4. RESULTS RELATED TO $P_m \wedge P_n$ (m, n being positive integers):

When atleast one of m, n is one, clearly $P_m \wedge P_n$ is an empty graph. So, we consider m, n ≥ 2.

Further, if m = n = 2, then $P_2 \wedge P_2 = K_2 \wedge K_2$. This is discussed in **§** 2. So, we consider the case when m, n≥ 2 and atleast one of m, n is ≥ 3.

We assume that P_m, P_n are disjoint and $V(P_m) = \{u_1, u_2, …, u_m\}$ and $V(P_n) = \{v_1, v_2, …, v_n\}$.

Remarks 4.1 [4]:

a) P_m, P_n are simple, finite graphs with m, n vertices respectively and further

$$d_{P_m \Lambda P_n}\{(u_i, v)\} = \begin{cases} 0 \text{ if } i=1 \text{ or } m \text{ and } j= 1 \text{ or } n, \\ 2 \text{ if } i=1 \text{ or } m \text{ and } 2 \leq j \leq n\text{-}1 \text{ or } 2 \leq i \leq m\text{-}1 \text{ and } j = 1 \text{ or } n, \\ 4 \text{ if } 2 \leq i \leq m\text{-}1 \ \& \ 2 \leq j \leq n\text{-}1. \end{cases}$$

b) Both P_m and P_n are connected graphs and none of them contains an odd cycle (infact, any cycle) by result (1.7) it follows that $P_m \wedge P_n$ is a disconnected graph with exactly two components (further each component is bipartite).

Result 4.2 [4]: For $n \geq 3$, $P_2 \Lambda P_n$ (which is isomorphic to $P_n \Lambda P_2$) is a graph which is a union of two (vertex) disjoint paths of length (n-1) each. So, by the Consequence (1.3)(b), it follows that Szeged index of each component is $(n-1)(n^2 - 2n)/6 = n(n-1)(n-2)/6$.

Remark 4.3: As the graph $P_m \wedge P_n$ is disconnected, the problem of discussing its Szeged index does not arise when both m, $n \geq 3$.

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